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Application of interval fields to geometric output quantities of Finite Element models

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Abstract: In uncertainty quantification, interval arithmetics provide an appropriate procedure when little knowledge is available on the nature of the probability distribution of uncertain or imprecise quantities. This commonly occurs in engineering applications due to subjective knowledge or incomplete availability of test data. Intervals are by definition unable to take into account dependent input and output quantities, which forces the assumption of independency when applying them. This is a severe limitation on the accuracy of the analysis as dependency is always present to some extent. The concept of interval fields (IF) provides a solution by defining non-deterministic fields using interval parameters. Applying interval fields can capture dependency among input or output parameters, while still using a possiblistic concept, thereby keeping the advantages of such an approach. This paper introduces a method to construct an output interval field based on a surrogate model of the input-output relation. The field functions are drawn directly from the coefficients of this model, leading to an output interval field with independent interval variables and deterministic field functions.

Keywords: possiblistic analysis, interval analysis, interval fields

NOMENCLATURE

x,y,r,$\mathbf{r}$ = geometric coordinate indications. Bold letters indicate coordinate vectors in 2D or 3D. subscript indicate corresponding domain, m.
Ω = domain over which a geometric input parameter is defined.
Υ = domain over which a geometric input parameter is defined.
ϕ_i = field functions of an input field description.
μ_i = field functions part of an output field description corresponding to main effects.
ν_i = field functions part of an output field description corresponding to interactions.
$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$: gradient of f.
superscript I = interval parameter, or an interval field if the spatial domain is explicitly mentionned.
w= vertical deflection, m

INTRODUCTION

In engineering applications, some degree of uncertainty is always present. To ensure a certain design objective is met, having knowledge of this uncertainty is very important. Uncertainty analysis (UA) has therefore established itself as an extensive field of research, aiming to accurately identify, quantify and assess uncertainty wherever it is present.

Often numerical models are used to acquire knowledge on the application's behaviour, partially substituting the use of expensive physical prototypes. The use of UA in numerical models can provide valuable information on the variation of this behaviour. In particular, uncertainty quantification (UQ) is that part of uncertainty analysis that attempts to quantify the uncertainty on the model's relevant output, given information on the uncertainty on the model's input and/or parameters. Based on the nature of this information, two approaches can be distinguished. *Probabilistic UQ* uses stochastic parameters (such as mean and deviation) to represent uncertainty at both input and output and attempts to quantify these. The Monte Carlo Sampling method is most commonly used to accomplish this. However, use of this approach requires extensive knowledge on the input uncertainty in the form of probability distributions and correlation between parameters. The absence of complete knowledge enforces assumptions on the shape of the probability distribution which are often not sufficiently supported by the data available. Next to this, *possibilistic UQ* makes abstraction of the nature of the probability of in- and output, and focuses exclusively on the possible range of these quantities. Given information on the range of the input quantities, the ranges of the relevant output quantities have to be found. The range of a quantity can be simply modelled by an *interval* bounded by a lower and upper value. This requires less extensive knowlegde on the uncertainty and provides a lower threshold to perform UA. In addition, knowledge of the bounds of certain output quantities often proves to be valuable information. This paper focuses on the application of possibilistic UQ in the field of structural dynamics. The Finite Element (FE) method is commonly used to construct models for numerical analysis.

Many parameters of such an FE model have a geometric aspect among them, such as component thicknesses, stiffness parameters and boundary conditions. The modelling process discretises these quantities to the level of the elements. Often these quantities can vary in the spatial domain, which for UA is important to acknowledge. However, physics dictate that the values in different elements are correlated in space. In probabilistic UQ, the theory of random fields (Vanmarcke, 1993) can be used to take into account quantities correlated in space. However, a well-established theory with similar properties is lacking in possibilistic UQ. Because of their simple definition, intervals are incapable of taking into account interdependent quantities.

As an answer to this, Moens et al. (2009) introduced the *interval field* to account for dependent variables in possiblistic context. Verhaeghe et al. (2013) elaborated the concept and applied it to a static FE problem. The objective of this paper is to introduce a new definition of the interval field specifically tailored for output quantities of static FE analysis, and apply it to a simple static FE problem with uncertain geometric parameters. The analysis will use a formerly introduced interval field at the input/parameter side to simulate uncertainty at this level. The objective of the interval field at the output side will be to accurately capture the resulting uncertainty at the output, *taking into account the dependency in the spatial domain*.

THE EXPLICIT INTERVAL FIELD

mathematical equation

Much like the random field, an interval field attempts to capture field quantities in a non-deterministic way. A random field is a function $H(\mathbf{r}, \theta_i), \mathbf{r} \in \Omega : \mathbb{R}^n$ of certain random parameters θ_i over a certain domain Ω. According to the Karhùnen-Loève Expansion (KLE) (Ghanem, Spanos, 2003), such a field can be written as

$$H(\mathbf{r}, \theta_i) = \sum_{i=1}^{\infty} \theta_i \cdot \phi(\mathbf{r}). \tag{1}$$

In Eq. 1, $\phi(\mathbf{r})$ are shape functions drawn from the definition of the correlation function of the field. θ_i are random uncorrelated variables. As a result, the shape functions incorporate information on the dependency, whereas the coefficients represent the variability. The *explicit interval field* follows a similar definition, where the random coefficients are replaced by interval parameters α_i^I. This leads to:

$$u^I(\mathbf{r}) = H(\mathbf{r}, \alpha_i^I) = \sum_{i=1}^{\infty} \alpha_i^I \cdot \phi(\mathbf{r}). \tag{2}$$

For the KLE, it has been proven that the random coefficients are uncorrelated, even fully independent in the case of gaussian distributions. However, simply changing the coefficients to intervals does not guarantee independence in any way, even if the basis functions from the KLE are used.

Dependency within the field

Like the random field, the shape functions of an explicit interval field are deterministic and incorporate the dependency. However, in possibilistic context, the correlation is of little meaning as it is derived from the conditional probability function within the field. As possibilistic analysis makes abstraction of probability distributions, a different measure of dependency will be used. A straightforward measure of dependency that does not involve correlation is the maximum allowable gradient of the field variable. If each element can take its value completely independent from the other, maximum and minimum values can theoretically occur in adjacent elements, giving a possible derivative equal to infinity. Instead, if the maximum derivative is zero, the entire field has to have one constant value, representing full dependence. Previous work from the authors (Imholz et al., 2015) discussed an explicit interval field definition for the input/parameter side of an FE-analysis. The measure of dependency in this case is the *maximum norm of the gradient of the field*, $\|\nabla u(\mathbf{r})\|_{max} = \sqrt{\sum_{i=1}^{n} (\frac{\partial u}{\partial r_i})^2}\Big|_{max}$. This leads to a set of locally defined shape functions (illustrated in Fig. 1) placed in each element of the FE mesh, which is illustrated in Fig. 2. The following section will discuss the construction of the output field based on this explicit interval field definition of the input.

GENERAL MODELS OF CONTINUOUS FIELD VARIABLES

Usually, a specific output function $y = f(u_1, ..., u_i, ..., u_n)$ is of interest that expresses the behaviour of the output as a function of the inputs u_i. Often, this relationship is represented by a certain model that approximates the real physics. In structural dynamics, these are often Finite Element models. In industrial applications, FE models can grow to very large dimensions with millions of elements. Evaluation of such a model may be very time-consuming. In this case a *surrogate model* can be constructed that approximates the accurate, high-fidelity model. The surrogate model usually shows less accuracy, but is much faster to evaluate than the underlying model. Commonly used approximating functions are polynomial functions, exponential functions or sine expansions. For a discrete problem, the general expression of a

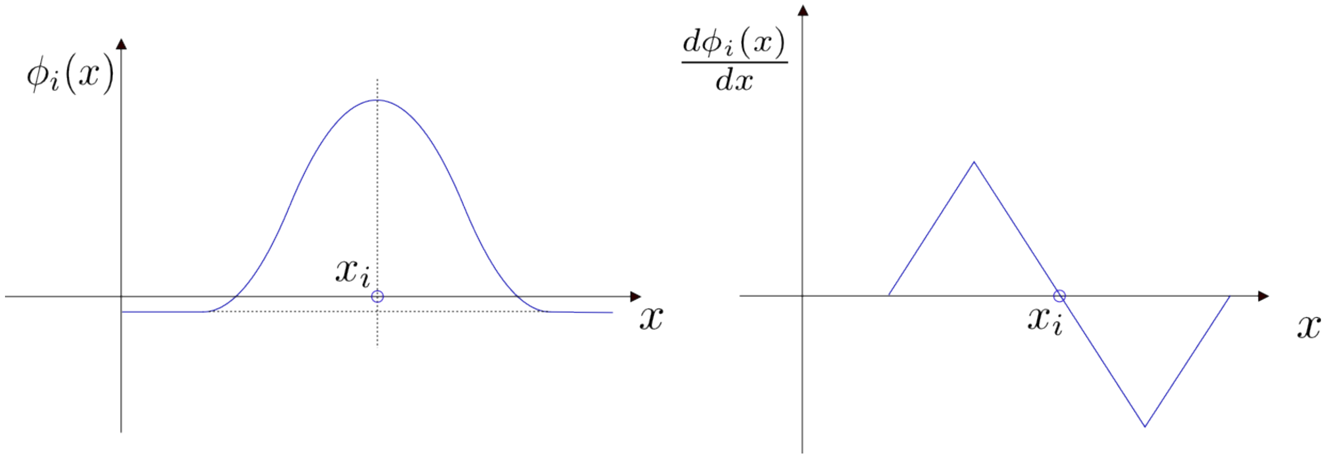

Figure 1 – Definition of a basis function and its first derivative that corresponds to the first derivative as measure for the dependency within the field.

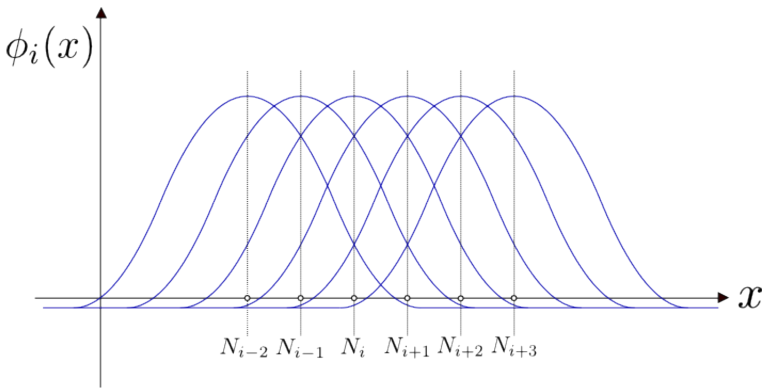

Figure 2 – One basis function is placed in each element of the discretised field mesh (1D example)

numerical model can be:

$$y = y_0 + \sum_{i=1}^{n} f(u_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} g(u_i, u_j) + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} h(u_i, u_j, u_k) + ... \tag{3}$$

In Eq. 3, function f represents the *main effect* of each single parameter on the output. The functions g and h represent input parameter *interactions* and express the combined effect of two or more varying inputs on the value of the output. In principle, up to n-way interaction terms are part of a full model, but usually, higher order interactions are of decreasing importance to the output, so they can commonly be ignored from a certain point on.

In this paper, we do not consider pure discrete parameters. Instead, we consider continuous field variables $u(\mathbf{r}_\Omega), \mathbf{r}_\Omega \in \Omega : \mathbb{R}^3$ as input, defined over a certain physical domain Ω. In theory, fields are an infinitely large collection of discrete inputs (i.e. de value in each point of the domain), but obviously, such a description using is not applicable in practice. Moreover, since we consider physical field variables, the field $u(\mathbf{r}_\Omega)$ is a continuous function of $\mathbf{r}_\Omega$, which enables us to write $u(\mathbf{r}_\Omega)$ in terms of $\mathbf{r}_\Omega$ and some function parameters c_i, which e.g. can be the coefficients of a polynomial or exponential function in $\mathbf{r}_\Omega$. When applying this to the second term in Eq. 3, the summation becomes an integral, and the spatial coordinate can usually be integrated to obtain a new function $F(c_i)$:

$$\sum_{i=1}^{n} f(u_i) \rightarrow \int_\Omega f(\mathbf{r}_\Omega, c_i) d\mathbf{r}_\Omega = F(c_i) \tag{4}$$

For the interaction terms, a similar operation is possible. Again, the spatial coordinate can be integrated out, leaving only the function parameters c_i as variables of the new functions $G(c_i)$ and $H(c_i)$.

We will illustrate this for the specific case that u is a 1D field with spatial coordinate r that equals:

$$u(r) = \sum_{i=1}^{n} c_i \cdot \phi_i(r) \tag{5}$$

In Eq. 5, $\phi_i(r)$ are fixed basis functions, e.g. hermite polynomials or sine functions, c_i are the coefficients. Suppose a prior analysis showed that relationship of u to a specific output $y(\mathbf{r}_\Upsilon)$ defined as a field variable over the output domain Υ, is equal to $a_0 + a_1 u + a_2 u^2$. Then, the continuous model will become:

$$y(\mathbf{r}_\Upsilon) = a_0(\mathbf{r}_\Upsilon) + \int_\Omega a_1(\mathbf{r}_\Upsilon, r) \sum_{i=1}^{n} c_i \cdot \phi_i(r) dr + \int_\Omega a_2(\mathbf{r}_\Upsilon, r) \left(\sum_{i=1}^{n} c_i \cdot \phi_i(r) \right)^2 dr \tag{6}$$

$$= a_0(\mathbf{r}_\Upsilon) + \sum_{i=1}^{n} c_i \mu_i(\mathbf{r}_\Upsilon) + \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \nu_{ij}(\mathbf{r}_\Upsilon) \tag{7}$$

with

$$\mu_i(\mathbf{r}_\Upsilon) = \int_\Omega a_1(\mathbf{r}_\Upsilon, r) \phi_i(r) dr \tag{8}$$

$$\nu_{ij}(\mathbf{r}_\Upsilon) = \int_\Omega a_2(\mathbf{r}_\Upsilon, r) \phi_i(r) \phi_j(r) dr \tag{9}$$

In Eq. 7, we have turned the output y into a field description as well, by defining a domain Υ over which y is defined. The output domain Υ can be the same Ω, it can be a part of it or they can be completely separated.

TRANSITION TO INTERVAL FIELD

We now introduce the uncertainty into the model. Suppose the input exhibits geomertic uncertainty: the value in each point may vary. To add this to the analysis, we define the input field as an explicit interval field (eq. 2) by simply turning the coefficients into interval parameters:

$$u^I(r) = \sum_{i=1}^{n} c_i^I \cdot \phi_i(r) \tag{10}$$

Following the previous section the resulting output can be easily derived by simply adding the I-superscripts in Eq. 7 as well.

$$y^I(\mathbf{r}_\Upsilon) = a_0(\mathbf{r}_\Upsilon) + \sum_{i=1}^{n} c_i^I \mu_i(\mathbf{r}_\Upsilon) + \sum_{i=1}^{n}\sum_{j=1}^{n} c_i^I c_j^I \nu_{ij}(\mathbf{r}_\Upsilon) \tag{11}$$

We emphasize that the amount of field terms in Eq. 11 increases with the number of field terms in the input and the order of the polynomial relation between in and output. Therefore, the approach is only feasible for a limited amount of field functions at the input and for input-output relations of limited complexity.

The following section will bring these definitions into practice by observing a simple FE model with uncertain boundary conditions, which are propagated to obtain the uncertainty on the static response of the model.

APPLICATION EXAMPLE: A CLAMPED PLATE MODEL

definition of the input field

The interval field mentionned above will be applied to a model of a clamped steel plate measuring $Lx \times Ly$ with isotropic properties. The material properties of the plate itself (i.e. density, thickness and stiffness parameters) are assumed deterministic and constant over the entire plate. We assume a steel plate (Young's modulus equal to 210 GPa and density equal to $8100\frac{kg}{m^3}$) with dimensions 1m x 1m and thickness 5 mm. The plate is imperfectly clamped at one side and statically loaded at the opposite side causing a vertical deflection. In this model, the clamping is assumed imperfect and modelled through a geometrically varying clamping stiffness $K_\theta(x)$, with x the position along the clamped edge. The geometric variation is however limited by a *maximum allowable first derivative* $\frac{\partial K_\theta(x)}{\partial x}$, accounting for the dependency over the spatial domain. This requirement is converted into shape functions as depicted in Fig. 1, one of which is placed in each element along the clamped edge. This representation ensures that each element still has it's own degree of freedom to keep the dimension of the uncertainty intact. However, the interval parameters can be considered independent while still obeying the maximum derivative constraint. For the 10x10 element mesh that is used in this case, this leads to an input interval field with 10 field terms and 10 independent interval parameters - the maximum derivative constraint is automatically met through this choice of shape functions.

definition of the output field

The behaviour of the vertical displacement w as a function of a constant clamping stiffness can be approximated by a power function, which after taking the logarithm equals eq. 12:

$$\log(w_0 - w) = a_0 + a \cdot \log K_\theta \tag{12}$$

with w_0 the displacement for a perfect clamping condition (defined as a negative number). Transition to input and output fields leads to:

$$\log(w_0(\mathbf{r}_\Upsilon) - w(\mathbf{r}_\Upsilon)) = a_0(\mathbf{r}_\Upsilon) + \int_{x=0}^{L_x} a(\mathbf{r}_\Upsilon, x) \cdot \log K_\theta(x) dx + \iint_{x=0}^{L_x} b(\mathbf{r}_\Upsilon, x_i, x_j) \log K_\theta(x_i) \log K_\theta(x_j) dx_i dx_j \tag{13}$$

In Eq. 13, we manually added an interaction term. We will now apply a specific field at the input which will lead to a corresponding output interval field. Define the field $K_\theta^I(x)$ in the following way:

$$\log K_\theta^I(x) = q_0 + q_1^I \cdot x^2, \tag{14}$$

Putting this in Eq. 13 leads to:

$$\log(w_0^I(\mathbf{r}_\Upsilon) - w^I(\mathbf{r}_\Upsilon)) = \mu_0(\mathbf{r}_\Upsilon) + \ q_1^I \cdot \mu_1(\mathbf{r}_\Upsilon) + \left(q_1^I\right)^2 \cdot \nu_{11}(\mathbf{r}_\Upsilon) \tag{15}$$

with

$$\mu_i(\mathbf{r}_\Upsilon) = \int_{x=0}^{L_x} a_1(\mathbf{r}_\Upsilon, x) x^2 dx \tag{16}$$

$$\nu_{ij}(\mathbf{r}_\Upsilon) = \iint_{x=0}^{L_x} b(\mathbf{r}_\Upsilon, x_k, x_l) x_k^2 x_l^2 dx_k dx_l \tag{17}$$

The output domain $\mathbf{r}_\Upsilon$ is equal to the (x, y)-domain, with $0 \leq x \leq L_x$ and $0 \leq y \leq L_y$. Figure 3 shows the 3 fields that are part of this output field. We see that allowing for spatial variabily adds a mild quadratic term to the model which is not present in de basic relationship of Eq. 12. However, the linear term appears to be dominant.

Next, we will examine the output interval field for the input field based on the maximum derivative constraint. Suppose it is known that the clamping stiffness varies around a certain mean value, but that the derivative of the clamping stiffness to the spatial coordinate along the clamped edge never exceeds a certain boundary. In this case, the uncertainty on the input $K_\theta(x)$ is given by the following interval field:

$$\log K_\theta^I(x) = q_0 + \sum_{i=1}^{10} q_i^I \cdot \phi_i(x), \tag{18}$$

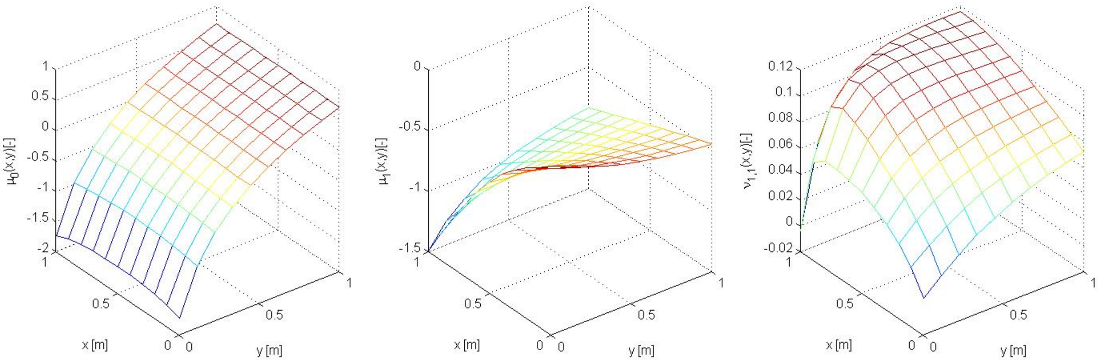

Figure 3 – Field functions $\mu_0(x,y)$, $\mu_1(x,y)$ and $\nu_{11}(x,y)$ propagated from the input field given in Eq. 14

$\phi_i(x)$ are piecewise second order polynomials (Fig. 1) that ensure a certain limit on the first derivative of the clamping stiffness is met. q_i^I are independent interval variables related to the local stiffness value in each element. If we introduce this in Eq. 12, the output interval field becomes:

$$\log(w_0(\mathbf{r}_\Upsilon) - w(\mathbf{r}_\Upsilon)) = \mu_0(\mathbf{r}_\Upsilon) + \sum_{i=1}^{10} q_i^I \cdot \mu_i(\mathbf{r}_\Upsilon) + \sum_{i=1}^{10}\sum_{j=i}^{10} q_i^I q_j^I \cdot \nu_{ij}(\mathbf{r}_\Upsilon) \tag{19}$$

with

$$\mu_i(\mathbf{r}_\Upsilon) = \int_{x=0}^{L_x} a_1(\mathbf{r}_\Upsilon, x)\phi_i(x)dx + 2q_0 \iint_{x=0}^{L_x} b(\mathbf{r}_\Upsilon, x_k, x_l)\phi_i(x_l)dx_k dx_l \tag{20}$$

$$\nu_{ij}(\mathbf{r}_\Upsilon) = \iint_{x=0}^{L_x} b(\mathbf{r}_\Upsilon, x_k, x_l)\phi_i(x_k)\phi_j(x_l)dx_k dx_l \tag{21}$$

As can be seen in Eq. 19, the total number of field functions amounts to 66, including 45 interaction terms, indicating the large contribution of these terms. From Fig. 4, we clearly see the symmetry of the model. The fields corresponding to interval parameters in points mirrored along the plane $x = 0.5$ should be mirrored as well. The field to the left and right indeed seem to be mirorred along the plane $x = 0.5$. By adjusting the width of the base functions $\phi_i(x)$, we can change the maximum allowable gradient to a lower value, and observe the effect on the output interval field. From comparing Fig. 4 and Fig. 5, we can see the effect of a lower maximum gradient: whereas the fields μ_i associated to main effects change only minorly, the fields ν_{ij} associated with interactions exhibit much smaller values, as can be expected. The main advantage of the field representation illustrated here is twofold:

1. Firstly, the field keeps the dependency present in the interval variables at the input side of the analysis. If these are chosen to be independent, the output interval parameters will be too, and a strict division between uncertainty (carried by the interval parameters) and dependency (carried by the field functions) is realised.

2. Secondly, the fields provide transparent information on the sensitivity of each point in the output domain to the different inputs. This is even more so for an explicit interval field with fixed basis functions that are not function of the value of the corresponding interval parameter. This characteristic allows the interval field to be used in early-on sensitivity analysis when not enough information is available for a thorough high-fidelity non-deterministic analysis.

CONCLUSION

The aim of this paper is to introduce a straightforward way of obtaining an output interval field given a certain input interval field with independent interval parameters. The approach is based on a surrogate model to replace an extensive and time-consuming model. The field functions are propagated using this surrogate model, while the interval variables are simply transferred to the output side. Thanks to this, independency of the interval parameters at the input is kept intact and the interval field does not introduce any conservatism when capturing the uncertain output region. Moreover, the fixed basis functions allow for rudimentary sensitivity analysis in an early design stage with limited data availability.

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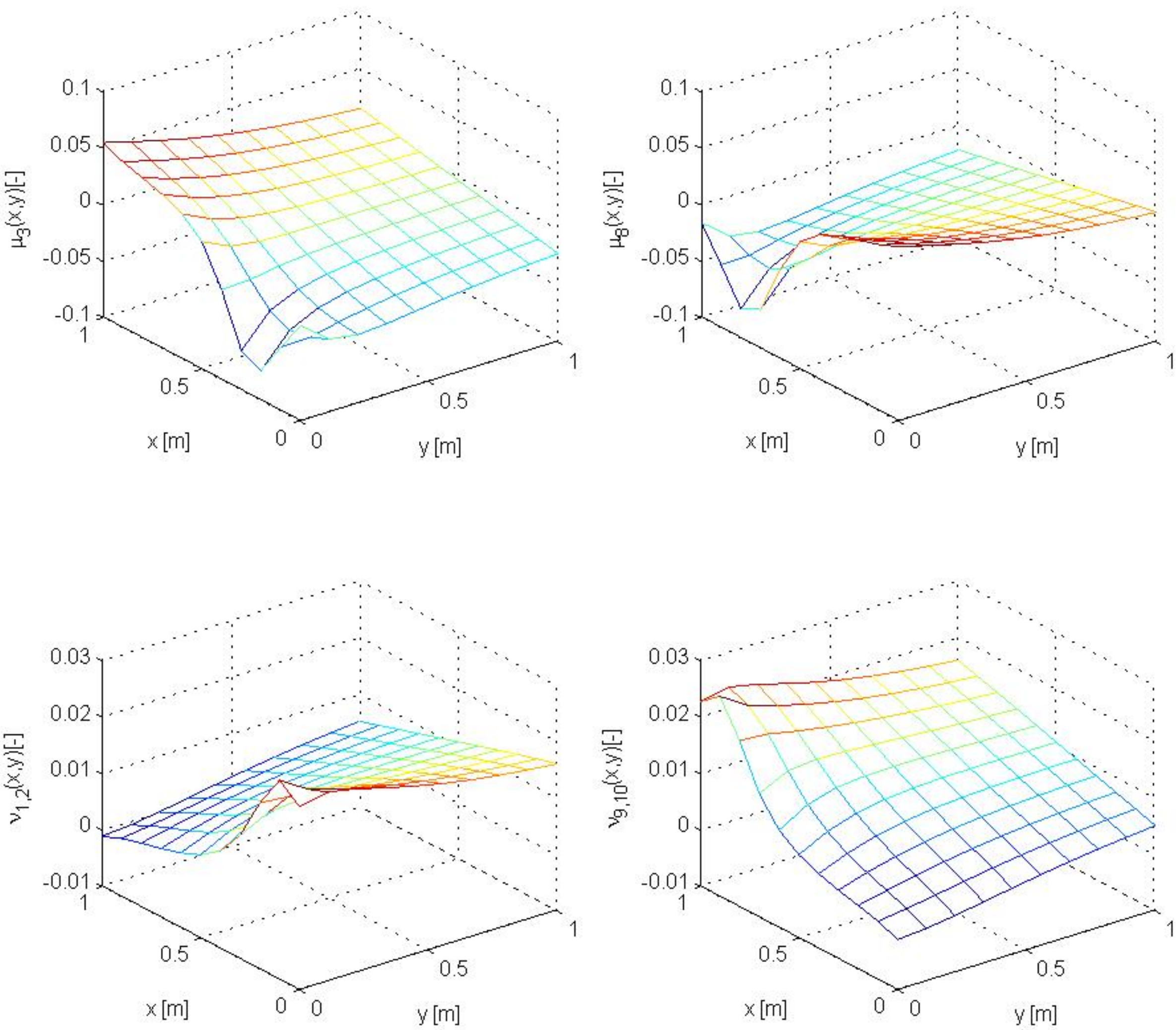

Figure 4 – some output field functions resulting from the input field of Eq. 18. From left to right and top to bottom: $\mu_3(x,y)$**,** $\mu_8(x,y)$**,** $\nu_{1,2}(x,y)$ **and** $\nu_{9,10}(x,y)$

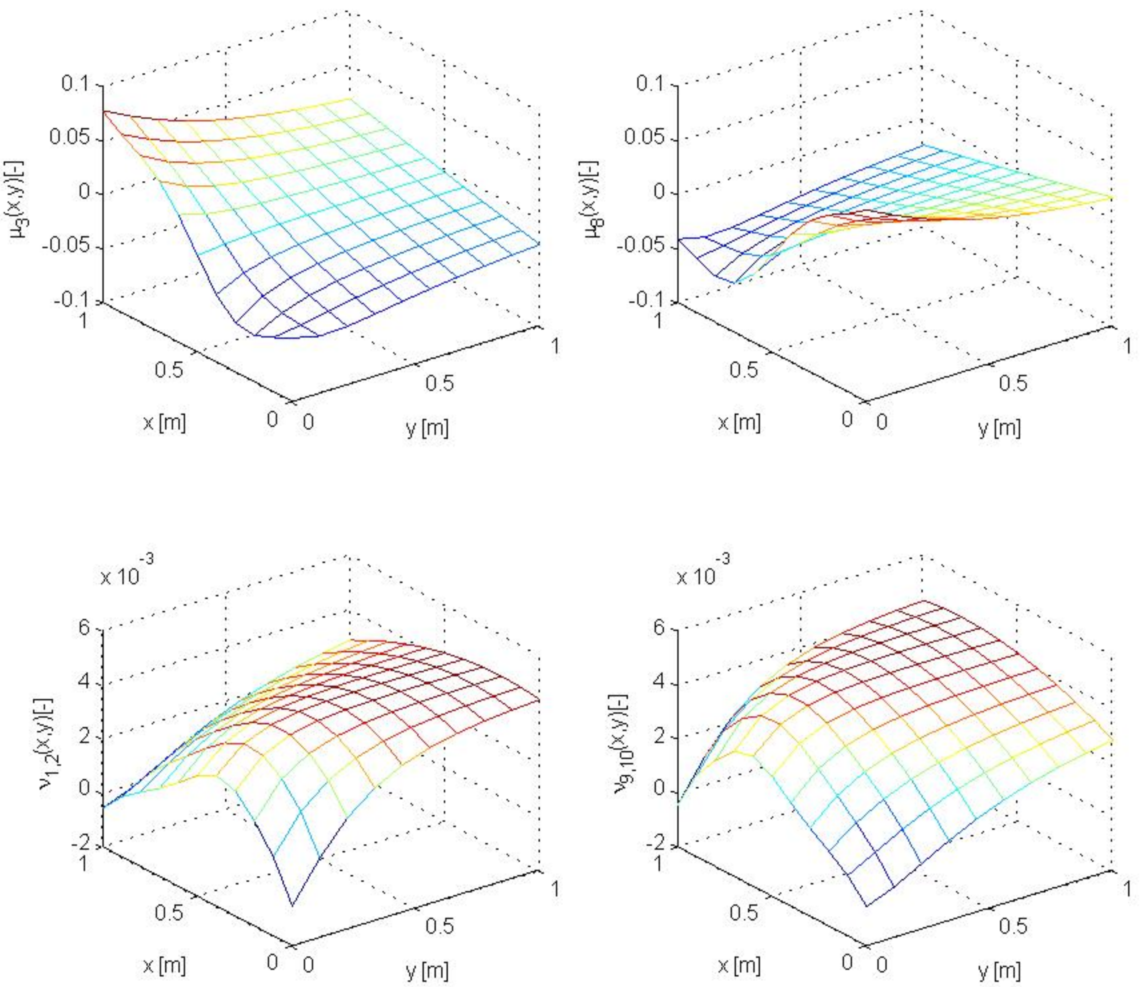

Figure 5 – some output field functions resulting from the input field of Eq. 18, but this time for a much lower maximum derivative (stronger dependence). From left to right and top to bottom: $\mu_3(x, y)$**,** $\mu_8(x, y)$**,** $\nu_{1,2}(x, y)$ **and** $\nu_{9,10}(x, y)$

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