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Distributed parameter estimation of a stochastic beam structure via sensitivity-based model updating using experimental FRFs

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Abstract: Structural parameter estimation is affected not only by measurement noise, but also by unknown uncertainties which are present in the system model. Deterministic structural model updating methods minimize the difference between experimentally measured data and computational prediction. Sensitivity-based methods are very efficient in solving structural model updating problems. The structural model includes parameters such as Poisson's ratio, Young's modulus, mass density, modal damping, etc. These parameters are usually considered homogeneous and deterministic along the structure. In this paper the distributed and non homogeneous characteristics of these parameters are considered in the model updating. The parameters are taken as spatially correlated random fields, and are expanded in a spectral KarhunenLoeve (KL) decomposition. Using the KL expansion, the spectral dynamic stiffness matrix of the beam structure is expanded as a series in terms of discretized parameters, which can be estimated using sensitivity-based model updating techniques. Numerical and experimental tests involving a beam structure with distributed bending rigidity and mass density are used to verify the proposed method. This extension to standard model updating procedures can enhance the dynamic description of structural dynamic models.

Keywords: Sensitivity-based model, Karhunen Loeve expansion, model updating, spectral element method

NOMENCLATURE

$\mathbf{U}(\mathbf{r}, t)$ = time dependent displacement variable
$\mathbf{r}$ = spatial pposition vector
$u_e(\mathbf{r}, \omega)$ = nodal displacement vector
$\mathbf{g}(\mathbf{r}, \omega)$ = shape function
$\mathbf{D}_0(\omega)$ = global dynamic spectral matrix
$\mathbf{K}_0(\omega)$ = stiffness spectral matrix
$\mathbf{M}_0(\omega)$ = mass spectral matrix
$\mathbf{C}_0(\omega)$ = damping spectral matrix
EI = bending stiffness
ρA = mass per unit of length
$v(x, t)$ = transverse fluxual displacement t = time domain
x = space domain
$\Re$ = real number
$\Im$ = imaginary number

k = wavenumber
E = Young's modulus
I = inertia moment
A = cross-section area
i = Unit imaginary number,$i = \sqrt{-1}$
L = length
$C_{\varpi}(\mathbf{r}_1, \mathbf{r}_2)$ = covariance function
b = correlation length
a = bounded domain with length
$\mathbf{H}$ = frequqncy responce function
$\mathbf{S}_j$ = $j-$th sensitivity matrix
$\mathbf{W}_e$ = positive define weighing matrix
$\mathbf{W}_p$ = parameter weighing matrix

Greek Symbols
ω = circular frequency
ρ = mass density
$\mathbf{\Theta}$ = slope of the beam element
$\xi(\theta)_j$ = $j-$th random variable
$varpi(\mathbf{r}, \theta)$ = random field
θ = element of the sample space Ω
λ_j = $j-$th eigenvalues
φ_j = $j-$th eigenvactor
δ=residual of determained parameter
ε = error

Subscripts
0 = mean values
$(\bullet)^T$ = Matrix transpose of $(\bullet)$
$(\bullet)^{-1}$ = Matrix inverse of $(\bullet)$
$diag$ = A diagonal matrix
$\bullet'$ express the spatial partial derivative

INTRODUCTION

Quantifying uncertainty in numerically simulated results is not recent. However, during the last few years, this research area has undergone remarkable development, in special for dynamic systems. The method commonly used is Monte Carlo (MC) simulation (Sobol', 1994). Otherwise, non sampling approaches such as the Perturbation Method may be used. It consists in expanding random fields in a truncated Taylor series around their mean (Xiu, 2010). The Direct Method consists in applying the moment equations to obtain the random solutions. The unknowns are the moments and their equations are derived by taking averages over the original stochastic governing equations.

A very powerful method in computational stochastic problems is the Stochastic Finite Element Method (SFEM) (Ghanem and Spanos, 1991). SFEM is an extension of the classical deterministic FE approach to the stochastic framework, i.e., to solve static and dynamic problems with stochastic mechanical, geometric, or loading properties (Stefanou, 2009). Adhikari (Adhikari, 2011) presented a doubly Spectral Stochastic Finite Element Method, where the Spectral Element Method (SEM) is given a stochastic treatment. Both techniques, SFEM and doubly Spectral SFEM, are formulated in a context of random fields. A method with wide application when considering random fields is the Karhunen-Loève (KL) expansion (Ghanem and Spanos, 1991; Papoulis and Pillai, 2002; Xiu, 2010). The KL expansion can be used to discretize the random field by representing it by scalar independent random variables and continuous deterministic functions. By truncating the expansion, the number of random variables become finite and numerically treatable. Many authors use the KL expansion to model Gaussian random processes, but it is possible to extend the KL expansion to non-Gaussian processes (Poirion and Soize, 1999; Puig and Soize, 2002; Sakamoto and Ghanem, 2002; Phoon K. K. and Quek, 2005).

The spectral element method (SEM) (Doyle, 1997; Lee, 2004) is based on the analytical solution of the displacement wave equation, written in the frequency domain. The element is tailored with the matrix ideas of FEM, but in SEM the interpolation function is the exact solution of the wave equation. This technique can be addressed by different denominations, such as dynamic finite element method (Hashemi et al., 1999; Hashemi and Richard, 2000), dynamic stiffness method (Paz, 1980; Banerjee and Williams, 1985; Banerjee, 1989; Banerjee and Williams, 1992; Banerjee and Fisher, 1992; Ferguson and Pilkey, 1993a,b; Banerjee and Williams, 1995; Manohar and Adhikari, 1998; Banerjee, 1997; Adhikari and Manohar, 2000), and spectral finite element method Doyle (1989); Gopalakrishnan et al. (2007). The advantage for built-up structures with geometrically uniform members is that these structural members can be modeled by a single spectral element, so that a significant reduction of the total number of degrees is obtained compared to FEM. It can be shown that one spectral element is equivalent to an infinite number of finite elements. Furthermore, since the method is based on the wave equation, it performs well at medium and high frequencies. However, there are still some drawbacks, such as difficulties to model non-uniform members and to apply arbitrary boundary conditions to 2D (shells) and 3D (solid) structures.

Model updating methods in dynamic structural analysis is basically a process of minimizing the differences between the numerical model predictions and measured responses obtained in experimental tests using a parameter estimation procedure (Mottershead and Friswell, 1993; Friswell and Mottershead, 1995). Most methods use FEM models (Zienkiewicz and R.L.Taylor, 2000) together with experimental modal parameters Ewins (1984); Maia and Silva (1997) for model updating. The book by Friswell and Mottershead (Friswell and Mottershead, 1995) compiles different model updating techniques.

Recently, some authors proposed stochastic model updating techniques (Khodaparast and Mottershead, 2008; Khodaparast et al., 2008; ArdaVanli and Jung, 2013). In this stochastic approach, random parameters are identified using measured modal parameters with estimated random distribution. Adhikari and Friswell (Adhikari and Friswell, 2010) developed a distributed parameter estimation method based on eigen-sensitivity. In the field of structural dynamics, the authors traditionally use modal parameters in model updating. In a structural dynamic test it is a common practice to measure the data in the form of Frequency Response Functions (FRF). Natke (Natke, 1977) presented an updating procedure using measured FRF instead of modal parameters. After that, a growing number of researchers focused on model updating algorithms using the measured data directly (Cottin et al., 1984; Arruda and DosSantos, 1993; Grafe, 1998). The most widely used technique in updating models of engineering structures based on vibration test data is the sensitivity method (Mottershead et al., 2011). This procedure was described in detail by Natke (Natke, 1992) and Friswell & Mottershead (Friswell and Mottershead, 1995).

The main goal of this paper is to use sensitivity-based model updating with measured FRFs to estimate spatially distributed parameters. The distributed parameters are assumed stochastic, which is more realistic given the variability caused by the manufacturing process. The study uses a simple beam structure where the uncertainty is included in the flexural bending (EI) and mass per unit of length (ρA). Such distributed deviations are unknown *a priori* and therefore can be considered to be samples from a random field, which is discretized into random variables using the KL expansion. The implemented technique is validated in a numerical simulation and then applied to experimental data for a polymer beam manufatured by 3D printing.

1 SPECTRAL ELEMENT METHOD FOR STOCHASTIC SYSTEMS

Supposing a linear damped distributed parameter dynamic system without any external force is governed by a linear differential equation Meirovitch (1997)

$$\rho_0 \frac{\partial^2 \mathbf{U}(\mathbf{r},t)}{\partial t^2} + L_{10} \frac{\partial \mathbf{U}(\mathbf{r},t)}{\partial t} + L_{20}\mathbf{U}(\mathbf{r},t) = 0 \tag{1}$$

where $\mathbf{U}(\mathbf{r},t)$ is the time dependent displacement variable, $\mathbf{r} \in \mathbb{R}$ is the spatial position vector, and t is time specified in some domain $\mathcal{D}$.

$$-\omega^2 \rho_0 \mathbf{u}(\mathbf{r},\omega) + i\omega L_1 0\{\mathbf{u}(\mathbf{r},\omega)\} + L_2 0\{\mathbf{u}(\mathbf{r},\omega)\} = 0 \tag{2}$$

Similar to FEM, the frequency-dependent displacement within an element is interpolated from the nodal displacements $u_e(\mathbf{r},\omega) = \mathbf{g}(\mathbf{r},\omega)^T \hat{\mathbf{u}}_e(\omega)$, where $u_e(\mathbf{r},\omega)$ is the nodal displacement vector and $\mathbf{g}(\mathbf{r},\omega)$ is the vector of frequency-dependent shape functions represented by

$$\mathbf{g}(\mathbf{r},\omega) = \mathbf{\Gamma}(\omega)\mathbf{s}(\mathbf{r},\omega) \tag{3}$$

One of the advantages in using SEM is that only one finite element is required for a homogeneous structural member. Therefore, the global dynamic spectral matrix for a deterministic system can be described as

$$D_0(\omega) = -\omega^2\mathbf{M}(\omega) + i\omega\mathbf{C}(\omega) + \mathbf{K}(\omega) \tag{4}$$

In a weak form, frequency-dependent $n \times n$ complex stiffness, mass and damping matrices can be expressed as

$$K(\omega) = \int_{\mathcal{D}} k_s(\mathbf{r})\mathbf{g}''(r,\omega)^T\mathbf{g}''(r,\omega)dr \tag{5}$$

$$M(\omega) = \int_{\mathcal{D}} \rho(\mathbf{r})\mathbf{g}(r,\omega)^T\mathbf{g}(r,\omega)dr \tag{6}$$

and

$$C(\omega) = \int_{\mathcal{D}} c(\mathbf{r})\mathbf{g}'(r,\omega)^T\mathbf{g}'(r,\omega)dr \tag{7}$$

In this present work a spectral element for a straight homogeneous beam is used (Doyle (1997); Lee (2004); Adhikari (2011)) and expanded for a stochastic treatment.

1.1 Spectral Beam Element

The fundamental equations for the flexural motion of a beam structure are briefly described. A more extensive formulation can be found in (Doyle, 1997; Lee, 2004). Figure 1 shows an elastic two-node beam element with an uniform rectangular cross-section subjected to dynamic forces at both ends. In this section all parameters are assumed to be deterministic variables.

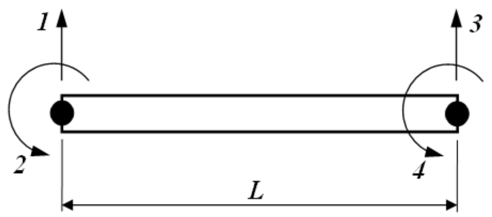

Figure 1: Two-node beam spectral element

The equation of motion of a damped Euler-Bernoulli beam under bending vibration may be writen as (Adhikari, 2011),

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 v(x,t)}{\partial x^2} +\right] + \rho A(x)\frac{\partial^2 v(x,t)}{\partial t^2} = 0 \tag{8}$$

where EI is the bending stiffness, ρA is the mass per unit length, $v(x,t)$ is the transverse flexural displacement. A hysteretic structural damping is introduced into the beam formulation by adding a damping term in the flexural bending. It is defined as $EI = EI(1+i\eta)$, where η is the hysteretic structural damping and $i = \sqrt{-1}$.

By considering the homogeneous differential equation with constant properties along the beam length, the spectral form becomes:

$$\frac{d^4\hat{v}}{dx^4} - \beta^4\hat{v} = 0 \tag{9}$$

Equation (9) can be split into a product of two terms which must vanish. Now, it is assuming a solution of the type $v(x)e^{i\omega t} = e^{k\omega}e^{i\omega t}$, where k (wavenumber) is given by:

$$k^4 - \beta^4 = 0 \quad or \quad k = \pm i\beta \backslash \pm \beta \tag{10}$$

for

$$\beta^4 = \frac{\rho A_0 \omega^2}{EI + (EI_0 i\eta)} \tag{11}$$

where ω is the circular frequency, E is the Young's modulus, A is the cross-section area, ρ is the mass density, I is the inertia moment, and $v(x)$ is the transverse flexural displacement. For the spectral Euler-Bernoulli beam element of length L, the general solution of $v(x)e^{i\omega t} = e^{k\omega}e^{i\omega t}$ can be then obtained in the form of

$$v(\mathbf{x},\omega) = a_1e^{-ikx} + a_2e^{-kx} + a_3e^{-ik(L-x)} + a_4e^{-k(L-x)} = \mathbf{s}(x,\omega)\mathbf{a} \tag{12}$$

where

$$\mathbf{s}(x,\omega) = \left[e^{-ikx}, e^{-kx}, e^{-ik(L-x)}, e^{-k(L-x)}\right]$$
$$\mathbf{a}(x,\omega) = \{a_1, a_2, a_3, a_4\}^T \tag{13}$$

The spectral nodal displacements and slopes of the beam element can be related to the displacement field at the two nodes ($x = 0$ and $x = L$), by

$$\mathbf{d} = \begin{bmatrix} v_1 \\ \Theta_1 \\ v_2 \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} v(0) \\ v'(0) \\ v(L) \\ v'(L) \end{bmatrix} \tag{14}$$

By substituting Eq.(12) into the right-hand side of Eq.(14) gives

$$\mathbf{d} = \begin{bmatrix} s(0,\omega) \\ s'(0,\omega) \\ s(L,\omega) \\ s'(L,\omega) \end{bmatrix} \mathbf{a} = \mathbf{\Gamma}(\omega)\mathbf{a} \tag{15}$$

where

$$\mathbf{\Gamma}(\omega) = \begin{bmatrix} 1 & 1 & e^{-ikL} & e{-}kL \\ -ik & -k & ie^{-ikL}k & e{-}kLk \\ e{-}ikL & e{-}kL & 1 & 1 \\ -ie{-}ikLk & -e{-}kLk & ik & k \end{bmatrix} \tag{16}$$

The frequency-dependent displacement within an element is interpolated from the nodal displacement vector $\mathbf{d}$ by eliminating the constant vector $\mathbf{a}$ from Eq.(14) and using Eq.(15) it can be expressed as

$$v(x,\omega) = \mathbf{g}(x,\omega)\mathbf{d} \tag{17}$$

where the shape function can be expressed as

$$\mathbf{g}(\mathbf{x},\omega) = \mathbf{s}(\mathbf{x},\omega)\mathbf{\Gamma}^{-1}(\omega) \tag{18}$$
$$= \begin{bmatrix} \frac{-2\cos(kx)-2\cosh(kx)+(1-i)(\cos(k((1+i)L-x))+i\cos(k((1+i)L-ix))+\cosh(k((1+i)L-x))+i\cosh(k((1+i)L-ix)))}{4\cos(kL)\cosh(kL)} \\ -\frac{2\sin(kx)+2\sinh(kx)+(1+i)(\sin(k((1+i)L-x))-\sin(k((1+i)L-ix))+\sinh(k((1+i)L-x))-\sinh(k((1+i)L-ix)))}{4k(\cos(kL)\cosh(kL)-1)} \\ \frac{\cos(k(L-x))-\cos(kx)\cosh(kL)+\cosh(k(L-x))-\cos(kL)\cosh(kx)+\sin(kx)\sinh(kL)-\sin(kL)\sinh(kx)}{2-2\cos(kL)\cosh(kL)} \\ \frac{\sin(k(L-x))-\cos(kx)\sinh(kL)+\cosh(kx)(\sinh(kL)-\sin(kL))+\cosh(kL)(\sin(kx)-\sinh(kx))+\cos(kL)\sinh(kx)}{2k(\cos(kL)\cosh(kL)-1)} \end{bmatrix}$$

In the case of the Euler-Bernoulli beam, a generalized transverse displacement at an arbitrary point can be expressed as,

$$v(x) = g_1(x)v_1 + g_2(x)\Theta_1 + g_3(x)v_2 + g_4(x)\Theta_2 \tag{19}$$

The dynamic stiffness matrix for the two-node spectral beam element is expressed in Eq.(4), the damping is assumed hysteric structural one. For this reason only the mass and stiffness matrices will be determined in a weak form:

$$\mathbf{K}_0(\omega) = EI_0 \int_0^L \mathbf{g}''(x)^T \mathbf{g}''(x) dx \tag{20}$$

and

$$\mathbf{M}_0(\omega) = \rho A_0 \int_0^L \mathbf{g}(x)^T \mathbf{g}(x) dx \tag{21}$$

where $'$ express the spatial partial derivative. The stochastic beam spectral element is formulated as a random process expanded in a spectral KL decomposition.

1.2 Karhunen-Loève expansion

Since the equations of motion for the beam spectral element are written as partial differential equations, it would be very difficult to apply random fields directly to them. To overcome this complexity the random field is discretized in terms of random variables. By doing this, many mathematical procedures can be used to solve the resulting discrete stochastic differential equations. The procedure applied here is a random field spectral decomposition using the KL expansion. The random field is described by various points expressed by random variables. Therefore, a large number of points is required for a good approximation. This concept is similar to the Fourier series expansion. Assuming that the covariance function is finite, symmetric and positive definite, it can be represented by a spectral decomposition. By using this concepts a random field can be expressed like a generalized Fourier series as,

$$\varpi(\mathbf{r},\theta) = \varpi_0(\mathbf{r}) + \sum_{j=1}^{\infty} \xi_j(\theta)\sqrt{\lambda_j}\varphi_j(\mathbf{r}) \tag{22}$$

where $\varpi(\mathbf{r},\theta)$ is a random field with the covariance function $C_{\varpi}(\mathbf{r}_1,\mathbf{r}_2)$. Here θ denotes an element of the sample space Ω, so that $\theta \in \Omega$; $\xi_j(\theta)$ are uncorrelated random variables. The subscript 0, $\varpi_0(\mathbf{r})$, implies the corresponding deterministic part. The constants λ_j and functions $\varphi_j(\mathbf{r})$ are, respectively, eigenvalues and eigenfunctions satisfying the integral equation:

$$\int_{\mathcal{D}} C_{\varpi}(\mathbf{r}_1,\mathbf{r}_2)\varphi_j(\mathbf{r}_1)d\mathbf{r}_1 = \lambda_j\varphi_j(\mathbf{r}_2) \quad \forall j = 1,2,.... \tag{23}$$

In this paper one dimensional spaces are considered. Since a Gaussian random field is representative of many physical systems and closed form expressions for the KL expansion exist, a Gaussian autocorrelation function with exponential decay will be used here. It can be expressed as,

$$C(x_1,x_2) = e^{-|x_1-x_2|/\mathtt{b}} \tag{24}$$

where $\mathtt{b}$ is the correlation length, which is an important parameter to describe the random field. A random field can be expanded in a finite basis of deterministic functions and random variables if the correlation length is very large compared with the domain under consideration. An analytical solution in the interval $-\mathtt{a} < x < \mathtt{a}$ where it is assumed that the mean is zero, produces a random field as,

$$\varpi_1(x,\theta) = \sum_{j=1}^{\infty} \xi_j(\theta)\sqrt{\lambda_j}\varphi_j(x) \tag{25}$$

Defining that $\mathtt{c} = 1/\mathtt{b}$, the corresponding eigenvalues and eigenfunctions for odd j are given by,

$$\lambda_j = \frac{2\mathtt{c}}{\omega_j^2+\mathtt{c}^2}; \quad \varphi_j(x)\frac{\cos(\omega_j\frac{L}{2})}{\sqrt{\mathtt{a}+\frac{\sin(2\omega_j\mathtt{a})}{2\omega_j}}} \qquad \text{where} \quad \tan(\omega_j\mathtt{a}) = \frac{\mathtt{c}}{\omega_j} \tag{26}$$

and for even j are given by,

$$\lambda_j = \frac{2\mathtt{c}}{\omega_j^2+\mathtt{c}^2}; \quad \varphi_j(x)\frac{\sin(\omega_j\frac{L}{2})}{\sqrt{\mathtt{a}-\frac{\sin(2\omega_j\mathtt{a})}{2\omega_j}}} \qquad \text{where} \quad \tan(\omega_j\mathtt{a}) = \frac{\omega_j}{-\mathtt{c}} \tag{27}$$

These eigenvalues and eigenfunctions will be used to obtain the stochastic dynamic stiffness matrices for beam spectral elements.

For practical applications, the Equation (25) is truncated with M numbers of terms, which could be selected based on the amount of information to be kept. Its value is also related with the correlation length and the number of eigenvalues kept, provided that they are arranged in decreasing order.

A widely made assumption is that the processes underlying a given physical behaviour is Gaussian. This is often made because of its simplicity (Vio et al., 2001). However, this assumption cannot be applied in many cases. A random stiffness, for example, should not be assumed to be Gaussian, as it cannot have negative values. Also, when the physical system is characterized by a non-linear behaviour the Gaussian assumption cannot be made. For this reason, in this work a stationary non-Gaussian random process is adopted using a memoryless transformation (Weinberg and L.Gunn, 2011) of an underlying standard Gaussian random process (Poirion and Soize, 1999; Puig and Soize, 2002; Sakamoto and Ghanem, 2002; Phoon K. K. and Quek, 2005).

1.3 Stochastic beam spectral element

In this work the flexural bending ($EI(x)$) and mass per unit of length ($\rho A(x)$) are considered as spatially distributed random variables. Therefore, the flexural bending is assumed as a random field of the form:

$$EI(x,\theta) = EI_0[1+\varepsilon_1\varpi_1(x,\theta)] \tag{28}$$

and the mass per unit of length is assumed a random field as

$$\rho A(x,\theta) = \rho A_0[1+\varepsilon_2\varpi_2(x,\theta)] \tag{29}$$

The subscript 0 indicates the mean value, $0 < \varepsilon_i \ll 1 (i = 1,2,...)$ are deterministic constants and the random field $\varpi_i(x,\theta)$ is taken to have zero mean, unit standard deviation and covariance $R_{ij}(\xi)$. Since, $EA(x,\theta)$ and $\rho A(x,\theta)$ are strictly positive, $\varpi_i(x,\theta)(i=1,2,...)$ is required to satisfy the probability condition $\mathrm{P}[1+\varepsilon_i\varpi_i(x,\theta)\leq 0]=0$. This requirement excludes the use of Gaussian models for these random fields. However, for small ε_i, it is expected that Gaussian models still can be used if the primary interest of the analysis is to estimate the first few response moments and not the response behaviour near tails of the probability distributions. Expanding the random fields $\varpi_1(x,\theta)$ and $\varpi_2(x,\theta)$ in a KL spectral decomposition one obtains the stochastic dynamic stiffness matrix written as,

$$\begin{aligned}\mathbf{D}(\omega,\theta) &= \mathbf{D}_0(\omega)+\Delta\mathbf{D}(\omega,\theta)\\ &= -\omega^2\left[\mathbf{M}_0(\omega)+\Delta\mathbf{M}(\omega,\theta)\right]+\left[\mathbf{K}_0(\omega)+\Delta\mathbf{K}(\omega,\theta)\right]\end{aligned} \tag{30}$$

where the deterministic part is given by the Eq.(4) and the random part $\Delta\mathbf{D}(\omega,\theta)$ is related to the coefficients $\Delta\mathbf{K}(\omega,\theta)$ and $\Delta\mathbf{M}(\omega,\theta)$, stiffness and mass, respectively, expanded in a KL decomposition of the form

$$\Delta\mathbf{K}(\omega,\theta)=\varepsilon_1\sum_{j=1}^{N_K}\xi_{Kj}(\theta)\sqrt{\lambda_{Kj}}\mathbf{K}_j(\omega) \tag{31}$$

and

$$\Delta\mathbf{M}(\omega,\theta)=\varepsilon_2\sum_{j=1}^{N_M}\xi_{Mj}(\theta)\sqrt{\lambda_{Mj}}\mathbf{M}_j(\omega) \tag{32}$$

where N_k and N_M are the numbers of terms kept in the KL expansion; $\xi_{Kj}(\theta)$ and $\xi_{Mj}(\theta)$ are uncorrelated Gaussian random variables with zero mean and unit standard deviation. The constant matrices $\mathbf{K}_j(\omega)$ and $\mathbf{M}_j(\omega)$ can be expressed as

$$\mathbf{K}_j(\omega)=EI_0\int_0^L\varphi_{Kj}(x_e+x)\mathbf{g}''(x)^T\mathbf{g}''(x)dx \tag{33}$$

$$\mathbf{M}_j(\omega)=\rho A_0\int_0^L\varphi_{Kj}(x_e+x)\mathbf{g}(x)^T\mathbf{g}(x)dx \tag{34}$$

Substituting Equation (26) and (27) in Equation (33) and (32) the closed-form expressions for the random part of the stiffness and mass matrices for the rod spectral element in odd j can be expressed as

$$\mathbf{K}_j(\omega)=\frac{EI_0}{\sqrt{\mathsf{a}+\frac{\sin(2\omega_j\mathsf{a})}{2\omega_j}}}\left[\int_0^L\cos(\omega_j(x_e+x))\mathbf{g}''(x)^T\mathbf{g}''(x)dx\right] \tag{35}$$

$$\mathbf{M}_j(\omega)=\frac{\rho A_0}{\sqrt{\mathsf{a}+\frac{\sin(2\omega_j\mathsf{a})}{2\omega_j}}}\left[\int_0^L\cos(\omega_j(x_e+x))\mathbf{g}(x)^T\mathbf{g}(x)dx\right] \tag{36}$$

and for even j it is given by

$$\mathbf{K}_j(\omega)=\frac{EI_0}{\sqrt{\mathsf{a}-\frac{\sin(2\omega_j\mathsf{a})}{2\omega_j}}}\left[\int_0^L\sin(\omega_j(x_e+x))\mathbf{g}''(x)^T\mathbf{g}''(x)dx\right] \tag{37}$$

$$\mathbf{M}_j(\omega)=\frac{\rho A_0}{\sqrt{\mathsf{a}-\frac{\sin(2\omega_j\mathsf{a})}{2\omega_j}}}\left[\int_0^L\sin(\omega_j(x_e+x))\mathbf{g}(x)^T\mathbf{g}(x)dx\right] \tag{38}$$

2 SENSITIVITY-BASED UPDATING METHOD USING FRF

The objective of sensitivity based parameter estimation methods is to improve the correlation between the measured and predicted responses. The correlation is determined by an objective function involving modal or dynamic response data. In general, they are non-linear functions with respect to the model parameters, and so an iterative procedure is required with the possible associated convergence problems (Friswell and Mottershead, 1995). The parameter estimation non-linear least squares method uses a truncated Taylor series expansion of the dynamic response in terms of the unknown parameters, often limited to the first two series terms, yielding the linear approximation:

$$\delta\mathbf{H}=\mathbf{S}_j\delta\xi, \tag{39}$$

where $\delta\mathbf{H}=\mathbf{H}_m-\mathbf{H}_j$ is the residual of the measured output, $\delta\xi=\xi-\xi_j$ is the perturbation in the parameters, and $\mathbf{S}_j$ is the sensitivity matrix. It contains the derivatives of the frequency response functions with respect to the chosen parameters to be varied ξ_j. The iteration is initialized with $\xi_0 = 0$ and it is assumed that there are more measured data than unknown parameters. Then, Equation (39) provides an over-determined set of simultaneous equations that can be solved using a least squares solution. Adopting the weight objective function:

$$J(\delta\xi) = \varepsilon^T \mathbf{W}_e \varepsilon, \tag{40}$$

where $\varepsilon = \delta\mathbf{H} - \mathbf{S}_j\delta\xi$ is the error in the predicted measurements based on the updated parameters and $\mathbf{W}_e$ is a positive definite weighting matrix. Substituting Eq.(40) in Eq.(39) leads to

$$J(\delta\xi) = \mathbf{W}_e\delta\mathbf{H}\delta\mathbf{H}^T - \mathbf{W}_e(\mathbf{S}_j\delta\mathbf{H}^T\delta\xi + \mathbf{S}_j^T\delta\mathbf{H}\delta\xi^T) + \delta\xi\mathbf{S}_j\mathbf{W}_e\mathbf{S}_j^T\delta\xi^T. \tag{41}$$

Minimizing J with respect to $\delta\xi$ is equivalent to:

$$\nabla J(\delta\xi) = 0 = -\mathbf{S}_j\mathbf{W}_e\delta\mathbf{H}^T + \mathbf{S}_j\mathbf{S}_j^T\mathbf{W}_e\delta\xi, \tag{42}$$

and solving Eq.(42) for $\delta\xi$ results,

$$\delta\xi = [\mathbf{S}_j^T\mathbf{W}_e\mathbf{S}_j]^{-1}\mathbf{S}_j^T\mathbf{W}_e\delta\mathbf{H}. \tag{43}$$

Thus, the updated parameter can be obtained from:

$$\xi_{j+1} = \xi_j + [\mathbf{S}_j^T\mathbf{W}_e\mathbf{S}_j]^{-1}\mathbf{S}_j^T\mathbf{W}_e(\mathbf{H}_m - \mathbf{H}_j). \tag{44}$$

The solution of Eq.(44) can be ill-conditioned. Titurus and Friswell (2008) presented a regularization treatment within the context of sensitivity-based FE model updating. The method to regularize gives the updated parameter vector as:

$$\xi_{j+1} = \xi_j + [(1-\gamma)\mathbf{S}_j^T\mathbf{W}_e\mathbf{S}_j + \gamma\mathbf{W}_p]^{-1}\{(1-\gamma)\mathbf{S}_j^T\mathbf{W}_e(\mathbf{H}_m - \mathbf{H}_j) - \gamma\mathbf{W}_p\xi_j\}. \tag{45}$$

The regularization parameter $\gamma \in [01]$ determines the relative weight between the regularized solution ($\|\xi_{j+1} - \xi_j\|$) versus the corresponding residual norm ($\|\S_j(\xi_{j+1} - \xi_j) - (\mathbf{H}_m - \mathbf{H}_j)\|$). The updated parameter is evaluated for an iterative process until convergence, which is determined when the change in parameters,$\|\xi_{j+1} - \xi_j\|$ is sufficiently small.

The choice of weighting matrices is a difficult subject, and estimated statistical properties can be employed (Friswell and Mottershead, 1995). Here we use a solution procedure presented by Grafe (Grafe, 1998) where no explicit statistical calculations of the weighting factors are required and the correlation coefficient $(X_s(\omega))$, is used directly as

$$[\diagdown \mathbf{W}_e \diagdown] = [\diagdown X_s(\omega) \diagdown] \tag{46}$$

The Correlation coefficient is based on the Modal Assurance Criterion (MAC), theory proposed by Allemang and Brown (1982); Allemang (2002). Where for any measured frequency point a correlation between the measured and predicted value is given by

$$X_s(\omega) = \frac{\left|\{H_m(\omega)\}^H\{H_a(\omega)\}\right|^2}{(\{H_m(\omega)\}^H\{H_m(\omega)\})(\{H_a(\omega)\}^H\{H_a(\omega)\})} \tag{47}$$

where $H_m(\omega)$ and $H_a(\omega)$ are the "measured" and predicted FRF vectors at matching excitation/response locations, respectively. A definition of parameter weighing matrix $(\mathbf{W}_p)$ was proposed by (Mottershead and Foster, 1991; Link, 1998). Similar to the approach Link method the parameter weighing matrix used here is expressed as

$$\begin{aligned} [w_e] &= [[\mathbf{S}][\diagdown \mathbf{W}_e \diagdown][\mathbf{S}]^T]^{-1} \\ [\diagdown \mathbf{W}_p \diagdown] &= \frac{\|[w_e]\|_2}{max(diag([w_e]))}[\diagdown diag([w_e]) \diagdown] \end{aligned} \tag{48}$$

2.1 Stochastic sensitivity of FRF

The sensitivity method is based on the linearisation of the non-linear relationship between measurable outputs, modal data or frequency response functions and the model parameters to be estimated (Mottershead et al., 2011). Several authors have presented eigen sensitivities. Considering that in practice the measured raw data obtained from the experimental test are the FRF, in this paper the sensitivity of FRF will be used. The coefficients of the KL expansion are assumed as uncertain parameters and will be estimated by Eq. (44). By following (Natke, 1977; Cottin et al., 1984; Arruda and DosSantos, 1993; Grafe, 1998) the deterministic FRF sensitivity relate to a general parameter vector φ can be written as:

$$\frac{\partial\mathbf{H}(\omega)}{\partial\varphi} = -\mathbf{H}(\omega)\frac{\partial\mathbf{D}(\omega)}{\partial\varphi}\mathbf{H}(\omega) \tag{49}$$

where $\mathbf{H}(\omega) = \mathbf{K}^{-1}(\omega)$ is inverse of the deterministic dynamic stiffness matrix. In the stochastic context, two techniques can be applied. The fist one estimates a random variable, $\varphi(\theta)$. The second one is associated with the parameter ξ_{Kj} of the KL expansion, which are the uncorrelated random variables of the te random field.s With the first approach Eq.(49) becomes:

$$\frac{\partial \mathbf{H}(\omega,\theta)}{\partial \varphi(\theta)} = -\mathbf{H}(\omega,\theta)\frac{\partial \mathbf{D}(\omega,\theta)}{\partial \varphi(\theta)}\mathbf{H}(\omega,\theta) \tag{50}$$

where $\mathbf{H}(\omega,\theta) = \mathbf{D}^{-1}(\omega,\theta)$, which is inverse of the stochastic dynamic stiffness matrix (Eq.30). In the second apporach, used in this paer, Eq.(49) is described by:

$$\frac{\partial \mathbf{H}(\omega,\theta)}{\partial \xi} = -\mathbf{H}(\omega,\theta)\left[\frac{\partial \mathbf{K}(\omega,\theta)}{\partial \xi_{Kj}} - \omega^2\frac{\partial \mathbf{M}(\omega,\theta)}{\partial \xi_{Mj}}\right]\mathbf{H}(\omega,\theta) \tag{51}$$

the derivative of $\mathbf{K}(\omega,\theta)$ and $\mathbf{M}(\omega,\theta)$ related to the parameter ξ_{Kj} produces:

$$\frac{\partial \mathbf{K}(\omega,\theta)}{\partial \xi_{Kj}(\theta)} = \varepsilon_1\sqrt{\lambda_{Kj}}\mathbf{K}_j(\omega) \tag{52}$$

and

$$\frac{\partial \mathbf{M}(\omega,\theta)}{\partial \xi_{Mj}(\theta)} = \varepsilon_2\sqrt{\lambda_{Mj}}\mathbf{M}_j(\omega) \tag{53}$$

Substituting Equation (51) and (52) in (53) it has,

$$\frac{\partial \mathbf{H}(\omega,\theta)}{\partial \xi(\theta)} = s_{ij} = -H(\omega)\left[\varepsilon_1\sqrt{\lambda_{Kj}}\mathbf{K}_j(\omega) - \omega^2\varepsilon_2\sqrt{\lambda_{Mj}}\mathbf{M}_j(\omega)\right]H(\omega) \tag{54}$$

In this paper the sensitivities have been calculated by finite differences and the receptance FRFs $(\mathbf{H}(\omega,\theta))$ were taken in dB scale (Arruda, 1992) with 1m/N reference. It can be shown to be

$$\frac{\partial(20log_{10})\left|\mathbf{H}(\omega,\theta)\right|}{\partial\xi(\theta)} \approx \left(\frac{\Re(\mathbf{H}(\omega,\theta))\frac{\partial(\mathbf{H}(\omega,\theta))}{\partial\xi(\theta)} + \Im(\mathbf{H}(\omega,\theta))\frac{\partial(\mathbf{H}(\omega,\theta))}{\partial\xi(\theta)}}{\Re(\mathbf{H}(\omega,\theta))^2 + \Im(\mathbf{H}(\omega,\theta))^2}\right) \tag{55}$$

The elements of the sensitivity matrix s_{ij} are given by Eq.55 and the $N_K + N_M$ dimensional vector of updating parameters ξ is

$$\xi = [\xi_{K_1}, \xi_{K_2}..., \xi_{K_{N_K}} \quad \xi_{M_1}, \xi_{M_2}..., \xi_{M_{N_M}}]^T \tag{56}$$

The elements of the vector ξ are sampled from independent and identically distributed standard Gaussian random variables (i.e., with zero-mean and unit standard deviation) from the KL expansion. The parameter vector ξ will be estimated from the "measured" FRF and used to reconstruct the $EA(x,\theta)$ and $\rho A(x,\theta)$ random fields. Once the parameters ξ are obtained, the estimated FRF can be calculated as

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{S}\xi \tag{57}$$

where $\mathbf{H}_0$ is the vector of deterministic FRF measured at a specified point, $\mathbf{S}$ is the sensitivity matrix calculated with Eq.55, and vector ξ is estimated parameter obtained of Eq.56.

3 NUMERICAL TEST RESULTS

In this numerical test a free-free beam is modelled with a two-node beam spectral element. Stochastic variability of the Gamma marginal type is considered for the beam flexural bending rigidity EI and for the mass per unit length ρA. The "measured" FRF was obtained by simulating a transfer receptance FRF with force excitation at node 1, and displacement response in node 1 and node 2. The unperturbed physical and geometrical properties of the beam are given by: $L = 0.33$ m, $h = 0.006$ m, $b = 0.018$ m, $\eta = 0.1$, and $= 1140$ kg/m^3.It is assumed that a variation of the unperturbed value of EI and ρA can be modelled by a homogeneous Gaussian random field with a memoryless transformation. For numerical calculations we considered 20% of variation with a correlation length of b $= L/3$.

A random field estimation of the beam flexural bending and mass per unit of length was performed. The data was generated using 4 terms in the KL expansion and the deviations from the initial value are distributed, simulating a physically realistic property. In this numerical test, we use the FRFs obtained at node 1 and 2 of the perturbed beam element as "measured". Two FRF's are considered in order to increase the information of the response and accurate the parameter estimation. The objective is to reconstruct the distributed flexural bending EI function and mass ρA from the "measured" FRFs obtained with a sample of a stochastic beam model.

The random field samples estimated using two FRFs with 4 parameters (ξ) of the KL expansion are shown in Fig.2. In both cases, the functions are smoother than the simulated functions that generated the "measured" data. This is due to the use of just small number of FRF, therefore with poor spatial information. The reconstructed random fields are used to calculate the FRF of the stochastic beam at each iteration of the optimization procedure. The comparison between the "measured", initial and updated FRF is shown in Fig. 7. The FRF with the estimated parameters is closer to the "measured" FRF, as expected.

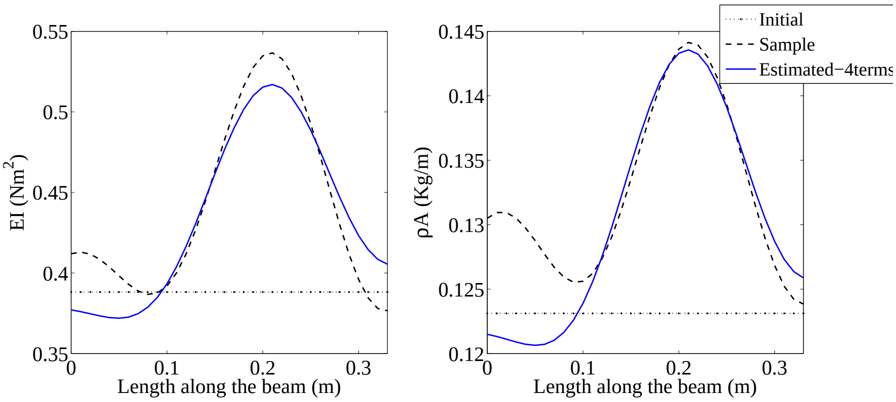

Figure 2: Initial, actual and reconstructed values of the flexural bending along the length (left-hand side) and mass per unit of length (right-hand side).

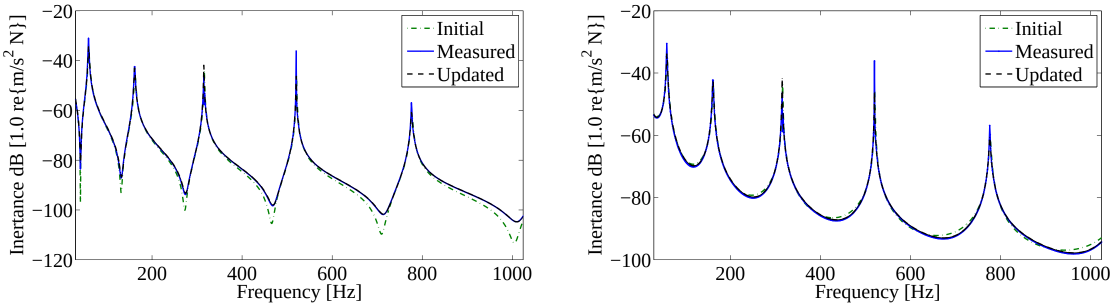

Figure 3: Comparison among an initial value, updated and the ”measured” FRF at node 1(left-hand side) and at node 2 (right-hand side)

3.1 Experimental test results

A beam made of polyamide (PA) with uniform rectangular cross-section was used in the experiment tests. The beam is 18mm wide, 6mm thick, with a mass per unit length of 0.02343kg/m. The average bending rigidity (EI) was obtained experimentally. The beam was manufactured using the Selective Laser Sintering (SLS) technology. As a consequence of the manufacturing process, a variability of the beam properties along its length can be expected. In order to verify the efficiency of the proposed method it was applied to a measured FRF and results were compared with measurements of the bending rigidity at many points along the beam using an ultrasound apparatus. The Young's modulus (E) was measured at 22 points along the beam with an ultrasonic pulse-echo device. The experimental setup is shown in Fig.(4). In this experiment a shear wave transducer $(U8403072/U8403071)$ was used. The signals were measured and analysed using an Olympus Panametrics NDT EPOCH 4 Ultrasonic Flaw Detector. The measured Young's modulus E along the beam is shown in Fig.(6), where it is compared with the predicted values using the KL expasion with 4 estimated parameters.

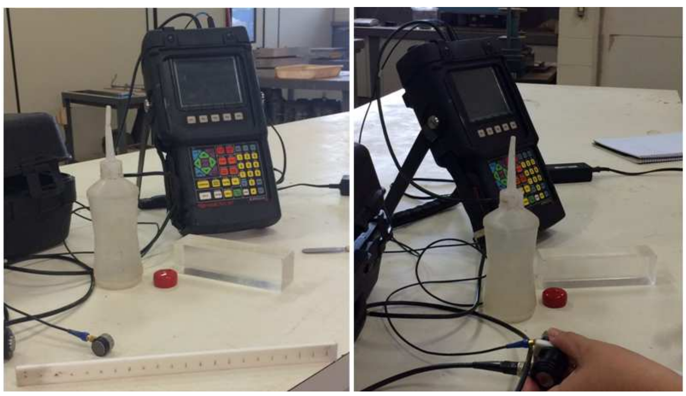

Figure 4: Procedure for experimental measure of Nylon beam properties.

Figure(5) shows the second experimental test setup, used to measure the FRF. The signals were acquired and analysed using LMS Test Lab. The FRFs were estimated with a bandwidth of 1024Hz and 1024 spectral lines. An impact hammer was used to excite the structure and a micro accelerometer to measure the response. The experimental FRFs were obtained by impact force excitation at node 1 and acceleration response at the point 1 and point 2.

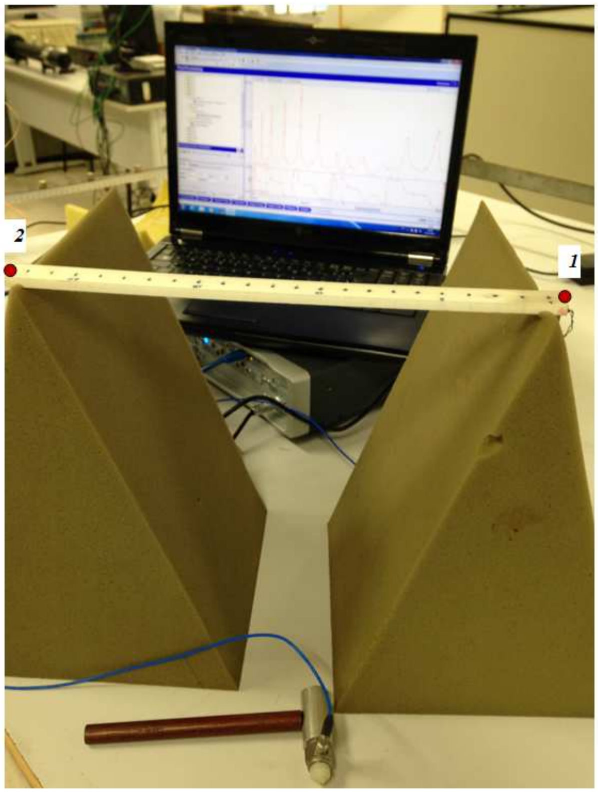

Figure 5: The test rig for the free-free beam.

The FRFs reconstructed with the estimated flexural bending with 4 terms (ξ) of the KL expansion is shown in Fig.7. The reconstructed $EI(x)$ presents a good agreement with the vales measured by ultrasound. The FRFs obtained with the initial constant value of EI, with the estimated $EI(x)$ and the measured FRFs are shown in Fig. 6. As the random field could not be reconstructed accurately, an acceptable difference between updated and measured FRFs can be observed.

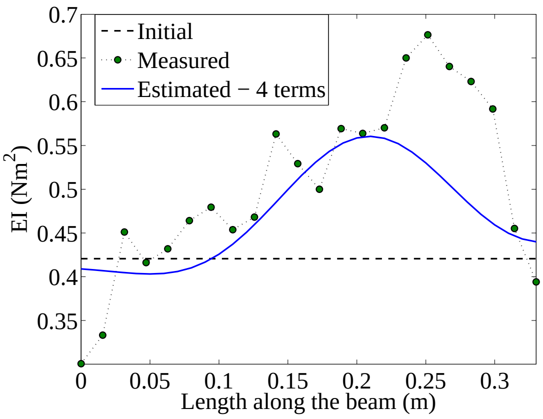

Figure 6: Initial, measured experientially and reconstructed values of the flexural bending $(EI(x))$ along the length

4 FINAL REMARKS

In the present work a technique to estimate spatially distributed parameters of samples of a stochastic structure using a KL expansion and sensitivity-based FRF model updating was proposed. Randomness was included in the flexural rigidity

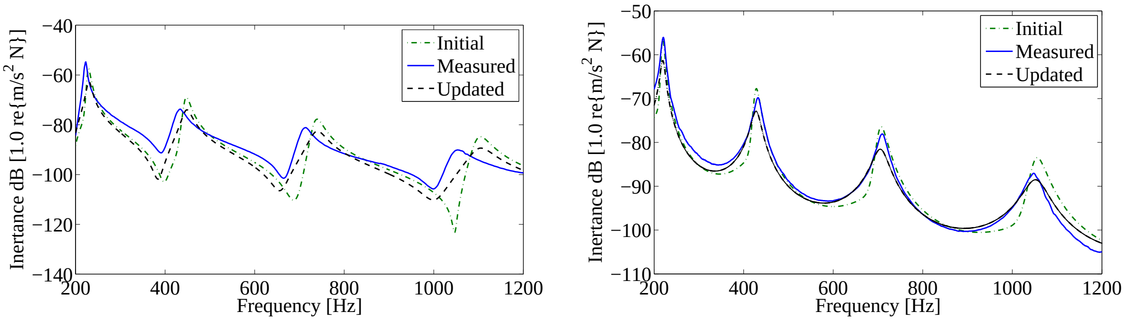

Figure 7: Comparison among an initial value, updated and the measured experientially FRF at node 1(left-hand side) and at node 2 (right-hand side)

(EI) and mass per unit length (ρA) of a beam structure. As a stochastic model is employed, the sensitivity-based method using FRF is also developed for a stochastic model based on spectral beam elements. To verify the efficiency of the presented technique a numerical test and an experimental test were performed. In the first case random field estimation of the beam flexural bending and mass per unit length was performed. This analysis tries to simulate a realistic situation where the true model parameters, EI and ρA can deviate from the baseline homogeneous values. The objective is to reconstruct the distributed random field from the "measured" FRF obtained for a sample of a stochastic model of the beam. The discretized variables (ξ) were estimated from the "measured" FRF through of a non-linear least squares curve fit procedure. A subset of these random variables can be considered as parameters to reconstruct the random field of the flexural bending and mass per unit of length. In the experimental test an experimentally obtained FRF was used. In this case only the flexural bending was predicted. An experimental measurement of the Young's modulus at 22 point along the beam was performed using ultrasound. By comparing the reconstructed and experimentally measured of $EI(x)$ the proposed method proved work reasonably well. Ongoing work consist of improving these preliminary results by curve fitting many measured FRF, instead of just one, to enrich the spatial information of the measured data.

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