

ROBUST ROTATIONAL STABILITY ANALYSIS OF A VERTICAL DRILL-STRING

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Abstract: We are interested in predicting rotational instability regions in the operation of a vertical drill-string, where stick-slip is likely to occur. The stick-slip is a severe type of self-excited vibration, which affects mainly polycrystalline diamond compact (PDC) bits, characterized by large fluctuations in the angular velocity of the bit. The stick-slip is one of the major causes of drill-string failures. We have modelled the drill-string as a shaft with piecewise constant cross section using Saint-Venant torsion theory with a non-linear boundary condition. Then, we converted the partial differential equation of movement to the discrete domain using the finite elements method. Finally, we represented the dynamical system in the state space and analysed its stability. We will introduce a new definition of stability suitable to predict the likelihood of stick-slip occurrence. Given the difficulty in characterizing some parameters of the model, uncertainties are present, and, to improve the predictability of the analysis, they will be taken into account.

Keywords: Stability, Drill-String, Vibration

INTRODUCTION

Motivation

The drilling of petroleum wells is performed by a drilling rig. The details of the drilling process of oil wells can be found, for instance, in Triggia *et al.* (2001) and Halliburton (1997). A rotary drilling rig (see the figure 1) in operation is subject to rotational, transversal and axial vibrations (Aadnoy *et al.* (2009)) (see the figure 2).

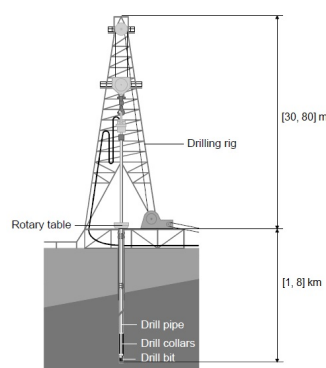


Figure 1 – Rotary drilling rig and its main components
(Serrarens *et al.* (1998))

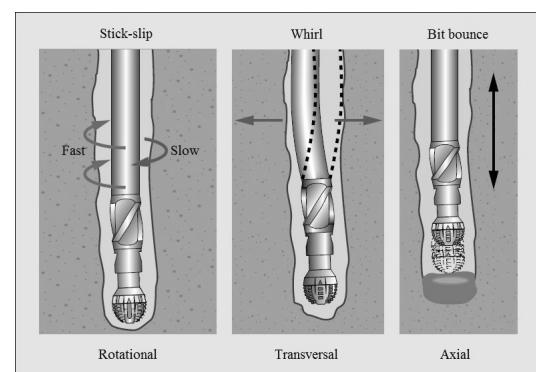


Figure 2 – Vibration modes of the drill-string
(Schlumberger (2010))

The rotational vibration occurs mainly with polycrystalline diamond compact (PDC) bits (Germaey *et al.* (2008)). In some extreme cases of rotational vibration, the bit experiences alternating phases of stick¹ and slip² (Liu *et al.* (2013)). The very repetitive alternating torsional load induced by the stick-slip makes the drill-string rapidly achieve its maximum number of cycles for fatigue failure (Wu *et al.* (2012)). The premature failure of the costly equipments usually employed in the oil drilling can turn the process economically non-viable. So, the avoidance of this type of vibration makes the study of the drill-string dynamics interesting for the oil industry. On the other hand, the difficulties in modeling the complicated dynamics of a drill-string make it a challenging applied math problem (Ritto (2010)).

¹Stick phase: the bit rotation stops for a finite instant of time.

²Slip phase: the bit rotation quickly accelerates up to a few times the imposed angular velocity.

Objective

Pavone and Desplans (1994) had already recognized the stick-slip as an unstable process. The stability map of a dynamical system is a graphical representation of all possible ordered n -uples of n input parameters of the dynamical system with different types of behavior of a given solution of the system (Strogatz (1994)). Each type of behavior of the given solution is represented by a region³ in the stability map.

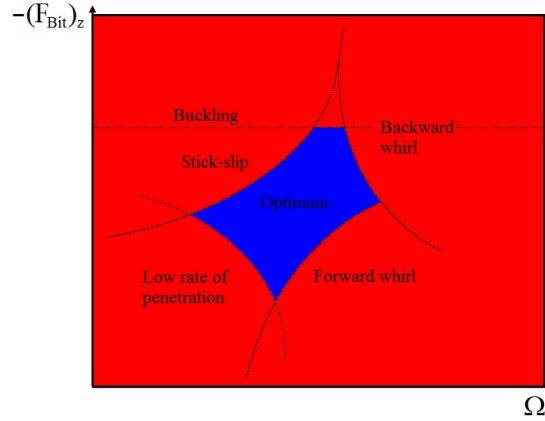


Figure 3 – Experimental stability map of a drill-string (Wu *et al.* (2010))

We can find in the literature experimental stability maps like the one depicted in the figure 3. In this map, the input parameters considered are Ω and $-(F_{Bit})_z$. Where Ω is the imposed angular velocity, $(F_{Bit})_z$ is the weight on the bit. We have put the minus sign in front of $(F_{Bit})_z$ in order to have positive values in the ordinate axis, since $(F_{Bit})_z$ points the opposite direction of the z -axis. The undesirable regions are colored in red. To ensure that the drill-string does not vibrate severely and the rate of penetration is economically viable, the driller should perforate in the optimum region⁴, the blue area in the stability map of the figure 3 (Percy (2014)). However, when drilling hard or abrasive rocks, the optimum region may even not exist (Wu *et al.* (2010)).

The objective of this article is to reproduce, at least qualitatively, the upper left corner of the figure 3 by means of computer simulation. Although, these abrupt transitions between the regions of the figure 3 can not be taken too seriously, due to the lack of knowledge of the exact values of the perforation parameters. Even if the driller operate within the optimal region but very close to the boundaries, he suffers high risk of having an unstable process. Put it another way, he needs what is commonly called in engineering as a safety factor. The probability of being stable is a mathematically elegant way to express this safety factor. We will perform a robust⁵ rotational stability analysis of the drill-string.

MATHEMATICAL MODEL

Strong form

There are several ways to model the nonlinear dynamics of a drill-string. Christoforou and Yigit (2003) use a fully coupled axial, lateral, and torsional lumped model. Tucker and Wang (2003) use the Cosserat⁶ theory. Richard *et al.* (2007) use a coupled axial and torsional two degrees of freedom model. Liu *et al.* (2013) use a fully coupled axial, lateral, and torsional eight degrees of freedom model, while Ritto *et al.* (2009) and da Cunha Junior (2015) use the Timoshenko⁷ beam theory combined with finite strain and coupled Saint-Venant⁸ torsion. We have addressed the rotational dynamics of the drill-string with Saint-Venant torsion theory. The strong form of our problem is (Meirovitch (1967)):

Find a function $\theta_z := (z, t) \mapsto \theta_z(z, t)$ with degree of continuity $C^2[0, L]$ in z and $C^2[0, \tau]$ in t where $z \in [0, L]$ and $t \in [0, \tau]$, such that

³Region: a graphical representation of a subset of all possible ordered n -uples of the n input parameters considered.

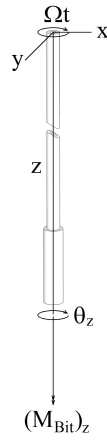
⁴Optimum region: the stable area excluding the low rate of penetration region.

⁵Robust: considering uncertainties.

⁶Eugène Maurice Pierre Cosserat (1866 – 1931): was a French mathematician and astronomer (Wikipedia (2014)).

⁷Stephen Timoshenko (1878 – 1972): was a ukrainian engineer (Wikipedia (2014)).

⁸Adhémar Jean Claude Barré de Saint-Venant (1797 – 1886): was a french engineer and mathematician (Wikipedia (2014)).



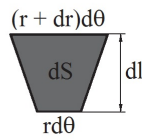
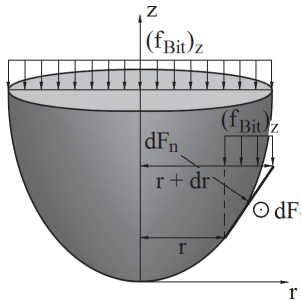
$$\begin{cases} \frac{\partial}{\partial z} \left(GJ \frac{\partial \theta_z}{\partial z} \right) + m_z = \rho J \frac{\partial^2 \theta_z}{\partial t^2}, & \text{if } z \in]0, L[\text{ and } t \in]0, \tau[\\ \theta_z = \Omega t, & \text{if } z = 0 \text{ and } t \in]0, \tau[\\ GJ \frac{\partial \theta_z}{\partial z} = (M_{Bit})_z, & \text{if } z = L \text{ and } t \in]0, \tau[\\ \theta_z = 0, & \text{if } z \in]0, L[\text{ and } t = 0 \\ \frac{\partial \theta_z}{\partial t} = 0, & \text{if } z \in]0, L[\text{ and } t = 0. \end{cases} \quad (1)$$

Where θ_z is the angular displacement of the drill-string cross section around its axis, z is the coordinate along the drill-string axis, t is the time, G is the shear modulus, J is the area moment of inertia of the drill-string cross section around its axis, ρ is the density (see the table 3), m_z is the axial torque per unit length, $(M_{Bit})_z$ is the torque on the bit. $]a$ denotes a real interval open in a , and $[a$ denotes a real interval closed in a . $C^m]a, b[$ denotes the set of functions m times differentiable in $]a, b[$ and with the m -th derivative continuous in $]a, b[$. Note that if a function has degree of continuity $C^m]a, b[$, its n -th derivative has degree of continuity $C^{m-n}]a, b[$. Particularly, a function has degree of continuity $C^0]a, b[$ if it is continuous in $]a, b[$. We usually omit the interval $]a, b[$ when it is the whole real line.

We attend only to the case in which m_z has degree of continuity $C^0]0, L[$ in z and $C^0]0, \tau[$ in t , where τ is some strictly positive real number such that the solution of the problem exists and is unique (Boyce and DiPrima (2006)). However, in our problem J has degree of continuity $C^{-1}]0, L[$ in z , so the equations (1) is not a well posed problem, since $\frac{\partial}{\partial z} \left(GJ \frac{\partial \theta_z}{\partial z} \right)$ does not exist in some $z \in]0, L[$. This motivates us to search for another form (see the subsection Weak form) of our problem with weaker assumptions about the continuity of θ_z . This other form is called weak, and is the form used in finite elements method.

Bit-rock interaction

We have approximated the bit by a solid of revolution (see the figure 4) whose generating line is a function $z := r \mapsto z(r)$, where r, θ, z are the cylindrical coordinates. The infinitesimal force dF_n normal to the surface is given by $dF_n = (f_{Bit})_z (\cos \phi)^2 dS$, where $(f_{Bit})_z$ is the vertical load per unit horizontal area⁹, ϕ is the slope angle of the tangent to $z(r)$ at the considered position, dS is the infinitesimal area in which dF_n is acting. From the geometry we have:



$$\begin{aligned} \cos \phi &= \frac{1}{\sqrt{1 + \left(\frac{dz}{dr} \right)^2}} \\ dl &= \sqrt{1 + \left(\frac{dz}{dr} \right)^2} dr \\ dS &= r d\theta dl \end{aligned} \quad (2)$$

Figure 4 – Idealized bit

Using the Coulomb¹⁰ law of dry friction, we have $dF_t = \mu dF_n \operatorname{sgn} \left(\frac{\partial \theta_z}{\partial t} \right)$, where dF_t is the infinitesimal force tangential to the surface, μ is the kinetic coefficient of friction, sgn is the sign¹¹ function. Substituting the equation for dF_n in the equation of the Coulomb law, using the geometric relations of the equation (2), and integrating over r and over θ , we obtain the torque relative to the z -axis $(M_{Bit})_z = 2\pi \bar{r} \mu (F_{Bit})_z \operatorname{sgn} \left(\frac{\partial \theta_z}{\partial t} \right)$. Where $\bar{r}$ is the lever-arm characteristic radius, which is defined by:

⁹The vertical load per unit horizontal area is $\frac{(F_{Bit})_z}{\pi (R_{Bit})^2}$.

¹⁰Charles-Augustin de Coulomb (1736 – 1806): was a french physicist (Wikipedia (2014)).

¹¹The sign function extracts the sign of its argument. Whenever $x \neq 0$ we define (Wikipedia (2014)): $\operatorname{sgn}(x) := \frac{x}{|x|}$. Otherwise $\operatorname{sgn}(x) := 0$.

$$\bar{r} := \frac{1}{\pi (R_{Bit})^2} \int_0^{R_{Bit}} \frac{r^2}{\sqrt{1 + \left(\frac{dz}{dr}\right)^2}} dr \quad (3)$$

Where R_{Bit} is the bit radius. Kovalyshen (2013) proposed four different idealized cut profiles. We calculated the $\bar{r}$ for each one of them (see the figure 5). In the right hand side of the figure 5 you can see a real cut profile of a PDC bit, which is quite similar to the second idealized cut profile of Kovalyshen (2013).

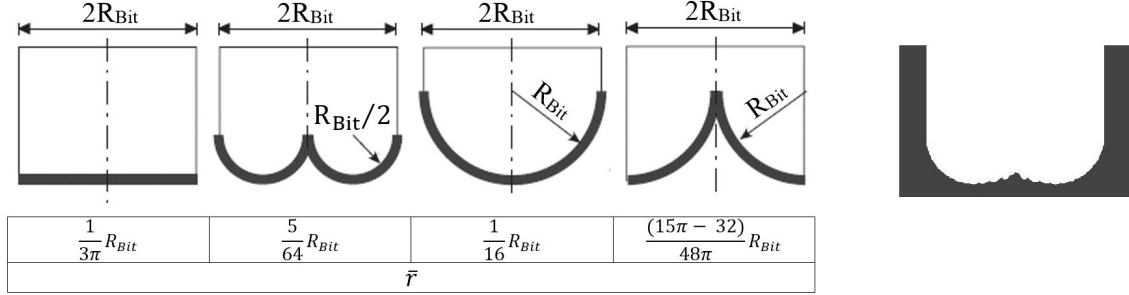
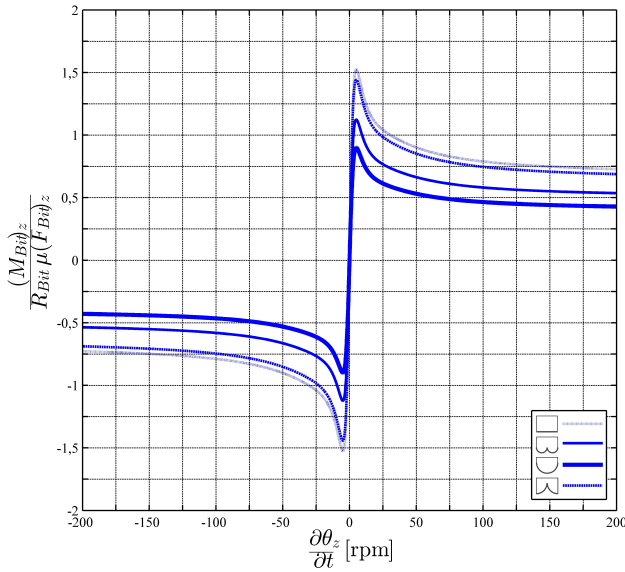


Figure 5 – Idealized cut profiles (Kovalyshen (2013)) with their corresponding $\bar{r}$ and a typical cut profile of a PDC bit (Weaver and Clayton (1993))

In order to avoid difficulties when differentiating $(M_{Bit})_z$, we have approximated it by the following C^1 function:



$$(M_{Bit})_z = 2\pi\bar{r}\mu (F_{Bit})_z \left(\tanh\left(\alpha_0 \frac{\partial\theta_z}{\partial t}\right) + \frac{\alpha_1 \frac{\partial\theta_z}{\partial t}}{1 + \alpha_2 \left(\frac{\partial\theta_z}{\partial t}\right)^2} \right) \quad (4)$$

Figure 6 – Graph of the torque on the bit

Where $\alpha_0, \alpha_1, \alpha_2$ are the bit-rock interaction constants (see the table 3). The graph of the function (4) is shown in the figure 6 for different cut profiles. The nearly vertical stretches of the figure 6 do not exist in reality, in fact, the friction torque can be any value until it reaches the maximum static friction torque. The work of Brett (1992) corroborates previous findings that it is just this difference between the static and kinetic coefficients of friction inherent of PDC bits that causes stick-slip. It is worth to mention that Lima *et al.* (2015) concluded that the rotational stability of the drill-string is affected by how the $(F_{Bit})_z$ is applied. In this article, we will consider that the $(F_{Bit})_z$ is applied all at once.

Weak form

In order to weaker our assumptions about the continuity of θ_z , we define a set of weight functions:

$$\mathbb{W} := \left\{ W : [0, L] \mapsto \mathbb{R} \left| \left\langle \frac{dW}{dz}, \frac{dW}{dz} \right\rangle_B < \infty \text{ and } W(0) = 0 \right. \right\} \quad (5)$$

And manipulate the equations (1) using the standard method *multiply by a weight function and integrate by parts* (Hughes (1987)). Thus, we obtain the so called weak form of our problem:

Find a function $\theta_z := (z, t) \mapsto \theta_z(z, t)$ with $\langle \frac{\partial \theta_z}{\partial z}, \frac{\partial \theta_z}{\partial z} \rangle_B < \infty$ and degree of continuity $C^2[0, \tau[$ in t where $t \in [0, \tau[$, such that

$$\begin{cases} \left\langle \rho J \frac{\partial^2 \theta_z}{\partial t^2}, W \right\rangle_B + \left\langle G J \frac{\partial \theta_z}{\partial z}, \frac{dW}{dz} \right\rangle_B = \langle m_z, W \rangle_B + (M_{Bit})_z W(L), & \text{if } t \in]0, \tau[\\ \theta_z = \Omega t, & \text{if } z = 0 \text{ and } t \in]0, \tau[\\ \langle \theta_z, W \rangle_B = 0, & \text{if } t = 0 \\ \left\langle \frac{\partial \theta_z}{\partial t}, W \right\rangle_B = 0, & \text{if } t = 0 \end{cases} \quad (6)$$

for all function $W \in \mathbb{W}$.

Where $\langle f, g \rangle_B$ denotes $\int_0^L f(z)g(z)dz$. B stands for the drill-string body. This form is called weak because its solutions can have lower degree of continuity than the solutions of the strong form. The attempt to solve the weak form instead of the strong form is called method of weighted residuals.

Fem approach

The system for the finite element B_i is written in shorthand notation:

$$\mathbf{M}_{B_i} \ddot{\mathbf{U}}_{B_i} + \mathbf{K}_{B_i} \mathbf{U}_{B_i} = \mathbf{F}_{B_i} \quad (7)$$

Where $\mathbf{U}_{B_i}$ is called finite element displacement vector, $\mathbf{M}_{B_i}$ is called finite element mass matrix, $\mathbf{K}_{B_i}$ is called finite element stiffness matrix, $\mathbf{F}_{B_i}$ is called finite element force vector.

In our problem, ρ , G , J are piecewise constant functions. If we choose the finite elements such that ρ , G , J do not vary inside them, $\mathbf{M}_{B_i}$ and $\mathbf{K}_{B_i}$ can be easily evaluated (Pilkey (2005)):

$$\mathbf{M}_{B_i} := \frac{\rho J l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \mathbf{K}_{B_i} := \frac{G J}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (8)$$

Where l is the finite element length.

In order to get rid of the internal torques (unknown) of the system (7), we can assembly the global finite elements matrices (Fish and Belytschko (2009)). Rather than considering a damping term direct in the equation (1), we will insert a damping matrix a posteriori in the global system of finite elements assuming a proportional damping of the modes (Meirovitch (2001)). In this article, we have considered the damping ratios of all modes of vibration to have the same value of the damping ratio of the first flexible mode ξ_1 . We should calculate the damping matrix two times. The first one before inserting the boundary conditions, only to obtain the unknown torque $(M_{RT})_z$ expended by the rotary table (or the top drive) to maintain the angular velocity at the top constant. The second one after inserting the boundary conditions (the first node displacement is Ωt), to actually solve the system.

STABILITY ANALYSIS

State space form

The state space form of a n order ordinary differential equation consists in representing the equation by a system of first order ordinary differential equations. This is the form used by MATLAB® R2013b ode45 routine to solve differential equations numerically and by us to analyse the stability. We can do that writing our system in an clever way:

$$\underbrace{\begin{bmatrix} \dot{\mathbf{U}} \\ \ddot{\mathbf{U}} \end{bmatrix}}_{\dot{\mathbf{X}}} = \underbrace{\begin{bmatrix} \mathbf{0} & \delta \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{U} \\ \dot{\mathbf{U}} \end{bmatrix}}_{\mathbf{X}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{F} \end{bmatrix}}_{\mathbf{B}} \quad (9)$$

Observe that the first line of the equation (9) is just an identity, and the second line is our second order system. Thus, we obtain the so called state space form of our problem:

Find a function $\mathbf{X} := t \mapsto \mathbf{X}(t)$ with degree of continuity $C^1[0, \tau[$ in t where $t \in [0, \tau[$, such that

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{B}(\mathbf{X}), & \text{if } t \in]0, \tau[\\ \mathbf{X} = 0, & \text{if } t = 0 \end{cases} \quad (10)$$

Where $\mathbf{X}$ is called state space vector, $\mathbf{A}$ is called state space coefficients matrix, $\mathbf{B}$ is called state space force vector.

Stability definition

Stability is a controversial topic. There is no consensus on the definition of stability, instead, we can find in the literature several different definitions (Nayfeh and Balachandran (2004)). The definition used here is a modification of Lagrange¹² stability, which, as will be seen, is suitable to predict the stick-slip occurrence in the long term.

Let Φ be the solution of the initial value problem (10), and let Ψ be a particular¹³ solution of the system (10) which we want to evaluate the stability:

We say that Ψ is stable if, and only if Φ converges to Ψ .

In our case, we have evaluated the stability of a particular solution of the form $\Psi := \begin{bmatrix} \Omega t \\ \Omega \end{bmatrix} + \begin{bmatrix} \theta_0 \\ 0 \end{bmatrix}$, where Ω is a n dimensional vector whose entries are Ω , θ_0 is a n dimensional constant vector representing the static torsion deformation, which can be obtained forcing Ψ to satisfy the system of differential equations (10). Ψ is the steady state solution, which corresponds to every node rotating with the same angular velocity Ω of the top, this is, without stick-slip. To draw the stability map, we have considered the initial value problem solution of the linearized system (10). This procedure is called linear stability analysis, and can be done by simple eigenvalues inspection, saving a lot of computational time. We have:

Ψ is stable if, and only if the real part of each eigenvalue of $\mathbf{A} + \nabla \mathbf{B}(\Psi)$ is less than zero.

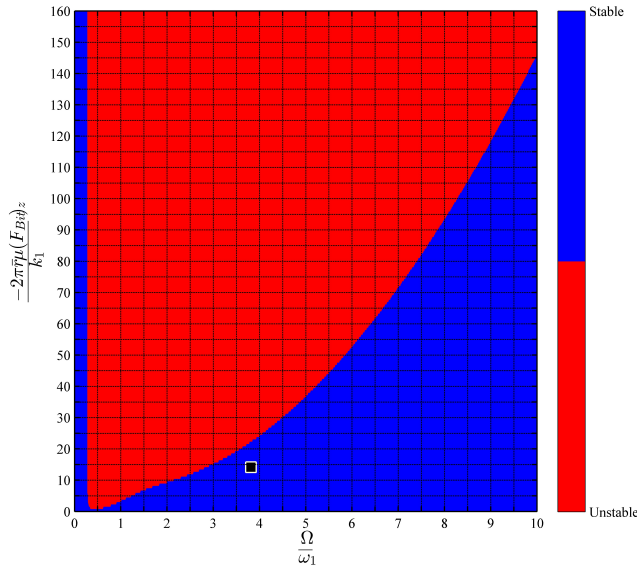


Figure 7 – Linear rotational stability map

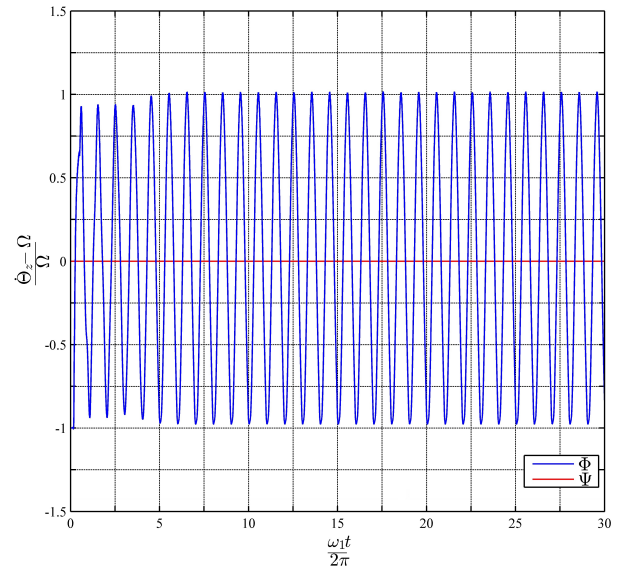


Figure 8 – Angular velocity of the test point

Pavone and Desplans (1994) and Kovalyshen (2013) noted that the predictions obtained using the linear stability analysis are in good agreement with experimental observations. However, the linear analysis should be used with caution, since it can predict a wrong behavior for the nonlinear system, specially in the transition regions near the stability curve. For instance, in the figure 7 is shown the rotational linear stability map for the drilling of limestone. Where k_1 is the first

¹²Joseph-Louis Lagrange (1736 – 1813): was an italian astronomer and mathematician (Wikipedia (2014)).

¹³Particular solution of a system of ordinary differential equations: a solution of the system which does not necessarily satisfy the initial condition (Boyce and DiPrima (2006)).

eigenvalue of the finite elements stiffness matrix $\mathbf{K}_B$ after inserting the boundary conditions, ω_1 is the natural frequency of the first flexible mode of vibration. Consider the test point (black square) of the figure 7 which is supposed to be stable. Its angular velocity is plotted in the figure , and it is clearly not stable at all. This is due to the numerical error introduced by the linearization process.

RESULTS

Modeling the bit-rock interaction is not trivial, and we have used a simple mathematical expression. So, we will consider uncertainty in the kinetic coefficient of friction μ . We will consider uncertainty, also, in the damping ratio ξ , since it is extremely difficult to measure the drill-string damping. Each one were considered separately. In order to obtain the probability density functions of the stochastic variables, we have used the principle of maximum entropy¹⁴, and to obtain the stochastic results, we have run several Monte Carlo¹⁵ experiments.

In the case of the kinetic coefficient friction μ the constraint is $a \leq \mu \leq b$. The probability density function of μ which maximizes entropy with this constraint is the uniform density (Kapur and Kesavan (1992)): $f_\mu(x) := \frac{1}{b-a}$. The constants a, b depend on the type of rock which is being drilled (see the table 1). In the figure 9 are shown the rotational stability maps for different mean values of μ . It can be noticed that an increase in the μ reduces the stable region. This statement is consistent with empirical findings that rough rocks are more susceptible to stick-slip.

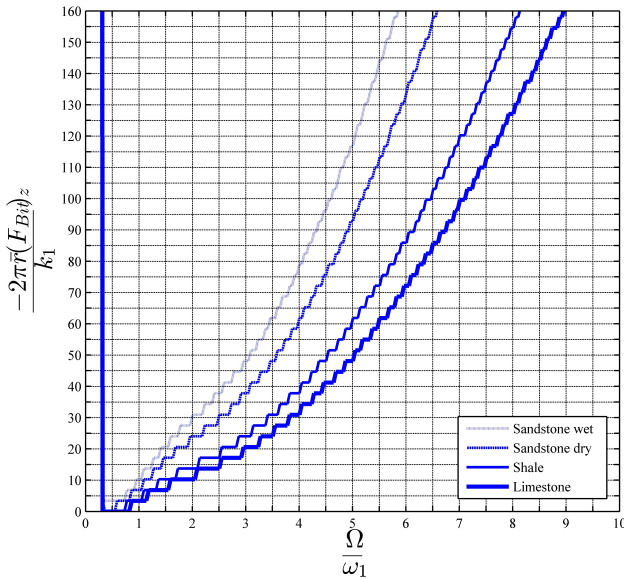


Table 1 – Kinetic coefficient of friction (Gaffney (1976); Kohli and Zoback (2013))

Rock	a	b
Shale	0.404	0.794
Sandstone	Wet	0.174
	Dry	0.353
Limestone	0.695	0.754

Figure 9 – Rotational stability maps for different μ

In the case of the damping ratio ξ the constraint is $0 < \xi \leq 1$. For computational reason, the following two constraints are also imposed $\mu(\ln(\xi))$, $\mu(\ln(1 - \xi))$. Where μ is the mean¹⁶ function. Now, the principle of maximum entropy gives us the beta density: $f_\xi(x) := \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$. Where B is the beta¹⁷ function. The constants a, b are evaluated as follows:

$$\mu(\xi) = \frac{a}{a+b} \quad \sigma(\xi, \xi)^2 = \frac{ab}{(a+b)^2(a+b+1)} \quad (11)$$

Where σ^2 is the variance¹⁸ function.

¹⁴The principle of maximum entropy states that the probability density function to be chosen is the one with largest entropy which respects the variable constraints (Wikipedia (2014)).

¹⁵The Monte Carlo method consists in generating samples of the stochastic variables and solving the deterministic problem for each value of the sample. And then make the statistics to find a unknown probability density function (Wikipedia (2014)).

¹⁶The mean function measures the central tendency of a stochastic variable. In the case of a continuous stochastic variable y , with probability density function $x \mapsto f_y(x)$, we define (Wikipedia (2014)): $\mu(y) := \int_{\mathbb{R}} x f_y(x) dx$. Not to be confused with the kinetic coefficient of friction.

¹⁷The beta function is a special function. Whenever $x, y > 0$ we define (Wikipedia (2014)): $B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt$.

¹⁸The variance function measures how much two stochastic variables change together. In the case of two stochastic variables x, y we define (Wikipedia (2014)): $\sigma(x, y)^2 := \mu((x - \mu(x))(y - \mu(y)))$.

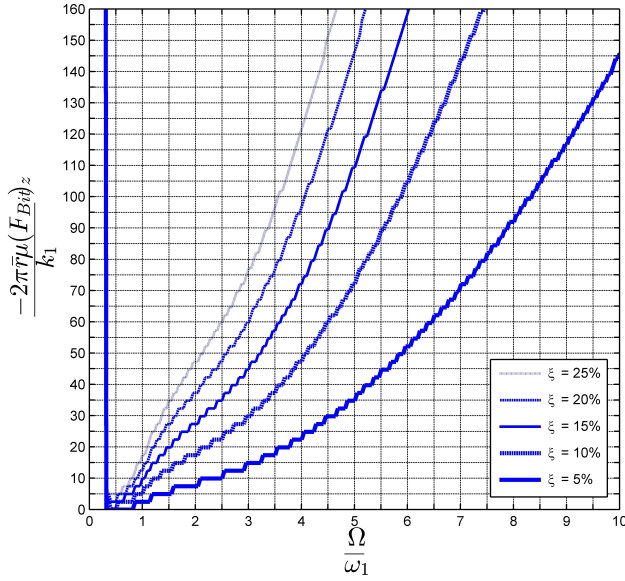
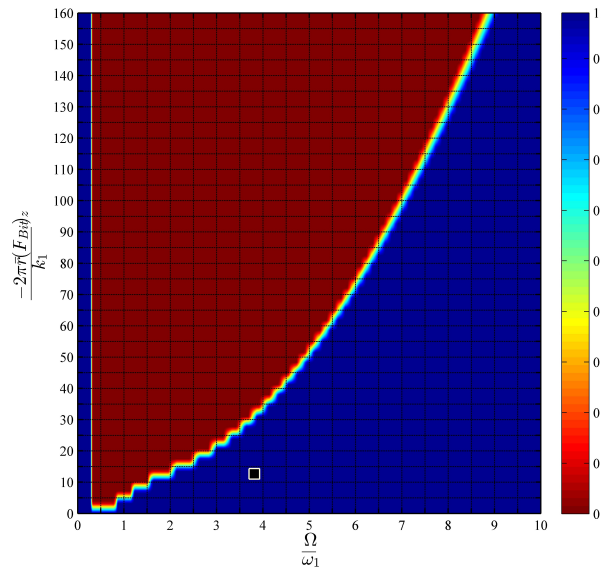
Figure 10 – Rotational stability maps for different ξ

Table 2 – Damping ratio

$\mu(\xi)$	$cv(\xi)$
0.05	1.00
0.10	1.00
0.15	1.00
0.20	1.00
0.25	1.00

In this article, we have specified the mean $\mu(\xi)$ and the coefficient of variation¹⁹ $cv(\xi)$, and then solved the relations (11) to find the constants a , b (see the table 2). In the figure 10 are shown the rotational stability maps for different mean values of ξ . As our intuition expects, the damping mitigates the stick-slip. The small steps in the figures 9 and 10 are consequence of the algorithm used to determine the stability curve.

Figure 11 – Rotational stability map considering uncertainty in μ

In the figure 11 is shown the rotational stability map considering uncertainty in the μ for the drilling of limestone. The color represents the probability of being stable. We can see that the map of the figure 11 is not too different from the map of the figure 7. We can still perceive a stability curve separating the stable and unstable areas. Furthermore, we can remove the dependence of the stability curve on the kinetic coefficient of friction multiplying the ordinate axis by μ . Thus, for reasonable values of uncertainty in the kinetic coefficient of friction, the rotational stability map does not change significantly, unless you do not know what type of rock you are drilling. In the figure 12 is shown the corresponding bit

¹⁹The coefficient of variation function measures the dispersion of a stochastic variable. In the case of a stochastic variable x we define (Wikipedia (2014)): $cv(x) := \frac{\sigma(x, x)}{\mu(x)}$.

angular position of a test point (black square) of the figure 11. The angular velocity of this same test point is plotted in the figure 13. The colors in these two last figures represent the probability density function.

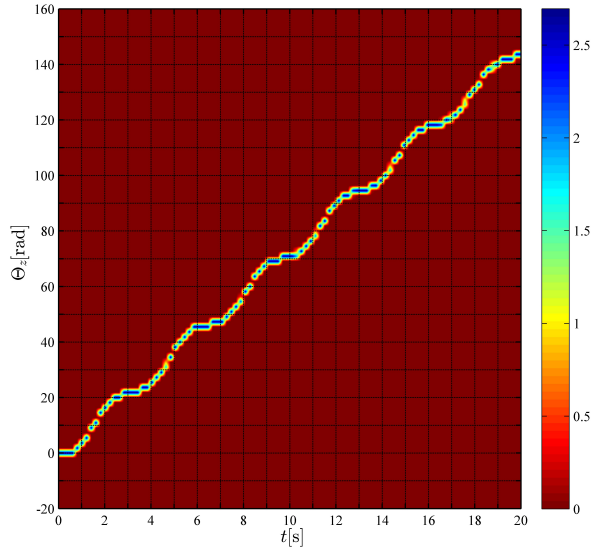


Figure 12 – Angular position of the test point of μ

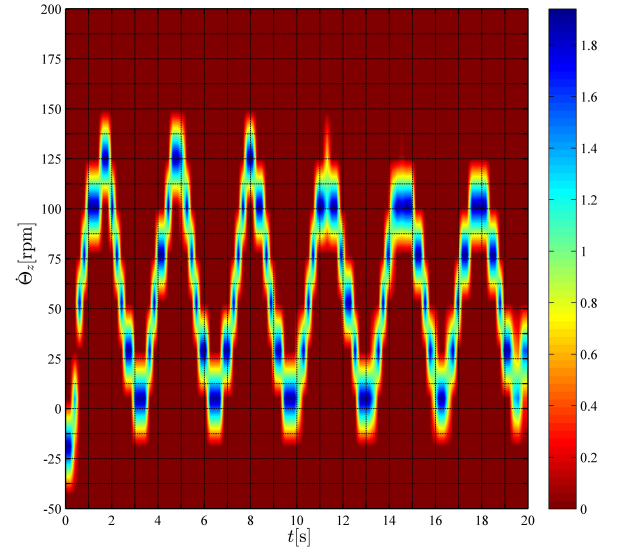


Figure 13 – Angular velocity of the test point of μ

The uncertainty in the damping ratio ξ has a much more interesting effect in the rotational stability map, as we can see in the figure 14. Instead of having a sharp curve between the areas, now we have a transition region with a non null probability of being unstable. The stable area becomes dramatically reduced. In the figure 15 is shown the corresponding bit angular position of a test point (black square) of the figure 14. The angular velocity of this same test point is plotted in the figure 16. The colors of these three last figures have analogous meaning of the corresponding figures for the kinetic coefficient of friction. To get these smooth gradients of colors of the stochastic results, we did a cubic interpolation of the mesh points.

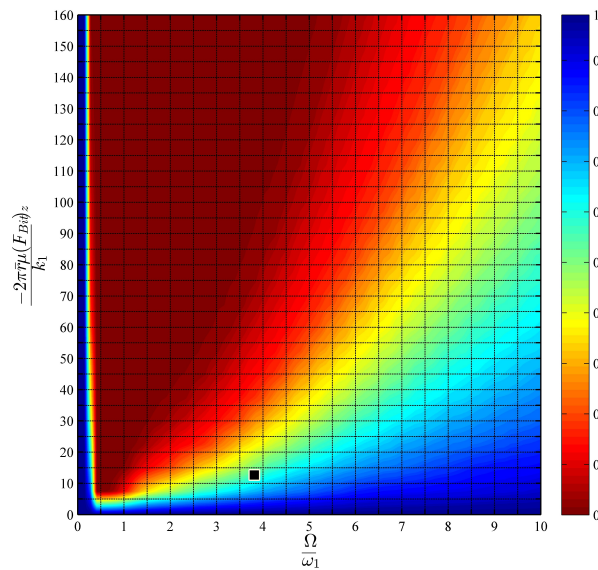
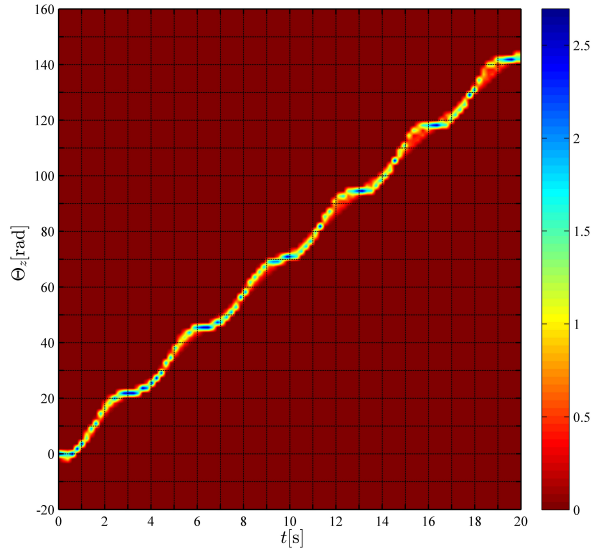
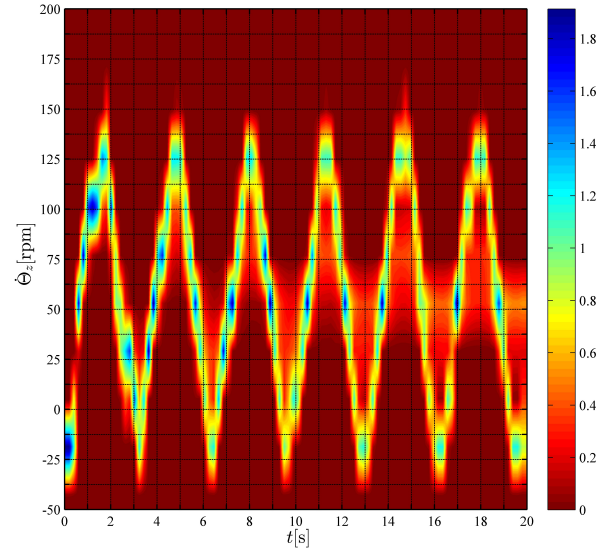


Figure 14 – Rotational stability map considering uncertainty in ξ

Figure 15 – Angular position of the test point of ξ Figure 16 – Angular velocity of the test point of ξ

FINAL REMARKS

We have used Saint-Venant theory to model the rotational dynamics of a drill-string with a bit-rock interaction which easily accommodates different bit shapes, and used this model to perform a linear stability analysis of the drill-string. Although the linear stability map reproduces qualitatively the experimental stability map, it predicts a wrong behavior near the stability curve. This drawback opens opportunity to search for new strategies of stability analysis which avoid solving the dynamical system. Particularly, the authors consider promising the strategy of using the linear stability as a preliminary analysis, and then solve the system for a small region offsetted from the stability curve.

In the stochastic analysis, we could observe that the uncertainty in the damping ratio is more important than in the kinetic coefficient of friction. It vanishes with the stability curve and reduces the stable area. To the authors knowledge, stability analysis considering uncertainties is a new result. Nevertheless, we have considered only parametric uncertainties, it would be interesting to consider also model uncertainties, and how to perform the stability analysis in this case.

Our model considers only the degree of freedom of rotation. One could improve it including the lateral and axial degrees of freedom, and then perform the stability analysis trying to reproduce the other regions of figure 3.

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APPENDIX. DATA USED IN SIMULATIONS

The values of the drill-string dimensions and material properties are the same used by Percy (2014). The values of the bit-rock interaction constants α_0 , α_1 , α_2 are obtained by least squares fit²⁰ from experimental data of Schlumberger.

²⁰The least squares fit is a method to approximate experimental data by a function with indeterminate constants. These constants are obtained minimizing the sum of the squares of the errors between each point in the experimental data and the value of the given function at this point.

Table 3 – Values of the deterministic variables

Variable	Value	Unit	Description
L_{DP}	1800	m	Drill pipe length
L_{BHA}	200	m	Bottom hole assembly length
$(R_{DP})_{out}$	0.060	m	Drill pipe outer radius
$(R_{DP})_{in}$	0.0475	m	Drill pipe inner radius
$(R_{BHA})_{out}$	0.075	m	Bottom hole assembly outer radius
$(R_{BHA})_{in}$	0.0475	m	Bottom hole assembly inner radius
$(F_{Bit})_z$	$-12 \cdot 10^3$	N	Weight on the bit
ρ	7850	kg/m ³	Density
G	$70 \cdot 10^9$	Pa	Shear modulus
Ω	70	rpm	Rotary table angular velocity
α_0	0.4775	s	Zeroth constant of the bit-rock interaction
α_1	8.7854	s	First constant of the bit-rock interaction
α_2	4.5595	s ²	Second constant of the bit-rock interaction
μ	0.723	—	Kinetic coefficient friction
ξ	0.05	—	Damping ratio

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