

Generalized Fourier Series for Representing Random Variables and Application for Quantifying Uncertainties in Optimization

M. Bassi¹, J. E. Souza De Cursi¹, R. Ellaia²

¹ Laboratory of Optimization and Reliability in Structural Mechanics LOFIMS, Mechanical Engineering Department, National Institute of Applied Sciences of Rouen, Box 08-76801, Saint Etienne du Rouvray Cedex, France

² Laboratory of Studies and Research in Applied Mathematics LERMA, Scientific Computing and Modeling Departement, Mohammadia School of Engineering, Ibn Sina Avenue, Box 765, Agdal, Rabat, Morocco

Abstract: *we present a new Hilbert expansion type method for quantifying uncertainties in optimization problems. A demonstration is made in a Bochner space to determine the conditions of using this approach which is based on Generalized Fourier Series Expansion of random variables. The main advantage of this technic is the approximation of a random variable without need to determine its joint probability distribution whit another random vector, which is one of the defects of the famous Wiener Chaos Expansion based methods. Moreover, our method is more flexible and its application proves its numerical efficiency.*

Keywords: *Optimization, uncertainties quantification, Hilbert expansion, Generalized Fourier Series, Bochner space.*

INTRODUCTION

The interest in Uncertainty Quantification (UQ) has deeply increased in last years. The attention of many researchers was brought to the needs of quantifying uncertainties in many knowledge fields, namely in Applied Mathematics, Physics, Engineering and Economical Science, so that researches were focusing on UQ since many decades. In reality, uncertainties govern the nature and human activities: motions of every being, climate, flows and every natural phenomenon, industrial processes, stock market indexes and economies of the countries.

To deal with uncertainties, intensive studies were done and researchers have proposed several methods in order to improve the quality and the efficiency of UQ and an extensive literature may be found. Among these methods, Monte Carlo Simulation (MCS) may be considered as the most known UQ approach. However, MCS may become computationally expensive, since large samples may be requested in order to get significant results “Souza de Cursi and Sampaio. (2015)” and the statistics generated by this method are poor in accuracy “Poles and Lovison. (2009)”. Improvements may be obtained by using adapted sampling, such as, for instance, the Latin Hypercube Sampling (LHS). This last approach is considered as generating a better accuracy “Poles and Lovison. (2009)”. However, it also presents some limitations synthetized by “Lebon *et al.* (2014)” as follows: *The error estimates may not be improved even if the number of samples increases [...]. When used for training response surfaces, a space-filling optimal design is interesting in order to sample the design space with a minimum number of response evaluations. When using LHS there is a risk that some of the random samples form a cluster to the detriment of some unexplored part of the design space. To circumvent these issues, some strategies may be applied using re-sampling strategy, optimization algorithm, or geometrical consideration. The two first methodologies are straightforwardly compatible with the proposed approach, but not investigated in the present paper.*

A different and competitive approach has been proposed by “Wiener. (1938), Xiu and Karniadakis. (2002) and Sudret. (2008)”, usually referred as Polynomial Chaos Expansion, where random variables are represented by multivariate series of Gaussian variables. This approach has shown to be less expensive and computationally more efficient than the MCS and LHS, but it was limited by the use of Gaussian variables and the need of a particular statistical information about the distributions under consideration: when approximating a random variable $\mathbf{X}$ by a random Gaussian vector $\boldsymbol{\xi}$, the knowledge of the joint probability distribution of the couple $(\mathbf{X}, \boldsymbol{\xi})$ is requested - what may introduce some difficulty. In addition, the use of independent variables leads to poor-quality approximations “Holdorf Lopez. (2010)” – in such a situation, the conditional mean $E(\mathbf{X}|\boldsymbol{\xi})$, which is the best approximation of $\mathbf{X}$ by an arbitrary function of $\boldsymbol{\xi}$ is a constant (see Theorem 8, Corollary 9 and the example following these results). Moreover, “Branicki and Majda. (2013)” stated that *PCE has significant limitations in systems with intermittent instabilities or parametric uncertainties in the damping [...] in such important dynamical regimes, PCE performs, at best, similarly to the vastly simpler Gaussian moment closure technique utilized earlier by the authors in a different context for UQ.*

In practice, it seems as useful to consider a different approach: let the uncertainties on a system be modelled by a random vector $\mathbf{U}$ and the system’s response be denoted by $\mathbf{X}$. In such a context, it is natural to assume that $\mathbf{X}$ is a function of $\mathbf{U}$, $\mathbf{X} = \mathbf{X}(\mathbf{U})$ and that statistical information is available on the couple $(\mathbf{X}, \mathbf{U})$ – such as, for instance, a sample of the couple – in the sequel, we will establish that this condition may be weakened and that convenient approximations may be generated when this information is missing. Since $\mathbf{X} = \mathbf{X}(\mathbf{U})$, we may consider a Hilbertian

approach: assuming that $\mathbf{X} = \mathbf{X}(\mathbf{U}) \in V$, where V is a separable Hilbert space for the scalar product $(\bullet, \bullet)$, we can choose an adequate Hilbert basis $\{\varphi_i\}_{i \in \mathbb{N}^*}$ and look for a representation:

$$\mathbf{X} = \sum_{i \in \mathbb{N}^*} x_i \varphi_i(\mathbf{U})$$

In the framework of random variables, a convenient scalar product is

$$(\mathbf{Y}, \mathbf{Z}) = E(\mathbf{Y}\mathbf{Z}) .$$

Recalling that $\{\varphi_i\}_{i \in \mathbb{N}^*}$ is orthonormal, the coefficients may be easily determined:

$$(\varphi_i, \varphi_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases} \Rightarrow x_i = (\mathbf{X}, \varphi_i) .$$

If the family is not orthonormal, we look for a sequence of orthogonal projections:

$$\mathbf{X} \approx P_n \mathbf{X} = \sum_{1 \leq i \leq n} x_i \varphi_i(\mathbf{U}) ,$$

where $\mathbf{x} = (x_1, \dots, x_n)^t$ is the solution of the linear system

$$\mathcal{A}\mathbf{x} = \mathcal{B} , \quad \mathcal{A}_{ij} = (\varphi_i, \varphi_j) , \quad \mathcal{B}_i = (\mathbf{X}, \varphi_i) .$$

In this situation, we expect that $P_n \mathbf{X} \rightarrow \mathbf{X}$ for $n \rightarrow +\infty$.

This approach has been successfully used in recent work, namely in the Wilson-Askey framework (see, for instance, “Szabłowski (2014)”). It has shown to be flexible and efficient in practical situations, with ability to furnish expansions involving variables having non-gaussian distributions, namely when disconnecting scalar products and probability distributions (see, for instance, “Kewlani, Crawford, and Iagnemma (2012), Sampaio and Souza de Cursi (2015)”).

However, up to this date, **some fundamental questions are still with us, namely the conditions of the existence and of the convergence of such an expansion**. In addition, a practical question concerns the situation where the knowledge about $\mathbf{U}$ is incomplete: for instance, when the joint distribution $(\mathbf{X}, \mathbf{U})$ cannot be determined or $\mathbf{U}$ is a hidden parameter.

These fundamental questions are the object of this work. This paper addresses a mathematical analysis of this approximation and its application in UQ. The study is established in the framework of a Bochner space, i.e., a space of functions taking their values on a Banach space. The main tools are Generalized Fourier Series, representations of Riesz-Radon-Nikodym and integrals of Bochner and McShannon. We show that, for a large class of random variables, on the one hand, Hilbert basis $\{\varphi_i\}_{i \in \mathbb{N}}$ may be considered and, on the other hand, such a representation exists and numerical approximations may be generated. As usual, the existence of a Hilbert basis is connected to the separability of the space of random variables under consideration. Separability implies also that the expansion has a meaning and that finite-dimensional approximations may be generated. In order to manage situations where the knowledge about $\mathbf{U}$ is incomplete, an alternative consists in using other distances or pseudo-distances, such as, for instance, those connected to the moment approach. We examine these points in the sequel of this work.

The main result

In order to establish a formal result, let us consider an interval $I \subset \mathbb{R}$ (or a regular domain $I \subset \mathbb{R}^n$) and denote by $L^2(I)$ the classical space of the real-valued square-summable functions defined on I . Since $L^2(I)$ is separable “Zeidler. (1999)”, we may consider a Hilbert basis $\{\varphi_i\}_{i \in \mathbb{N}} \subset L^2(I)$ – this means that any element $\mathbf{Z} \in L^2(I)$ may be represented as $\mathbf{Z}(\mathbf{t}) = \sum_{i \in \mathbb{N}} z_i \varphi_i(\mathbf{t})$. Naively, we may think as follows: if $\mathbf{u}$ is a function defined on a set Ω , taking its values on I , we may take $\mathbf{t} = \mathbf{u}(\omega)$ and generate an expansion in terms of $\mathbf{t}$. From the mathematical standpoint, this simple idea involves two difficulties: on the one hand, we must establish that the resulting function remains an element of $L^2(I)$ and, on the other hand, that separability holds when the measure involved in the analysis is not the Lebesgue measure, but a probability measure. In other words, since the random variable $\mathbf{u}$ induces a measure on I , the resulting space must be separable in order to apply the procedure.

Recall that a probability on a sample space Ω is a particular finite measure μ defined on Ω , so that it is generated – as usual in measure theory – by a σ -algebra Σ on the sample space: in general, a measure space involves a triplet (Ω, Σ, μ) . However, in many practical situations, $\Omega \subset \mathbb{R}^n$ and Σ is the classical Borel algebra generated by Cartesian products of intervals. In this case, μ is a transformation of the Lebesgue measure and it is usual to drop Σ and

to denote simply (Ω, μ) . Recall that such a notation implicitly refers to Borel algebra and transformations of the Lebesgue measure.

The Lebesgue measure belongs to the class of *countably generated measures*, *id est*, measures generated by using a countable partition of subsets: indeed, a Lebesgue measure on an interval satisfies this condition, since it may be generated by intervals having both the extremities as rational numbers (see, for instance, “Kantorovich (1982)”). Thus, regular transformations of a Lebesgue measure on an interval and measures defined on the countable partition of intervals generated by the rational numbers are also countably generated. These properties will be used in the sequel: a countably generated measure generates a functional space that is separable. In particular, measure spaces generated by regular transformations of a Lebesgue measure remain separable.

As a first step, let us enounce a formal result stating that countable generated measures generate separable spaces.

Theorem 1

Let (Ω, Σ, μ) be a measure space and let p satisfy $1 \leq p \leq +\infty$. If μ is σ -finite and Σ is countably generated, then $L^p(\Omega, \Sigma, \mu)$ is separable. ■

Proof: see “Cohn. (2013)”. ■

In fact, the topological properties of Ω imply separability:

Theorem 2

Let (Ω, Σ, μ) be a measure space with $\mu(\Omega) < \infty$. Then the Banach space $L^p(\Omega, \Sigma, \mu)$ is separable if and only if Σ with metric $\rho(A, B) = \mu(A \Delta B)$ is separable. ■

Proof: See “Bruckner, Bruckner and Thomson. (2008)”. ■

As previously observed, the regular transformations of a Lebesgue measure generate separable spaces:

Theorem 3

Let (Ω, Σ, μ) be a measure space with $\mu(\Omega) < \infty$. (Ω, Σ, μ) is separable and non-atomic if and only if there exists an isomorphism between (Ω, Σ, μ) and $([0,1], \Lambda, \lambda)$, where λ is the Lebesgue measure. ■

Proof: See “Lacey. (1974)”, “Corbae et al. (2009)”, “Bogachev (2007)” or “Fabec (2000)”. ■

In order to apply these results, let us examine the change of variable between ω and U , which is justified by the following result

Theorem 4

Let $(\Omega, \mathbf{P}_\omega)$ be a probability space. Assume that $\mathbf{P}_\omega > \mathbf{0}$ on Ω . Let $U: \Omega \rightarrow \mathbb{R}$ be a random variable having as associated measure $\mathbf{P}_U$ (for instance, the measure defined by its probability density). Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be such that

$$Y(\omega) = \psi(U(\omega)) \Rightarrow E(|Y|) < \infty.$$

Then, on the one hand,

$$\psi(U) \text{ is } \mathbf{P}_U - \text{measurable if and only if } \psi(U(\omega)) \text{ is } \mathbf{P}_\omega - \text{measurable}$$

and, on the other hand,

$$E(\psi(U(\omega))) = \int \psi(U(\omega)) \mathbf{P}_\omega(d\omega) = \int \psi(U) \mathbf{P}_U(dU). \blacksquare$$

Proof: For the first assertion, see “Lelong. (2009)”. For the second one, see “Caumel. (2011)”. ■

Theorem 4 shows that:

Corollary 5

Let $X = X(U(\omega))$ and $Y = Y(U(\omega))$ be two elements from $L^2(\Omega, \mathbf{P}_\omega)$. On the one hand,

$$E(XY) = \int X(U(\omega))Y(U(\omega))P_{\omega}(d\omega) = \int X(U)Y(U)P_U(dU) ;$$

on the other hand,

$$E(|X|^2) = \int [X(U(\omega))]^2 P_{\omega}(d\omega) = \int [X(U)]^2 P_U(dU) . \blacksquare$$

This result implies that

$$\int [X(U(\omega))]^2 P_{\omega}(d\omega) < \infty \Leftrightarrow \int [X(U)]^2 P_U(dU) < \infty .$$

Thus, the first difficulty is solved: $X(U)$ is an element of $L^2(I, P_U)$ if and only if $X(U(\omega))$ is an element of $L^2(\Omega, P_{\omega})$.

The separability of the space $L^2(I, P_U)$ yields from the separability of $L^2(I)$: in fact, whenever P_U is generated by transformations of a Lebesgue measure, the resulting space is separable.

These results ensure that,

- On the one hand, $L^2(\Omega, P_{\omega})$ is separable whenever $\Omega \subset \mathbb{R}^n$ and P_{ω} is generated by an isomorphism of the Lebesgue measure. In particular, $L^2(\Omega, P_{\omega})$ is separable when P_{ω} is a uniform probability.
- On the other hand, $L^2(I, P_U)$ is separable whenever $I \subset \mathbb{R}^n$ and P_U is generated by an isomorphism of the Lebesgue measure. In particular, $L^2(I, P_U)$ is separable when P_U is a uniform probability.

For instance, let us assume that $P_U(du) = \eta(u)du \geq \nu(u)du$. In this case, we may consider a Hilbert basis $\{\varphi_i\}_{i \in \mathbb{N}} \subset L^2(I, \nu)$ and

$$X \in L^2(I, P_U) \Leftrightarrow X = \sum_i x_i \varphi_i(U) .$$

Indeed,

Theorem 6

Assume that $L^2(I, \nu)$ is separable, that $X \in L^2(I, \mu)$, $0 \leq \nu \leq \mu$. Then, there exists $\{x_i\}_{i \in \mathbb{N}}$ such that

$$X = \sum_i x_i \varphi_i(U) . \blacksquare$$

Proof

$$\int_I [X(u)]^2 \nu(u) du \leq \int_I [X(u)]^2 \mu(u) du < \infty \Rightarrow X \in L^2(I, \mu) . \blacksquare$$

As a consequence,

Theorem 7

Assume that there exists $K \in \mathbb{R}$ such that $|U'| \leq K$ and $\int_I [X(U(\omega))]^2 d\omega < \infty$. Then, for any Hilbert basis $\{\varphi_i\}_{i \in \mathbb{N}} \subset L^2(I)$,

$$X = \sum_i x_i \varphi_i(U) . \blacksquare$$

Proof

$$\int_I [X(u)]^2 du = \int_I [X(u(\omega))]^2 u'(\omega) d\omega \leq \int_I [X(u(\omega))]^2 |u'(\omega)| d\omega$$

$$\leq K \int_I [X(u(\omega))]^2 d\omega < \infty \Rightarrow X \in L^2(I). \blacksquare$$

Finally, we observe that Hilbert basis may be replaced by total families in all these results: expansions in terms of general total families may be obtained in all the situations considered above.

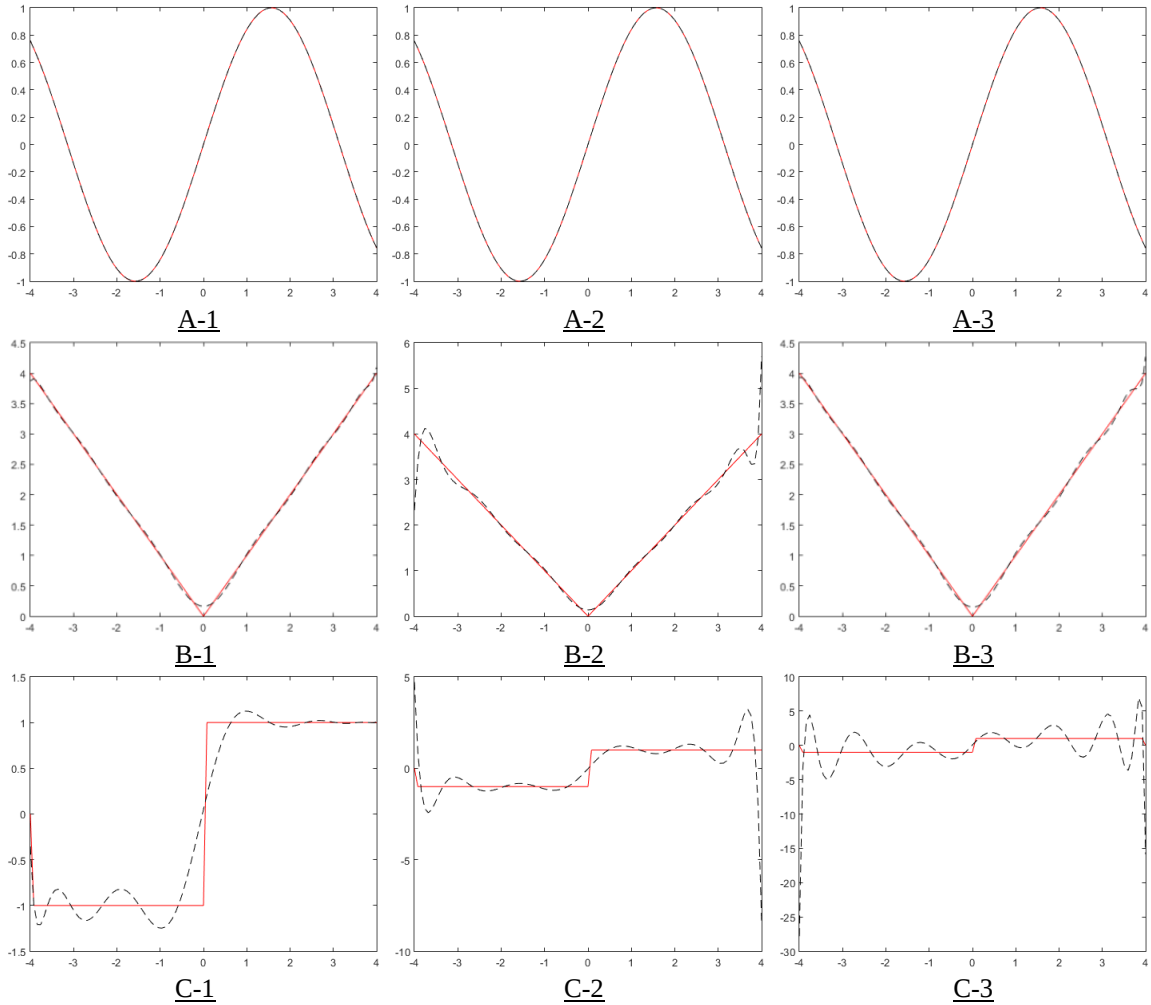
Numerical examples

Now, we apply our approach by considering a polynomial basis $\{\varphi_i\}_{i \in \mathbb{N}}$ in a first time and a trigonometric one in a second time and in the two situations we make varying both the probability distribution function (PDF) of the random variable $\mathbf{U}$ and the expression of $\mathbf{X} = \mathbf{X}(\mathbf{U})$. We consider $I = [a, b] = [-4, 4]$ and the PDFs of $\mathbf{U}$ and $\mathbf{X} = \mathbf{X}(\mathbf{U})$ presented in the table below.

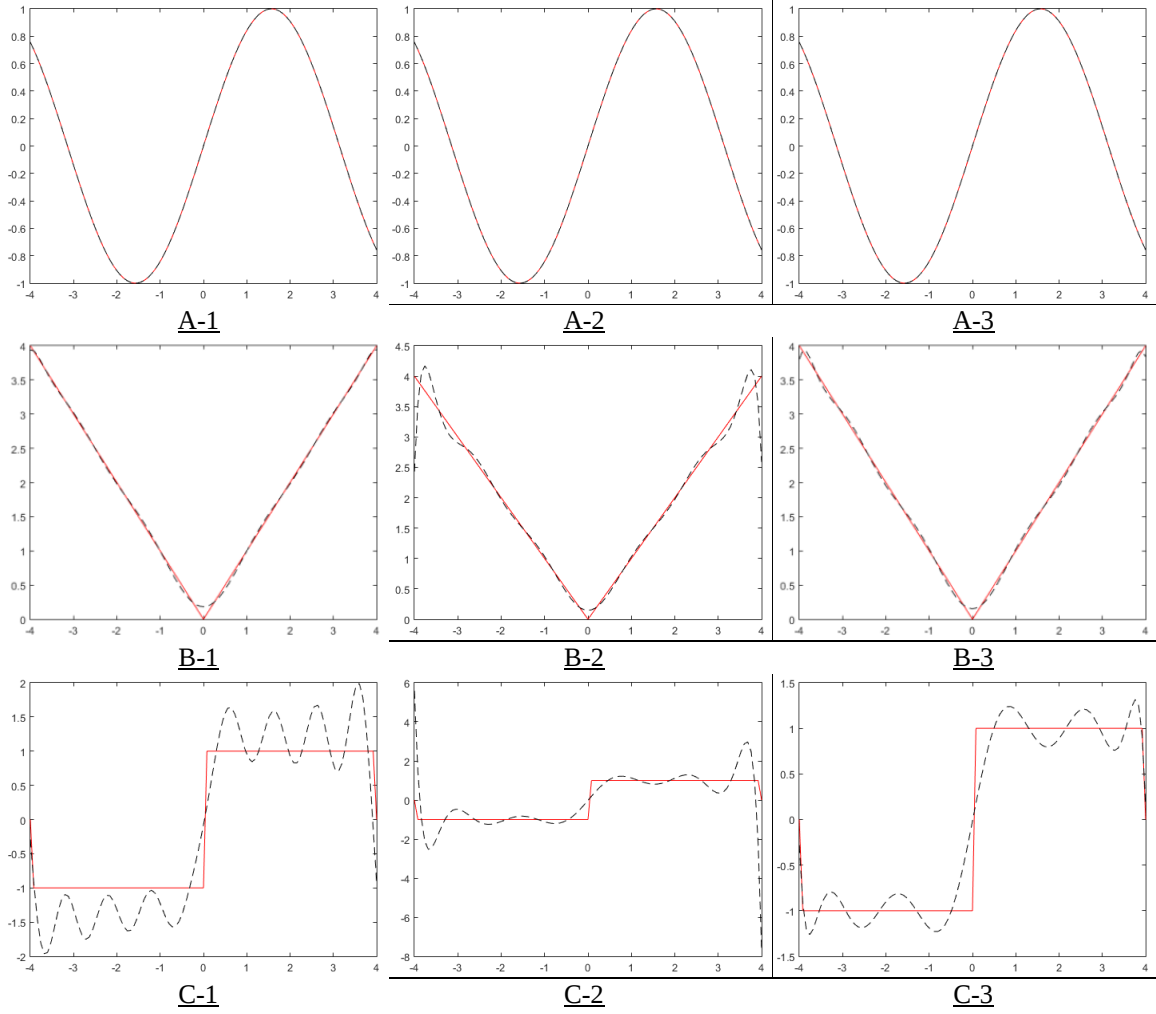
$\mathbf{X} = \mathbf{X}(\mathbf{U})$		Distribution of $\mathbf{U}$	
A	$\sin(\mathbf{U})$	1	$\mathcal{U}([-4, 4])$
B	$ \mathbf{U} $	2	$\mathcal{N}(0, 1)$
C	$\begin{cases} -1; \text{if } -4 \leq \mathbf{U} \leq 0 \\ 1; \text{if } 0 \leq \mathbf{U} \leq 4 \\ 0; \text{otherwise} \end{cases}$	3	$\mathcal{C}(0, 1)$

We have performed extensive experimentation involving these variables and distributions, but we present only a part of the results, by limitation of the available room.

Results furnished by the Polynomial basis



Results furnished by the Trigonometrical basis



For all the examples above we have noticed that many factors are involved in the quality of the approximation of the random variable, particularly the degree of the series that we use in the expansion. The best results are obtained for a range of degrees from 11 to 14. For lower values the approximation quality is not as good as we have obtained and for higher values a sort of leakage appears as follows, where we have chosen a polynomial basis with $X(U) = |U|$ and $U \sim \mathcal{C}(0,1)$. We made an expansion of degrees 5, 13 and 25.

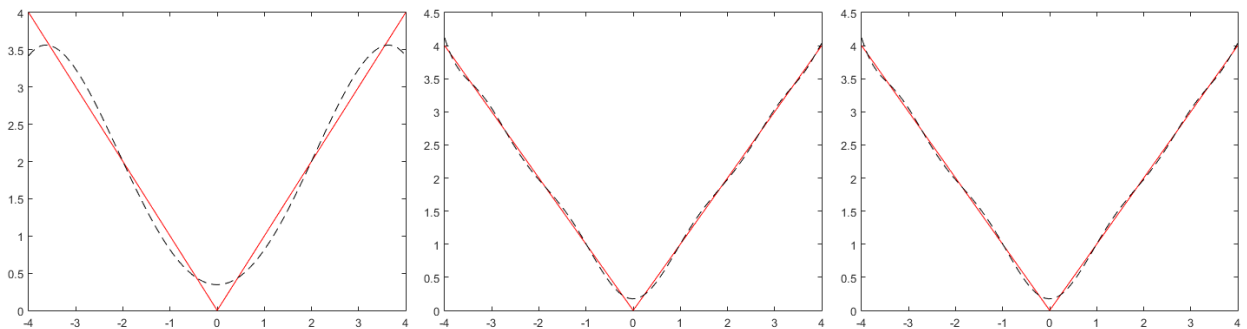


Figure 1 - Expansion of 5th, 13th and 25th degree

The case where the knowledge of U is incomplete: expansions using an artificial variable A .

In many practical situations, the knowledge of U is incomplete. Namely, its distribution or the joint distribution of the couple (X, U) may be unknown. In an extreme situation, it may arise that U is not known and only samples from X are available. In such a situation, we look for an explanation of the observed variability in terms of hidden variable. A practical solution may consist in considering an artificial random variable A and determining an expansion in terms of

functions of $\mathbf{A}$. For instance, we may determine an orthogonal projection $P_n \mathbf{X}$ from $\mathbf{X}$ onto $S_n = [\{\varphi_1(\mathbf{A}), \dots, \varphi_n(\mathbf{A})\}]$ - the vector space of the linear combinations of $\{\varphi_1(\mathbf{A}), \dots, \varphi_n(\mathbf{A})\}$:

$$\mathbf{X} \approx P_n \mathbf{X} = \sum_{1 \leq i \leq n} x_i \varphi_i(\mathbf{A}),$$

However, as previously observed, using an independent variable $\mathbf{A}$ leads to poor results:

Theorem 8

Let $(\Omega, \mathbf{P}_\omega)$ be a probability space. Let $\mathbf{X}, \xi \in L^2(\Omega, \mathbf{P}_\omega)$ be two independent variables. Then $E(\mathbf{X}|\xi) = E(\mathbf{X})$, id est, the conditional mean of $\mathbf{X}$ with respect to ξ is the constant random variable equal to the mean of $\mathbf{X}$ on the whole sample space. ■

Proof: Let $\mathbf{Z} = \psi(\xi) \in L^2(\Omega, \mathbf{P}_\omega)$. Since $\mathbf{X}$ and ξ are independent, we have $E(\mathbf{X}\psi(\xi)) = E(\mathbf{X})E(\psi(\xi))$, so that $E((\mathbf{X} - E(\mathbf{X}))\mathbf{Z}) = 0$. Since $\mathbf{Z}$ is arbitrary, we have $\mathbf{X} - E(\mathbf{X}) \perp S = \{\mathbf{Z} : \mathbf{Z} = \psi(\xi) \in L^2(\Omega, \mathbf{P}_\omega)\}$. But $E(\mathbf{X}) \in S$, so that $E(\mathbf{X}|\xi) = E(\mathbf{X})$. ■

This result shows that the best approximation of $\mathbf{X}$ as a function of ξ is a constant given by the mean of $\mathbf{X}$. As a consequence, we have

Corollary 9

Let $(\Omega, \mathbf{P}_\omega)$ be a probability space. Let $\mathbf{X}, \xi \in L^2(\Omega, \mathbf{P}_\omega)$ and $W = [\{\varphi_i(\xi)\}_{i \in \mathbb{N}}] \subset L^2(\Omega, \mathbf{P}_\omega)$ be the vector space formed by the finite linear combinations of elements from $\{\varphi_i(\xi)\}_{i \in \mathbb{N}}$. If the random variable $\mathbf{1}$, constant, equal to one on the whole sample space verifies $\mathbf{1} \in W$, then the orthogonal projection of $\mathbf{X}$ onto W is $E(\mathbf{X})$. Thus, the orthogonal projection of $\mathbf{X}$ onto $\overline{[\{\varphi_i(\xi)\}_{i \in \mathbb{N}}]}$ is $(\mathbf{X})$. ■

Proof: Observe that $W \subset S$ and $E(\mathbf{X}) \in W$, so that $E(\mathbf{X})$ is the orthogonal projection of $\mathbf{X}$ onto W . Thus, $E(\mathbf{X})$ is the orthogonal projection of $\mathbf{X}$ onto $\overline{W} = \overline{[\{\varphi_i(\xi)\}_{i \in \mathbb{N}}]}$. ■

Let us illustrate this situation with a simple example: assume that U is uniformly distributed on $(0,1)$ and $X = \sin(2\pi U)$. Let A be independent from U , uniformly distributed on $(0,1)$. Then

$$E(X(U)\varphi_i(A)) = E(X(U))E(\varphi_i(A)) = 0, \quad \forall i \in \mathbb{N}^*.$$

Thus, $P_n X = 0, \forall n \in \mathbb{N}^*$.

In order to overcome this difficulty, we must generate some correlation between the variables $\mathbf{X}$ and $\mathbf{A}$. This may be performed by moment fitting, collocation and increasingly ordering both the samples – what generates a positive correlation between the variables.

The approach by moment fitting is generated by using other distances, different from the hilbertian distance between the random variables: recall that the orthogonal $P_n \mathbf{X}$ corresponds to the element of $S_n = [\{\varphi_1(\mathbf{A}), \dots, \varphi_n(\mathbf{A})\}]$ that minimizes the distance $d(\mathbf{X}, S_n) = \min\{d(\mathbf{X}, \mathbf{Y}) : \mathbf{Y} \in S_n\}$. The equations above correspond to the distance $d(\mathbf{X}, \mathbf{Y}) = \|\mathbf{X} - \mathbf{Y}\| = \sqrt{(\mathbf{X} - \mathbf{Y}, \mathbf{X} - \mathbf{Y})}$. We notice that the orthogonal projection $P_n \mathbf{X}$ corresponds to the global minimum of

$$J(\mathbf{x}) = d\left(\mathbf{X}, \sum_{1 \leq i \leq n} x_i \varphi_i(\mathbf{A})\right).$$

When U is unknown and an artificial variable $\mathbf{A}$ is used, it may be interesting to use a distance or pseudo-distance disconnected from the scalar product: for instance, we may consider a pseudo-distance measuring the distance between moments:

$$d(\mathbf{X}, \mathbf{Y}) = \sum_{p=1}^k |M_p(\mathbf{X}) - M_p(\mathbf{Y})| \quad \text{or} \quad d(\mathbf{X}, \mathbf{Y}) = \sum_{p=1}^k \left| \frac{M_p(\mathbf{X}) - M_p(\mathbf{Y})}{M_p(\mathbf{X})} \right|, \quad M_p(\mathbf{Z}) = E(\mathbf{Z}^p).$$

The first one gives the sum of the errors between the first k moments, while the second one gives the sum of the relative errors between the same k first moments. Minimizing d corresponds to find the coefficients such that these moments are the closest as possible to the empirical ones – notice that the minimization of d is a **global optimization problem**, since J is a **non convex functional**. This approximation is suggested by the following variation of Lévy's theorem:

Theorem 10

Let $(\Omega, \mathbf{P}_\omega)$ be a probability space. Let $\{X_n: n \in \mathbb{N}\}$ be a sequence of random variables on $(\Omega, \mathbf{P}_\omega)$, $\Phi_n(t) = E(e^{itX_n})$ be the characteristic function of X_n . Let X be a random variable on $(\Omega, \mathbf{P}_\omega)$, $\Phi(t) = E(e^{itX})$ be its characteristic function. Then:

$$X_n \rightarrow X \text{ in distribution} \Leftrightarrow \Phi_n(t) \rightarrow \Phi(t) \text{ pointwise on } \mathbb{R}. \blacksquare$$

Proof: see “Stroock (1994)”, “Souza de Cursi (1991)”. ■

Let us illustrate this approach by considering again the situation where U is uniformly distributed on $(0,1)$, $X(U) = \sin(2\pi U)$, A uniformly distributed on $(0,1)$, A and U independent. The results are shown in the Figure 2 below. We observe that the orthogonal projection leads to a poor quality approximation, while the use of alternative distances leads to better results. In these experiments, the global optimization problem has been solved by combining intrinsic Matlab procedures *fminsearch* and *fminunc*.

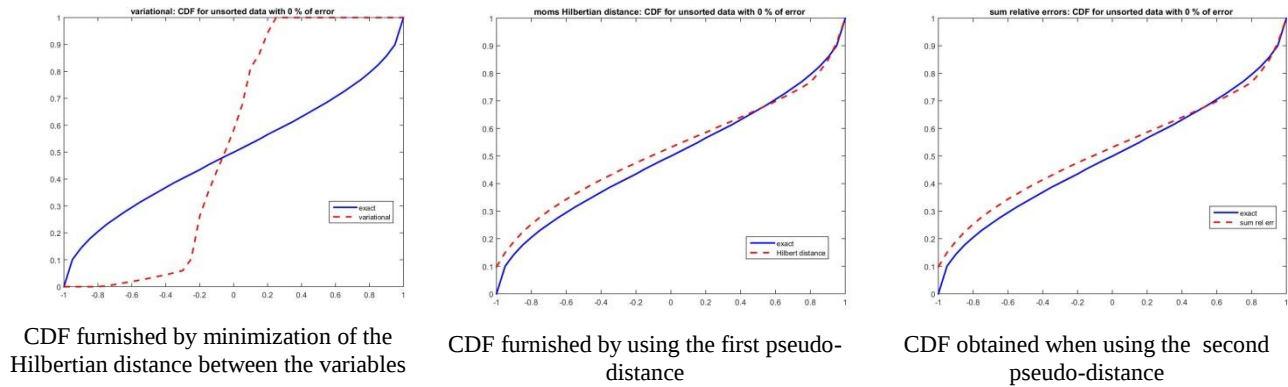


Figure 2 - Expansion using an artificial variable (same data set for all the figures)

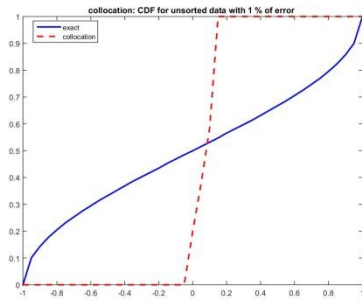
An alternative approach in order to generate dependence between the variables is collocation: when a sample $(X_1, \dots, X_{ns})$ from X is available, we may consider a sample $(A_1, \dots, A_{ns})$ from A and determine the coefficients of the finite expansion by solving the system

$$P_n X(A_i) = X_i, i = 1, \dots, ns.$$

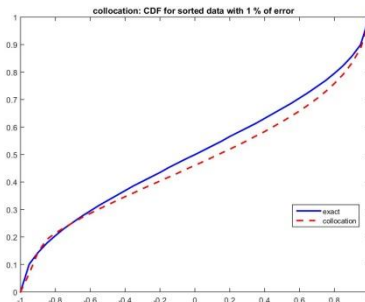
These equations correspond to the linear system

$$\mathbf{M}\mathbf{x} = \mathbf{N}, \quad M_{ij} = \varphi_j(A_i), \quad N_i = X_i$$

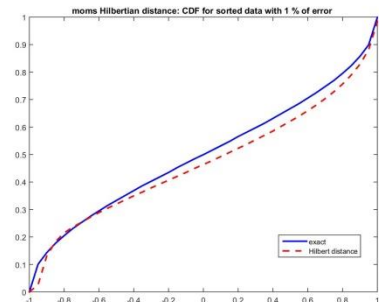
which is overdetermined for $ns > n$ (least squares solution is often used). This approach is competitive in terms of computational cost, but it is, on the one hand, more sensible to measurement errors in the construction of the sample from X and, on the other hand, may lead to an ill-conditioned system. Analogously to the preceding, it is more efficient if the data is correlated. For instance, we may ascending sort both the samples in order to create a positive correlation. Using a sample of $ns = 100$ variates from $X(U) = \sin(2\pi U)$ and collocation by a polynomial of degree 4, results for sorted and unsorted data are exhibited in Figure 3. We observe that the quality of the approximation of the distributions furnished by collocation is poor for unsorted data. Concerning the two pseudo-distances previously introduced, sorted and unsorted data furnish the same result. As in the preceding example, the global optimization problem has been solved by combining intrinsic Matlab procedures *fminsearch* and *fminunc*.



CDF furnished by unsorted data

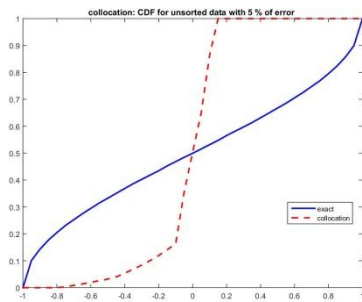


CDF furnished by sorted data

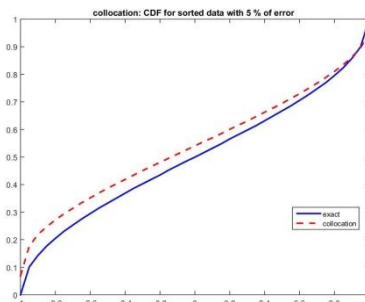


CDF obtained for sorted data(first pseudo-distance)

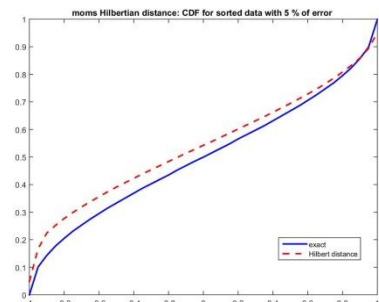
Figure 3 – Results obtained using an artificial variable and collocation (1 % measurement errors, same data set for all the figures)



CDF furnished by unsorted data



CDF furnished by sorted data



CDF obtained for sorted data (first pseudo-distance)

Figure 4 – Results obtained using an artificial variable and collocation (5 % measurement errors, same data set for all the figures)

Concluding Remarks

We addressed a mathematical analysis of hilbertian expansions of random variables in a framework corresponding to UQ applications. The main result is stated in the framework of a Bochner space and establishes the separability of the basic probability space $L^p(\Omega, \Sigma, \mu)$ when μ is a regular transformation of the Lebesgue measure. This result gives a mathematical framework for the popular expansions which have shown to be efficient in practice and, thus, gives a formal result corresponding to empirical results. In addition, it furnishes elements for the analysis of systems involving hidden parameters. Future work will be dedicated to the confirmation of the preliminary results exposed in this work.

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