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On the Uncertainties of Master Curves for Viscoelastic Materials

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Abstract: The dynamic properties of viscoelastic materials (VEM) show highly frequency-temperature dependency. The present work is concerning to the modeling, identification and validation of Thermoreologically Simple (TRS) materials. The thermoreologic behavior of the material is refered by two models. The first concerns to the frequency dependence of the VEM which is represented by a fractional derivative model, and the second is related to the temperature dependence of the material and it is based on the Frequency Temperature Superposition Principle (FTSP). An experimental set-up was proposed to obtain the complex shear modulus at several temperatures using Dynamic Mechanical Analysis (DMA). The goal of the present work is to propose a model calibration strategy for thermoreologically simple materials using a Bayesian framework, where the parameter identification was performed using Markov Chain Monte Carlo (MCMC) methods. The results are quite compelling due to the fact that the proposed approach is easy-handed. Furthermore, this approach could be applied on any generic constitutive model using the FTSP. The agreement between the model and its predictive capabilities were quantitatively assessed using area validation metric using some scenarios that provide favourable analysis for agreement of the model and the set of available experimental data.

Keywords: Viscoelasticity, Thermoreologically Simple materials, Metric Validation, Bayesian identification

1 INTRODUCTION

Nowadays, viscoelastic materials (VEM) have been used in the aerospace, electromotive, electronics industries in order to damp mechanical vibration due to its energy dissipation characteristics. Hence, the knowledge of the dynamic properties is necessary in order to a full representation of these properties where the constitutive models play a crucial role to describe the dynamical behavior of viscoelastic materials (VEM). Nevertheless, the VEM have complex dynamic characteristics in which the response depends on several environmental variables, such as the temperature, frequency, preload, amplitude, which can affect the final representation of the response [Jrad, H. et al., 2013].

Several models have been introduced to represent the dynamic response of VEM. In linear viscoelasticity problems are considered arrangements of springs and dash-pots elements to represent the response of the VEM [Wineman, A.S. and Rajagopal, K.R., 2000]. On the other hand, models based on the concept of internal variable state are based on the concept of irreversible process. In these models, the number the elements establish the number material parameters, therefore, it is necessary a large number of parameters to produce a good fit [Coleman, B.D. and Gurtin, M.E., 1967, Castello, D.A. et al., 2008]. Moreover, constitutive models based on the concept of fractional derivative operators [Bagley, R.L. and Torvick, J., 1983, Bagley, R.L., 1986, Lion, A., 1997, Pritz, T., 2003] have the feature that the fit can be made with a reduced number of parameters in comparison with the internal variables state models. It is should be noted that these models are defined for a only one temperature, that is, the model has not represent a temperature dependence, instead this models are defined for isothermal states.

Operational temperature can vary the dynamical response of VEM. Exists a particular class of polymers said *thermoreologically simple* (TRS) materials, where the relaxation mechanisms can be described by only one temperature function, and this temperature dependency can be described by the Frequency-Temperature Superposition Principle (FTSP) [Ferry, J.D., 1980]. The FTSP states a mutual correspondence between the frequency and temperature effects, that is, the change in a mechanical property induced by a variation on temperature can be identical to the one produced by a variation on frequency [Ferry, J.D., 1980, Christensen, R.M., 2003, Madigosky, 2006]. The FTSP is used by manufacturers or even by most of the laboratories like a strategy to generate the nomogram representations [Jones, D.I.G, 1978]. In fact, the most cases the FTSP is only used to adjust experimental data in a master curve and generate nomograms, in this sense, this approach does not take into account any constitutive model associated to the VEM.

Some authors have proposed a description for constitutive models to describe the dynamical behavior of VEM that incorporate both the frequency and the temperature dependency [Moreira, R.A.S. et al., 2010, Kim, S.Y. and Lee, D.H., 2009, Henry, T. C. et al., 2015, Madigosky, 2006]. Moreira, R.A.S. et al. [2010] uses the FTSP and a constitutive model based on fractional derivative operators. Kim, S.Y. and Lee, D.H. [2009], in turn, makes the identification of a model based on fractional derivative operators thought laminated sandwich beams with viscoelastic core.

In order to use the FTSP and obtain a comprehensive representation of the dynamical behavior for the VEM it is necessary take data over several frequencies and temperatures. One of the most useful experimental techniques to stablish the mechanical behavior in VEM is the Dynamic Mechanical Analysis (DMA) [Jrad, H. et al., 2013, Ferry, J.D., 1980, Menard. K.P., 2008]. The DMA allow us evaluate, determine and quantify experimentally the dynamic modulus of a VEM over a range of temperature and frequency, where this experimental data can be used to adjust a master curve via FTSP or even to do a model calibration.

Model calibration is accomplished through a inverse problem formulation, which associates the information both form model predictions and measured data. Several factors can affect the model calibration. Firstly, it is common knowledge that noise, the signal acquisition methods can affect the quality of the measurement. Furthermore, models are build based on assumptions and approximations are rarely tacking into account in the calibration process. For this reason, a probabilistic approach is more suitable to taking into account this kind of uncertainties in addition to it can deal properly treatment for them.

The general purpose of the present work is to present a strategy for model calibration based on a Bayesian framework for a VEM with the hypothesis of *thermoreologically simple* materials. The frequency dependence for the VEM is based on a fractional derivative operators and the thermorologic behavior is modeled with the Frequency-Temperature superposition principle. For this purpose an experimental set-up was properly designed to obtain the dynamical behavior of a VEM as a function of temperature such that we could obtain an environment with lower level of physical complexity. The predictive capabilities for the calibrated VEM model are evaluated according to the basic principles of validation metrics.

In the following sections, the mathematical formulation for the FTSP and the VEM model formulation based on fractional derivative operators with five parameters is first described, and it is followed by an inverse problem methodology based on a Bayesian approach and a verification framework. The experimental Set-up were specifically designed for this purpose in which material shear complex modulus can be measured for some temperatures of interest. The analyses of the models parameters as well as the validation metrics is presented and discussed, followed by concluding remarks.

2 MECHANICAL MODELING

The mechanical response of a VEM depends on temperature and frequency. Classical models used to describe the dynamical behavior of VEM are considered independent of the temperature, that is, the models as well as the experimental data are specified for isothermal states of temperature [Wineman, A.S. and Rajagopal, K.R., 2000]. This first section recalls the definitions for apply the FTSP which is the keypoint used in the present analysis, as well as the constitutive model used in the calibration of the VEM.

2.1 Termoreologically simple materials

In principle, the FTSP strictly applicable on a VEM called as *thermoreologically simple*. This is a characteristic of the rheological behavior, where the the distribution of relaxation times is independent of temperature, that is, all time constants (relaxation times) should have the same temperature dependence [Jrad, H. et al., 2013, Christensen, R.M., 2003, Madigosky, 2006]. In this sense, the FTSP can be useful to describe effects of temperature into the dynamic behavior of VEM.

In this sense, mechanical properties at different temperatures can be shifted horizontally and vertically to obtain a only curve denominated *master curve* which is defined at chosen reference temperature T_0 in a wide frequency range. When the *thermoreologically simple* hypothesis has been satisfied, then it is possible establish an equivalence between the mechanical properties measured at any frequency ω_i and temperature T_i and those at any other reduced frequency ω_R and reference temperature T_R, and vice versa [Ferry, J.D., 1980, Rouleau, L. et al., 2013]. To represent the FTSP in a general form, the shift factor can be defined as follows:

$$\omega_R = a_{T_0}(T_i)\,\omega_i \tag{1}$$

where $a_{T_0}(T_i)$ represents the horizontal temperature shift function. Therefore, using the FTSP for a mechanical property defined as $E(\omega, T)$ can be determined as follows:

$$E(\omega_R, T_0) = E(\alpha_{T_0}(T_0, T_i)\,\omega_i, T_i) \tag{2}$$

A common horizontal shift function to describe the FTSP for *thermoreologically simple* materials is the Williams-Landel-Ferry equation (WLF) and it gives an empirical expression for the temperature dependent horizontal shift coefficients[Madigosky, 2006, Fowler, B. and Rogers, L., 2004, Ferry, J.D., 1980, Moreira, R.A.S. et al., 2010, Rouleau, L. et al., 2013]. The WLF equation is based on the assumption that the fractional free volume increases linearly with temperature and may be derived from a modification of the Arrhenius equation from the theory of chemical reaction rates [Ferry, J.D., 1980, Christensen, R.M., 2003, Brinson, H.F. and Brinson, L. C., 2010, Shaw, M.T. and MacKnight, W.J., 2005]. The WLF can be expressed as:

$$log(\alpha_T(T_0, T_i)) = C_1 \frac{T_i - T_0}{C_2 + T_i - T_0} \tag{3}$$

where C_1 and C_2 are material coefficients constants that represent the FTSP and vary form one material to another.

2.2 Constitutive Model Formulation

Dynamical response of VEM is characterized by causal phenomenon where the stress field always precedes strain field [Wineman, A.S. and Rajagopal, K.R., 2000]. This behavior gives memory characteristics to the VEM, that is, the current stress depends on the current strain and its recent strain history. The constitutive laws to describe the relation between stresses and strains are formulated through integral hereditary or by a convolution form.

An alternative formulation can be done through fractional derivative equation [Pritz, T., 2003, Caputo, M. and Mainardi, F., 1971, Bagley, R.L., 1986, Bagley, R.L. and Torvick, J., 1983, Pritz, T., 1996, Mainardi, F., 1994, Friedrich, Ch. and Braun, H., 1992, Rogers, L., 1983, Mainardi, F., 2010], to cite a few. The general form for 1-D constitutive models for a isotropic VEM based on time domain fractional derivative operators can be expressed as follows [Bagley, R.L. and Torvick, J., 1983, Bagley, R.L., 1986]

$$\sigma(t) + b_1 \frac{d^{\beta_1}}{dt^{\beta_1}}\sigma(t) + b_2 \frac{d^{\beta_2}}{dt^{\beta_2}}\sigma(t) + \cdots + b_n \frac{d^{\beta_n}}{dt^{\beta_n}}\sigma(t) = a_0\varepsilon(t) + a_1 \frac{d^{\alpha_1}}{dt^{\alpha_1}}\varepsilon(t) + a_2 \frac{d^{\alpha_2}}{dt^{\alpha_2}}\varepsilon(t) + \cdots + a_m \frac{d^{\alpha_m}}{dt^{\alpha_m}}\varepsilon(t) \quad (4)$$

where $\{a_1, \dots a_m\}$ and $\{b_1, \dots b_n\}$ are parameters of the constitutive model, t is the time, $0 < \beta_r \leq 1$ and $0 < \alpha_s \leq 1$ are the fractional derivative orders for the stress $\sigma(t)$ and strain $\varepsilon(t)$ fields, respectively. In case all parameters $\{\beta_r, \alpha_s\}$ are integer numbers, Eq.4 corresponds to a classical constitutive model for a isotropic VEM [Wineman, A.S. and Rajagopal, K.R., 2000].

The fractional derivative operator d^γ/dt^γ used in Eq.4, it is defined as follows [Mainardi, F., 2010, Caputo, M. and Mainardi, F., 1971]

$$\frac{d^\gamma}{dt^\gamma}[v(t)] \equiv \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_0^t \frac{v(t-\tau)}{\tau^\gamma} d\tau \quad (5)$$

where $\Gamma(\cdot)$ is the Gamma function and γ is the order of the fractional derivative operator, $0 \leq \gamma \leq 1$ [Bagley, R.L., 1986]. Equation 5 clearly presents the hereditary characteristic of the fractional operator when uses the entire history of the the function $v(t)$ to compute its fractional derivative $\frac{d^\gamma}{dt^\gamma}[v(t)]$ at current time t. The Fourier transform $\mathcal{F}$ for the fractional derivative operator presented in the Eq.5, can be defined as:

$$\mathcal{F}\left\{ \frac{d^\gamma}{dt^\gamma}[v(t)] \right\} = (j\omega)^\gamma \mathcal{F}\{v(t)\} \quad (6)$$

A detailed analysis of fractional constitutive models for viscoelasticity from the point of view of the thermodymics of irreversible processes is presented by Alexander Lion Lion, A. [1997]. Pritz, T. [1996, 2003] presents analyses for two constitutive models derived from Eq.4 and he considers models with 4 and 5 constitutive parameters for which $\beta_r = 0$ for $r \geq 3$ and $\alpha_s = 0$ for $s \geq 2$. Concerning the constitutive model with five parameters [Pritz, T., 2003] it is able to provide an asymmetric loss factor peak and approximately constant loss factor at high frequencies. This model is presented in Eq.7

$$\sigma(t) + \tau^\beta \frac{d^\beta}{dt^\beta}\sigma(t) = G_0\varepsilon(t) + G_0\tau^\beta \frac{d^\beta}{dt^\beta}\varepsilon(t) + (G_\infty - G_0)\tau^\alpha \frac{d^\alpha}{dt^\alpha}\varepsilon(t) \quad (7)$$

where τ is the relaxation time, and G_0 and G_∞ are the static and dynamic modules, respectively. The model obtained when $\alpha = \beta$ in Eq.7 corresponds to the fractional Zener model [Pritz, T., 2003, Mainardi, F., 2010].

The constitutive model for the VEM in the Fourier domain is a complex function which only depends on frequency. Therefore, the complex modulus $\tilde{G}(j\omega)$ for the constitutive model in Eq.7 can be derived by the application of the Fourier transform to both sides of Eq.7 and recasting it as follows

$$\tilde{\sigma}(j\omega) = G_0 \left(1 + (d-1)\frac{(j\omega\tau)^\alpha}{1+(j\omega\tau)^\beta} \right) \tilde{\varepsilon}(j\omega) = \tilde{G}(j\omega)\, \tilde{\varepsilon}(j\omega) \quad (8)$$

where $d = G_\infty/G_0$ is the ratio between the dynamic and static modulus, $\tilde{V}(j\omega)$ denotes the Fourier transform of the function $v(t)$, ω is the circular frequency in rad/sec and $j = \sqrt{-1}$ is the imaginary number. The complex modulus is a frequency dependent operator and its constitutive parameters $\boldsymbol{\theta} = \{G_0, d, \tau, \alpha, \beta\}$ should be obtained by appropriate inverse analysis [Kim, S.Y. and Lee, D.H., 2009, Zhang, E. et al., 2013, Hernández, W.P. et al., 2015].

3 INVERSE PROBLEM: PARAMETER ESTIMATION

Many researchers have used a probabilistic approach in estimation and validation problems, and there is an evident need for a treatment related to VEM characterization. Recently, Oates, W. S. et al. [2013] and Haario, H. et al. [2014] use a Bayesian approach to explore and quantify the uncertainty in a constitutive model of VEM based on the concept of internal state variables. Zhang, E. et al. [2013] proposed the calibration of a model using a Bayesian approach using experimental data from the frequency response function for a sandwich beam.

3.1 Direct and inverse Problem

In principle, the FTSP can be applied on VEM independent of the existence of a constitutive model of the material, instead the FTSP is used on the experimental data to plot the master curve. Furthermore, constitutive models are based on the hypothesis of thermorheologically simple materials which can represent the frequency dependence of the complex modulus data for isothermal states. Hence, in this article uses the FTSP as a tool in the constitutive material formulation.

Therefore, it is consider for the direct problem the set of parameters of the constitutive model based on fractional derivative (Eq.8) and the model for the thermorheologically simple materials (Eq.3) in order to recover a frequency and temperature dependency.

Then, based on these two models the inverse problem is aimed to estimate the following set of constitutive parameters:

$$\boldsymbol{\theta} = \{G_0, d, \tau, \alpha, \beta, C_1, C_2\}^T \tag{9}$$

3.2 Bayesian Framework

In the Bayesian approach, the unknown quantities as well as the measurements are considered as random variables, where the knowledge state about the uncertainties associated to these random variables are described by means of their probability density functions (PDF). In the inverse problem inference, the Bayesian framework consists in inferring all possible information about unknown quantities based on all knowledge available about the measurements, model structures and the information obtained prior to the measurement [Kaipio, J. and Somersalo, E., 2004].

Let's consider a measurement of an system response quantity $\mathbf{Y}$, and also assume that the relation between measurements and model predictions may be described by the additive error observation model.

$$\mathbf{Y} = A(\boldsymbol{\theta}, \mathbf{x}) + \boldsymbol{\varepsilon} \tag{10}$$

where $A(\boldsymbol{\theta}, \mathbf{x})$ corresponds to an operator that provides model predictions for system response given a vector model parameters $\boldsymbol{\theta}$, an input vector $\mathbf{x}$ and a set of initial and boundary conditions. In Eq.10 the variable $\boldsymbol{\varepsilon} \backsim \mathcal{N}(0, \sigma_e^2)$ corresponds to the additive measurement noise whit zero mean and variance σ_e. Using the Bayes' formula, the posterior density of the unknown parameters $\boldsymbol{\theta}$ given the measured data $\mathbf{Y}$, namely $\pi(\boldsymbol{\theta}|\mathbf{y})$ is describe as follows [Kaipio, J. and Somersalo, E., 2004]

$$\pi(\boldsymbol{\theta}|\mathbf{Y}) = \frac{\pi(\mathbf{Y}|\boldsymbol{\theta})\,\pi(\boldsymbol{\theta})}{\pi(\mathbf{Y})} \tag{11}$$

where $\pi(\mathbf{Y}|\boldsymbol{\theta})$ represents the likelihood model, $\pi(\boldsymbol{\theta})$ is the prior model for the unknown parameters $\boldsymbol{\theta}$ and $\pi(\mathbf{Y})$ corresponds to the density of measured data. In fact $\pi(\mathbf{Y})$ is hardly available inasmuch as it would require a large number of realisations for measured data $\mathbf{Y}$, in the Eq.11 it acts as scaling factor and it is of little importance. The prior model $\pi(\boldsymbol{\theta})$ express the our current state of knowledge about the unknown parameters. Concerning to the likelihood model, assuming that model parameters $\pi(\boldsymbol{\theta})$ and the additive noise $\boldsymbol{\varepsilon}$ are mutually independent, it cast as follows [Kaipio, J. and Somersalo, E., 2004]

$$\pi(\mathbf{Y}|\boldsymbol{\theta}) = \pi_e(\mathbf{Y} - A(\boldsymbol{\theta}, \mathbf{x})) \tag{12}$$

where $\pi_e(\mathbf{Y} - A(\boldsymbol{\theta}, \mathbf{x}))$ corresponds to the probability density of the additive noise $\boldsymbol{\varepsilon}$.

Generally speaking, the Bayes' rule (Eq.11) corresponds to the complete probabilistic model for the inverse analysis, and it allows us make the inference of the distribution $\pi(\boldsymbol{\theta}|\mathbf{Y})$ which enables to compute various estimates for $\boldsymbol{\theta}$ as well as a posterior uncertainty for these estimates.

It is important to emphasize that the posterior $\pi(\boldsymbol{\theta}|\mathbf{Y})$ often is a complex and implicit function of the parameter set $\boldsymbol{\theta}$ and explore this distribution can be difficult. Moreover, due to the operator $A(\boldsymbol{\theta}, \mathbf{x}))$ involves nonlinearities between predictions and parameters we do not have an analytical solution for the posterior. Henceforth, some sampling techniques, such as Markov Chain Monte Carlo (MCMC) methods, have been used to avoid multidimensional integrations in order to infer information from the posterior $\pi(\boldsymbol{\theta}|\mathbf{Y})$ [Robert, C. and Casella, G., 2005]. The goal of MCMC methods is generating samples of a set of parameters which according to a stationary distribution, in this case the posterior PDF. The samples are drawn sequentially such that the distribution depends on the last sample drawn. As for the implementation of issues, MCMC methods are conceptually simple and the algorithms are easy to be described; the authors refers the interested readers to the reference books by Robert, C. and Casella, G. [2005] and Gelman A. et al. [2003] for detailed information concerning the algorithms.

The Metropolis-Hastings (MH) algorithm is a simple procedure to make samples of an arbitrary PDF. In the MH algorithm, samples are generates as states of a special Markov chain whose limiting stationary distribution corresponding to the target PDF, that is, the PDF of the Markov chain sample θ_j simulated at the j^{th} Markov step tends to the target PDF. In this case, the sequence of $\boldsymbol{\theta}_j$ starts in an initial state $\boldsymbol{\theta}_0$ and for each step j is drawing a candidate $\boldsymbol{\theta}_j$ from the proposal distribution $T(\boldsymbol{\theta}_j|\boldsymbol{\theta}_{j-1})$ which only depends on the previous state $\boldsymbol{\theta}_{j-1}$. The new state candidate $\boldsymbol{\theta}_j$ is accepted or rejected according to some probability level. In the MH algorithm, the proposal PDF is usually chosen such that it is easy to draw samples from $\boldsymbol{\theta}^* \sim T(\boldsymbol{\theta}^*|\boldsymbol{\theta}_{j-1})$, and a commonly choice is a Gaussian distribution, that is $\boldsymbol{\theta}^* \sim \mathcal{N}(\boldsymbol{\theta}_{j-1}, \Sigma)$, where Σ corresponds to the covariance matrix.

4 VALIDATION METRICS

Although calibration of a model may indicate the model's data fitting ability, it does not assure its predictive capabilities [AIAA, 1998, ASME, 2006]. In this sense, a validation process can be interpreted as the procedure to determining the degree to which a model is an accurate representation of the real world from the perspective of the intended uses of the model [AIAA, 1998, ASME, 2006]. In other words, quantifying the credibility of a model to represent some phenomena of interest is the primary purpose of validation processes. Predictive capabilities of a computational model should be assessed based on quantitative comparisons between model predictions and measured quantities [Oberkampf, W.L. and Barone, M.F., 2006].

Oberkampf, W. L. and Roy, C.J. [2010] recommended some features to define metrics to quantify the agreement between computational and experimental responses, which are not absolute or unique. The authors proposed a validation metric based on the statistical concept of confidence interval in order to take into account uncertainties in the experimental data and in model predictions whenever it is possible.

A validation metric can be consider as mathematical operator that uses the experimental measurements and the model predictions to quantify the accuracy of the model at the same conditions of the experimental measurements. [Oberkampf, W.L. and Barone, M.F., 2006] establish that the functional form of the metric must make into account the uncertainties in the experimental data and in model prediction to quantify a good agreement between them. Therefore, to define a validation metric in an experimental set-up not only is necessary the measurement of a system response, but also establish, define and characterize all the uncertainty sources during the experiment.

The uncertainties can be categorized according to their essence in epistemic and aleatory [Oberkampf, W. L. and Roy, C.J., 2010]. The first are associated with the lack of knowledge about the model or processes and the second are due to the inherent variation of a quantity and it can be represented by PDFs. When the uncertainties in the model have been defined and characterized they must be propagated as inputs through the model in order to obtain a system response which will be compared with the experimental data. The key focus in the validation metrics, is that when the uncertainties are propagated, it is expected that the model predict the variability of the experimental measured response, which are associated to the variability of the inputs. Hence, any disagreement between the model response and the experimental data is attributed to the uncertainty. In processes associated to the measurement of experimental data there is always random uncertainty which can interfere in the calibration of its parameters, and hence in the expectation of the model. Generally speaking, this uncertainty can be represented by a increase of the variance in the experimental measurements.

There are many possible validation metrics, in this work we are focused in the implementation called as *area validation metric*, and it is based on the concept of cumulative Density functions (CDF). A significant advantage of the area validation metric is that it can be computed even when very few samples are available from the simulation or from experimental [Roy, C. J. and Oberkampf, W.L., 2011]. When the uncertainty of nature aleatory is propagated through the model, the expected response of the model can be represented by its CDF. In the same way, experimental measurement of the model can be used to construct the CDF associated to the experimental data. In this sense, the area validation metric $\mathbf{d_A}$ is referred as the area between these two CDFs [Roy, C. J. and Oberkampf, W.L., 2011].The area validation metric quantifies horizontal distances between CDFs of the model and experimental data expectations. Then, this metric is the smallest area between the expected response of the model CDF and the experimental CDF. The area validation metric and is given by

$$\mathbf{d_A}(F^M(x), S_n^E(x)) = \int_{-\infty}^{\infty} |F^M(x) - S_n^E(x)| dx \tag{13}$$

where $F^M(x)$ is the CDF from the expected response of the model, $S_n^E(x)$ the CDF from the experiment measurement, and x is an specific system response. In order to make reasonable comparisons among this validation metric, it is introduced the following scaling what leads to a dimensionaless metric [Borges, F.C.L. et al., 2015]

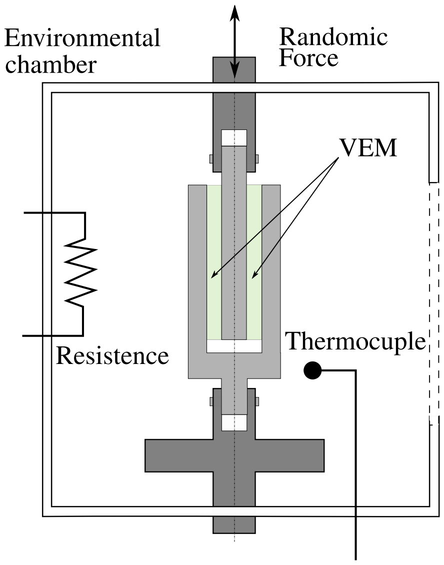
$$d_A^N = \frac{\mathbf{d_A}(F^M(x), S_n^E(x))}{S_n^E(\hat{x})} \tag{14}$$

in which $\hat{x}$ is the mean of the system response.

5 EXPERIMENTAL SET-UP

By DMA it is possible to obtain experimental data for several temperatures and Frequencies. In practise, due to the limitations associated with the measurement instruments the dynamic response of VEM are limited to a reduced frequency range [Rouleau, L. et al., 2013]. Hence, to obtain a comprehensive behavior of the dynamic properties, the FTSP is used on several sets of data at frequency and temperature whenever that the hypothesis of termoreologically simple materials is satisfied.

In this work to establish the complex shear modulus, it is used the Direct Method, where both applied force and system response are measured directly. Figure 1 shows the experimental set-up developed for the tests. Note that this configuration tend to favor the uniform shear stress in the VEM. The assembly consists of three aluminium boards, the middle board is attached to the other two through VEM. The VEM used in this study is a comercial double-sided tape produced by $\mathbf{3M}$® under code $\mathbf{VHB4955}$, which enables the link between the rigid boards. The length, width and thickness of the used VEM are 80 mm, 120 mm and 2 mm, respectively. The mass of the midel board is 1.54 kg. Aiming at measuring the dynamic response, a MTS press imposes prescribed vertical displacements to the middle aluminium board which promotes a predominant planar shear deformation as can be seen in Fig.1a. The vertical displacement was carefully tuned in order consider small deformations which would trigger non-linearities in the response.Random excitations with a frequency range between $[0-60]\ Hz$ were applied. This test was performed for a set of temperatures $T = \{22, 30, 35, 40\}$ where the tests were accomplished by means a chamber whit controllable temperature, as shown in Fig.1b.



(a) General view of the test

(b) General view of the test (temperature variation)

Figure 1: General set-up for the tests

Using the frequency response functions relating measured force and displacement response, the shear modulus and loss factor are determined from complex stiffness $K^*(\omega)$

$$K^*(\omega) = H_{xf}(\omega) + m\omega^2 \tag{15}$$

where m is the mass of the the middle board, ω is the angular frequency and $H_{xf}(\omega)$ is the experimental FRF. The complex shear modulus can be obtained thus as follow

$$G^*(\omega) = \frac{K^*(\omega h)}{2dL} \tag{16}$$

where h stands for the thickness, d is the width and L the length of the VEM tape. The constant 2 considers the VEM in each side of the middle board. The experimental results for the storage modulus $(Real\{G^*(\omega)\})$ and loss modulus $(Imag\{G*(\omega)\})$ are presented in Fig.2

6 RESULTS

The results of the parameter calibration after the inference process will be presented as follows. The parameter estimation considers exclusively uncertainties in the experimental data in order to analyse the influence of the aleatory uncertainty on the model calibration. This influence is evaluated for four models defined according to the set of data calibration and the reference temperature of the master curve. The results for area validation metric used to analyse the accuracy of the models with the experimental data is also shown.

6.1 Parameter calibration

The method based on a Bayesian approach for the identification for the set of parameters $\boldsymbol{\theta}$ defined for the models of the Eq.7 and Eq.3 in order to obtain a general thermoreologically simple representation for the complex shear modulus of the VEM. It should be highlighted that, this approach allows us to characterize the constitutive model for the VEM as well as the uncertainties in the form of PDFs of their parameters. The analysis not only focuses on calibration of parameters

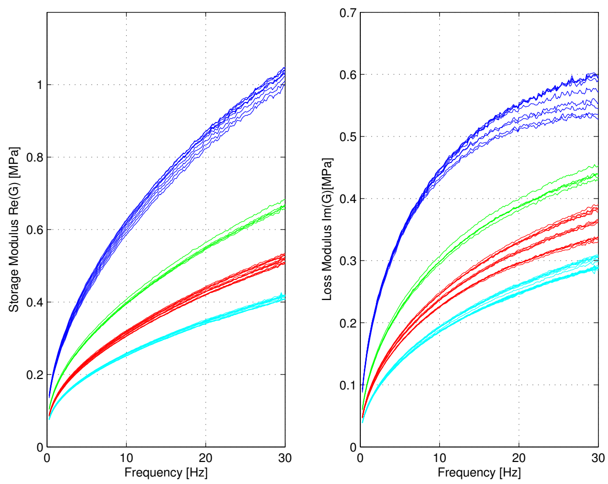


Figure 2: Dynamic modulus for the Experimental Data.(—) 22°C,(—) 30°C,(—) 35°C,(—) 40°C

but also study the influence of the experimental data used in to calibration process. Henceforth, let's consider 4 models defined by the calibration data sets S^i and the validation temperature of the model T_v^i. The sets of calibration data S^i and validation temperature T_v^i are summarized in the Table1. In the calibration process via MCMC were consider 1e5 samples in the Markov Chain. The identified parameters $\boldsymbol{\theta}$ and its respective standard deviation for the models are shown

Table 1: The sets of calibration data S^i and validation temperature T_v^i used in the calibration process.

Set of data Calibration	Validation Temperature T_v
$S^1 = \{22, 30, 35\}$	$T_v^1 = 40^oC$
$S^2 = \{22, 35, 40\}$	$T_v^2 = 30^oC$
$S^3 = \{30, 35, 40\}$	$T_v^3 = 22^oC$
$S^4 = \{22, 30, 40\}$	$T_v^4 = 35^oC$

in Tab.2, where the index $S^i,\ i = \{1, 2, 3, 4\}$ indicates the set of experimental data and $M@T_j, T = \{22, 30, 35, 40\}$ is the temperature de reference of the master curve which corresponds to the same temperature of the validation data sets.

Table 2: Estimates and standard deviation for the model parameters $\boldsymbol{\theta}$.

Model	G_0	σ_{G_0}	d	σ_d	τ	σ_{tau}	α	σ_α	β	σ_β	C_1	σ_{C_1}	C_2	σ_{C_2}
S1M@40	5,24	0,08	57,639	1,458	2,04E-04	7,31E-06	0,528	1,83E-03	0,528	1,93E-03	5,590	0,202	131,612	3,915
S2M@30	4,29	0,06	85,887	1,995	3,21E-04	1,21E-05	0,501	1,41E-03	0,500	1,41E-03	5,273	0,135	105,190	2,691
S3M@22	4,28	0,05	78,224	1,823	1,15E-03	4,56E-05	0,504	1,44E-03	0,502	1,74E-03	5,156	0,092	84,099	2,015
S4M@35	4,50	0,06	77,117	1,774	2,21E-04	7,87E-06	0,507	1,44E-03	0,506	1,51E-03	5,284	0,126	115,406	2,550

The analysis of the temperature-independent parameters referred as $\boldsymbol{\theta}_1 = \{G, d, \alpha, \beta\}$, it is possible highlight the following

- The expected value for the static modulus G_0 depends on the calibration data set S^j used to adjust the model. In particular, when the set S^1 is used, the values for G_0 as well as σ_{G_0} will get the higher values. This fact is due to the use of FTSP, for these reference temperature the set S^1 has no information about $T = 40^oC$ which is related to low frequency and hence with the static modulus. Moreover, the models that consider the temperature $T = 40^oC$ in the calibration data sets have close values for G_0, which shows are independent of the reference temperature T_0.
- The lowest values for d as well as σ_d are get using the calibration data set S^1. As in the previous case, it is due to use of the FTSP. On the other hand, the higher values for d are get using the set S^2 because the exist the higher range of temperature between the experimental data. Observe the values for d using the sets S^3 and S_4 are close.
- The order of the fractional derivative operators α e β do not show considerable variation between models, and it is important to mention that the set S^1 has the higher values.

In order to analyse the temperature-dependent parameters calibrated referred as $\boldsymbol{\theta}_2 = \{\tau, C_1, C_2\}$, it is possible highlight the following

- For this VEM with increasing of the reference temperature of the master curve, the value for C_1 decreases while the value for C_2 increases. Furthermore, the uncertainty for the parameter C_1 is always lower than the uncertainty for C_2, this fact is due to the nonlinear characteristic for the WLF equation.

- It is possible highlight that exists an influence of the calibration data sets S^j with C_2, and it is strictly related to the temperature range of the experimental data used in the calibration process. This fact can be found with the calibration data set S^1 and S^3 which generates the high and low values for C_2 respectively. In these sets, there is no information of the experimental data at $T = 40^oC$ and $T = 22^oC$ and it generates a closer range of temperature to the adjust of the WLF equation.

Figure 3 is shown the final data fitting for the model $S^1M@40$ using the calibration data set S^1 whit a reference temperature $T_0 = 40^oC$. Fig.3 shows the storage modulus and loss modulus for complex shear modulus $G^*(\omega)$. Observe that, applying the FTSP the range of frequency of the experimental data and the model was increased from $30Hz$ (Fig.2) to approximately $230Hz$ (Fig.3).

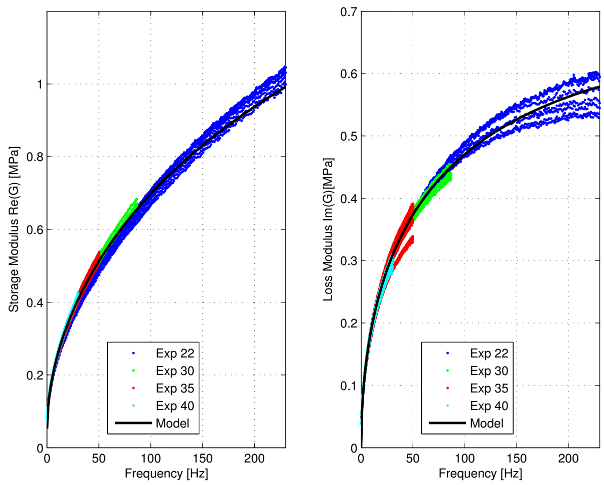


Figure 3: Adjusted model for S1@40 (—). Experimental data:(—) 22°C,(—) 30°C,(—) 35°C,(—) 40°C

6.2 Area Validation Metric

In order to accomplish the validation of the model, when the parameter uncertainties are quantified they are consider as aleatory. Those uncertainties are propagated through the model to obtain the CDF of the system response, and thus, it is possible apply the metric validation to analyse the accuracy of the model for a VEM with the experimental data. Due to the fact that the dynamical behavior of the complex modulus depends on the frequency, the present analysis will consider three frequencies to evaluate the storage and loss modulus as well as the loss factor. Those frequencies are defined at 10,50 e 90 % of the experimental frequency. In Figure 4 are shown the CDFs for the experimental data and the expected model responses for a reference temperature $T_0 = 30^oC$ using the calibration data set S^1 at the frequencies previously defined. Figure 5 shows the area validation metric for the models at three frequencies. The set of calibration data S^j used can affect the estimate of the expected model and the behavior of the complex shear modulus can be different at several frequencies. According to the Fig.5 the validation metric is influenced by the calibration data sets, and in this case, the use of an specific set can improve the expected dynamical characteristics at low, medium or high frequencies.

Figure 5 shows that the metric validation has a dependency between the calibration data sets used in the calibration and the expected model response. It is important to highlight that the CDFs for the storage and loss modulus as well as the loss factor in the expected model presents a lower dispersion than the CDFs for the experimental data. Following will be discussed some results.

- The magnitude of the area validation metric associated to the storage modulus is greater than the loss modulus.

- Observe that the area validation metric for the loss factor in all models at high frequency is higher in comparison with the low and medium frequency.

- At low frequencies the metric for the storage and loss modulus as well as the loss factor for the calibration data sets S^2 and S^4 have the lowest values. Note that this sets have the a wide range of temperature for the experimental data (22°C-40°C) where according to the FTSP the lower temperature (22°C) provides information about the high frequencies while the highest temperature (40°C) is related to the low frequencies.

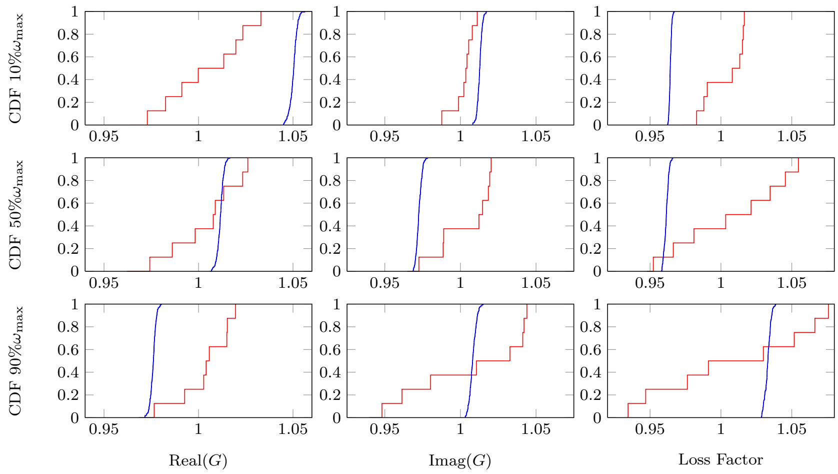


Figure 4: CDFs for the models adjusted S3@M22ºC. (—) Experimental data, (—) Expected Model S3M@22

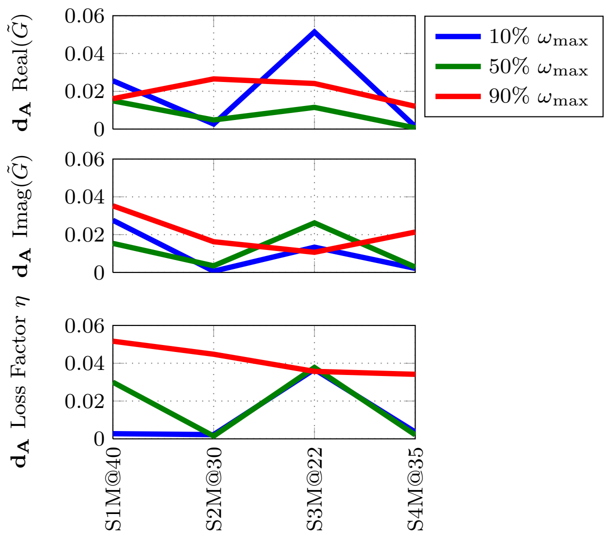


Figure 5: Area validation metric for the 16 selected models, (—) 10%ω, (—) 50%ω, (—) 90%ω.

- The same behavior occurs for the medium frequency. In all models it is shown that the metric has a good agreement for the storage modulus. However, the set S^3 fails to represent the loss modulus. This behavior shows that the range of temperature of the experimental data set can offer information at low and high frequencies and this is a essential factor for a good agreement of the model.

7 FINAL REMARKS

In the present work a calibration method based on a Bayesian approach was used to obtain a full representation of a VEM in the frequency-temperature domain. The model validation was developed on the principles of validation metrics suitable for the experimental data commonly used in structural dynamics.

The originality of this work lies to implement a methodology that allows at the same time the adjust of a master curve, and the calibration of the constitutive model thought a Bayesian approach, where it is possible quantify the parametric uncertainty of the model conditioned to experimental measurements.

The proposed approach presents some advantages. Firstly, this allows to include the thermodynamic constraints imposed on the constitutive model in the model of the temperature correspondence when it is satisfied the thermorheologically simple hypothesis. Secondly, the Bayesian strategy is addressing to represent the parameters as a random variable defined by a PDF, and this information can be used efficiently in a model verification process.

Finally, the agreement of the model and its predictive capabilities were quantitatively assessed using the Area validation metric. Based on the validation metric analysis it is possible to conclude that the range of temperature of the calibration data set is a key-point into the implementation of the FTSP. This was verified defining four models that provide favourable analysis for agreement of the model and the set of available experimental data.

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