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Analysis of a dry-friction oscillator driven by a stochastic base motion

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Abstract: In this paper the dynamics of a dry-friction oscillator driven by a stochastic base motion has been studied. The system consists of a simple oscillator (mass-spring) moving on a base with a rough surface. This roughness induces a dry-frictional force between the mass and the base which is modeled as a Coulomb friction. Besides this, the base has an imposed stochastic bang-bang motion which excites the system in a stochastic way. The non-smooth behavior of the dry-frictional force associated with the non-smooth stochastic base motion induces in the system stochastic stick-slip oscillations. The focus of the paper is to construct a stochastic model to the base motion and to analyze the stick-slip dynamics of the system from a probabilistic view point. Defined a time interval for analysis, the variables of interest are the number of time intervals in which stick and slip occur, the instants at which they begin and their duration. To make a stochastic analysis, these variables are modeled as stochastic objects and Monte Carlo simulations are employed to compute statistics of them.

Keywords: friction-induced vibration, stick-slip, non-smooth system, stick duration, stochastic quantification.

NOMENCLATURE

x = position of the mass over the base, m
m = mass weight, kg
k = spring stiffness, N/m
v = base speed, m/s
f = frictional force between mass and base, N
$f_{\max}$ = maximum friction force between mass and base, N
n = normal force exercised by the base on the mass, N
μ = friction coefficient, dimensionless
ω_n = natural frequency of the system, rad/s
t_a = duration chosen for analysis, s
λ = expected value of number of changes in the base speed per unit of time, 1/s
Q = random variable which represents the number of sign changes in the base speed, dimensionless
$P_1, \cdots, P_Q$ = random process which represents the instants of sign change of the base speed, s
$Y_1, \cdots, Y_Q$ = random process which represents the instants of sign change (sorted in ascending order) of the base speed, s
S_T = random variable which represents the number of sticks in the stochastic system response, dimensionless
S_L = random variable which represents the number of slips in the stochastic system response, dimensionless
$T_1, \cdots, T_{S_T}$ = random process which represents the instants in which the sticks begin, s
$D_1, \cdots, D_{S_T}$ = random process which represents the duration of the sticks, s
$L_1, \cdots, L_{S_L}$ = random process which represents the instants in which the slips begin, s
$H_1, \cdots, H_{S_L}$ = random process which represents the duration of the slips, s

1 INTRODUCTION

The analysis of stick-slip dynamics caused by dry-friction is not a new subject. The literature dealing with the problem is vast (see for instance "Berger, 2002", "Feeny et al., 1998" and "Awrejcewicz and Olejnik, 2007") and, reflects the economic interest in understanding the dynamic behavior of this type of non-smooth vibration ("Fidlin, 2006", "Vande Vandre, Van Campen and De Kraker, 1999" and "Galvanetto, U., 1999"). Dry-friction appears in several situations, as in drilling process and in mechanical gear systems . It could be the source of dynamic instability, noise and reduction of performance. In drilling, stick-slip oscillations happens due to the dry-friction force that exists between the bit and the rock. This friction force could be big enough to stuck the bit during some time intervals. As a constant velocity is imposed at the top of the drillstring and during the stick the bit at the bottom does not move, the drillstring is twisted. It accumulates energy in terms of torsion up to the moment that it is suddenly released. This phenomenon generates torsional vibrations and instabilities in system dynamics. If not controlled, can be harmful to the drilling process and cause a waste of energy. Furthermore, when the bit is stuck, there is no penetration. Due to all these reasons, in this case the stick-slip oscillations are undesirable and should be avoided.

Usually, stick-slip oscillations appears in mechanical systems in which uncertainties play an important role. For example in drilling, some of sources of uncertainties are the bit-rock interaction, the presence of impacts, and fluid-structure interaction. In gears, randomness arise from manufacturing, assembly errors, and random load. Beyond these sources of uncertainties, the dry friction force itself presents an inherent random behavior (Feng, 2003). The influence of ambient conditions in the properties of contact surfaces ("Bengisu and Akay, 1999") and the dependency on the relative velocity of the bodies in contact turn the dry friction force uncertain. Because of this, a stochastic approach is the ideal way to address the problem of dry-friction ("Lima and Sampaio, 2014" and "Lima and Sampaio, 2015b").

In "Lima and Sampaio (2015a)" a similar dry-friction oscillator to the one found in this paper was analyzed. This first work has allowed the duration of the stick-mode to be analyzed from a deterministic and from a stochastic viewpoint. The focus was to understand what are the parameters that control the duration of the stick-mode. Here the objectives are different. We do not want only to determine the duration of stick- and slip-modes, we want to fully characterize the system response from a a probabilistic view point. This includes the number of time intervals in which stick occur, the instants at which they begin and their duration. Since the analysis is done from a stochastic viewpoint, these information of interest are modeled as stochastic objects. The focus of the paper is to compute statistics of them, as mean, variance and joint densities functions. We believe that the theoretical knowledge on these instants can help in the development of robust control techniques to avoid the stick-mode. The idea is that predicting the instant in which a stick will start, the controller can anticipate an action on the system that will avoid the stick beginning or at least reduce its duration. One possible application is in the robust optimization of the drilling process.

The source of uncertainties of the dry-friction oscillator analyzed in this paper is the base motion. It is considered that the base has an imposed stochastic bang-bang motion which excites the system in a stochastic way.

This paper is organized as follows. Section 2 describes a most simple stick-slip oscillator. The construction of the probabilistic model of the uncertain base motion is given in Section 3. Statistics of the number of time intervals in which stick occur, the instants at which they begin, and their duration are presented in Section 4.

2 DYNAMICS OF THE STICK-SLIP OSCILLATOR

As explained in the introduction, the system analyzed in this paper is composed by simple oscillator (mass-spring) moving on a rough surface, as shown in Fig. 1. It is assumed that there is friction between the mass and the surface which

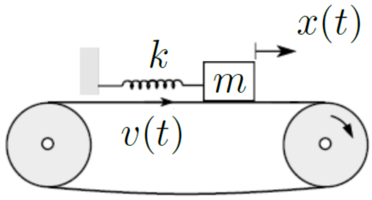


Figure 1 – Stick-slip oscillator.

is modeled as a Coulomb friction. Due to this friction model, the resulting motion of the mass can be characterized in two qualitatively different modes with a non-smooth transition between them. The first one is the stick-mode, in which the mass and base have the same velocity during an open time interval. The second one is the slip-mode, in which mass and base have different velocities. The position of the mass over the base is represented by x and its equation of motion is

$$m\,\ddot{x}(t) + k\,x(t) = f(t)\,, \tag{1}$$

where m is the mass, k is the spring stiffness and f is the frictional force between mass and base. The base speed is represented by v and it is assumed that f during the slip-mode depends on the slip velocity, $v - \dot{x}$, as

$$f(t) = n\mu\,\mathrm{sgn}(v(t) - \dot{x}(t))\,, \tag{2}$$

where n is the normal force exercised by the base on the mass and μ is the friction coefficient (Jordan and Smith, 2007). Since we use the simplest friction model, the friction coefficient is assumed to be constant. Thus, during the slip-mode, the value of the frictional force, f, is known. Its absolute value is constant and equal to the maximum friction force, $f_{\text{max}} = \mu n$. During the stick-mode, the mass velocity is equal to the base speed, i.e. $\dot{x} = v$, and the mass acceleration is equal to the base acceleration, i.e. $\ddot{x} = \dot{v}$. Thus, the equation of motion Eq. (1) can be rewritten as

$$m\dot{v} + k\,x(t) = f(t)\,.$$

The value of the frictional force during the stick-mode varies and it is confined to the interval $-f_{\text{max}} \leq f \leq f_{\text{max}}$. When the the base speed is constant in time, the system dynamics is very simple and has known analytical solutions. The literature dealing with this configuration is vast. During the stick-mode we have

$$\begin{array}{rcccl} -f_{\text{max}} & \leq & k\,x(t) & \leq & f_{\text{max}} \\ -n\mu & \leq & k\,x(t) & \leq & n\mu\,. \end{array}$$

Then, once in a stick-mode, the mass stays moving with the base until $x(t) = \frac{n\mu}{k}$ in case of positive base velocity, or until $x(t) = -\frac{n\mu}{k}$ in case of negative base velocity. Observe that during the stick-mode, the modulus of the elastic force increases up to the limit value $|f_{\max}|$, i.e. the modulus of maximum friction force. When it exceeds this value, the stick-mode ends and the mass will start a slip-mode. Because of this, considering that base speed is constant in time, knowing the mass position when a stick starts, it is possible to predict its duration. Remark that the duration of the stick-mode is limited and its maximum value is $d_{\max} = 2\frac{n\mu}{k\,v}$. When the base speed is not constant in time, the system dynamics is richer.

3 STOCHASTIC MODEL OF THE BASE MOTION

Considering that the dry-friction oscillator has an imposed stochastic bang-bang motion, we propose to model its velocity it as a random process in time constant by parts, represented by $\mathcal{V}$. We consider that $\mathcal{V}$ assumes only the two values $1,0$ m/s and $-1,0$ m/s. Beside this, defined an interval $[0\,,t_a]$ for analysis, the number of changes of the velocity sign of $\mathcal{V}$ in this interval is given by a random variable Q with Poisson distribution with parameter λt_a. Thus, for $q = 0,\ 1,\ 2,\ \ldots$, the probability mass function of Q is given by

$$\Pr(Q = q) = \frac{(\lambda t_a)^q\, e^{-\lambda t_a}}{q!}\,. \tag{3}$$

where λt_a is the mean. Figure 2 shows a the probability mass function of Q for different values of λt_a. Observe that

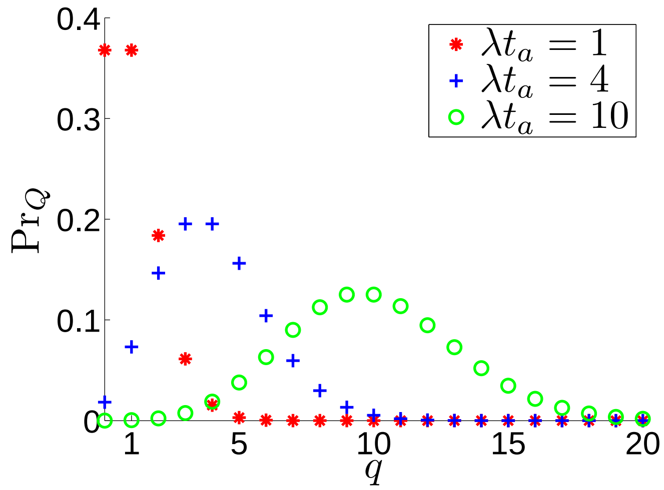


Figure 2 – Probability mass function of a Poisson random variable for different values of λt_a.

λ represents the expected value of number of changes per unit of time. Thus, given a time interval, the number of changes of the velocity sign has a Poisson distribution, with a parameter which is proportional to the length of the time interval. Note that, so far, nothing was said about the instants in which the changes occur. To determine them, we use the fact that we want our random process $\mathcal{V}$ to be stationary and to be the most uncertain as possible. Therefore, it is reasonable to distribute all instants of changes over the interval $[0\,,t_a]$ in a completely arbitrary way. Then, these instants of changes are modeled as independent and identically distributed random variables, $P_1,\ P_2,\cdots,P_Q$, each of them uniformed distributed over $[0\,,t_a]$. Given a realization of $P_1,\ P_2,\cdots,P_Q$, i.e. given the q-uple $(p_1,\ p_2,\cdots,p_q)$, to generate a realization of $\mathcal{V}$ on $[0\,,t_a]$, we need to sort the samples. We transform the q-uple $(p_1,\ p_2,\cdots,p_q)$ in $(y_1,\ y_2,\cdots,y_q)$ in a way that $y_1 \leq y_2 \leq \cdots \leq y_n$. This operation generates new random variables, $Y_1,\ Y_2,\cdots,Y_Q$, wherein $Y1 = \min_{1\leq i\leq n} P_1,\cdots,P_Q$.

The following three-step procedure is used to generate a realization of $\mathcal{V}$ on $[0\,,t_a]$:

1. draw a sample, q, of Q with Poisson distribution with parameter λt_a. This number represents the total number of changes on $[0\,,t_a]$.
2. draw one sample of $P_1,\ P_2,\cdots,P_Q$ with uniform distribution over $[0\,,t_a]$. This generates a q-uple $(p_1,\ p_2,\cdots,p_q)$ which represent the instants of change in the velocity sign.
3. sort the the q-uple $(p_1,\ p_2,\cdots,p_q)$, i.e. the instants of change, in ascending order to get $(y_1,\ y_2,\cdots,y_q)$.

The imposed discontinuities in the base velocity turns the system dynamics very rich. Each change of the velocity sign can be understood as a reboot of the system. The mass position and velocity in the instants of change of the velocity sign behave as initial conditions to the next time-interval in which the base velocity is constant. Besides of this, due to the bang-bang base motion, if the mass is in the stick-mode in the instant just before the belt discontinuity, it will be obligatorily in the slip-mode in the instant just after the discontinuity. Thus, the stick is interrupted by the discontinuities on the base velocity. Remark that with the bang-bang base motion, just knowing the mass position when a stick starts, it is not possible to predict its duration.

4 STATISTICAL ANALYSIS OF THE STICK-SLIP PROCESS

As it was assumed that the base motion is uncertain, the equation of motion of the system, Eq. 1, became a stochastic differential equation. Thus, the response of the stochastic stick-slip oscillator is a random process. Defined a time interval for analysis, the variables of interest are the number of time intervals in which stick and slip occur, the instants at which they start and their duration. They are modeled as stochastic objects:

- number of time intervals in which stick occur is represented by a random variable S_T;
- number of time intervals in which slip occur is represented by a random variable S_L;
- instants at which the sticks begin are represented by a discrete random process $T_1, \cdots, T_{S_T}$, where the subscripts indices $1, \cdots, S_T$ indicates the order that they occur, i.e., the instant in which starts the first stick, the second and so on up to the S_T-th stick;
- duration of the sticks are represented by a discrete random process $D_1, \cdots, D_{S_T}$, where again the subscripts $1, \cdots, S_T$ indicates the order that they occur;
- instants at which the slips begin are represented by a discrete random process $L_1, \cdots, L_{S_L}$, where $1, \cdots, S_L$ indicates the order that they occur;
- duration of the sticks are represented by a discrete random process $H_1, \cdots, H_{S_L}$, where $1, \cdots, S_L$ indicates the order that they occur.

Figure 3 shows a sketch of the sequence of sticks and slips in the system response. Observe that we count the first slip just after the first stick, i.e., we have $L_1 > T_1$. Besides this, if the system response ends during a slip, the number of sticks is equal or the number of slips, i.e. $S_T = S_L$. If the system response ends during a stick, then $S_T = S_L + 1$. Statistics

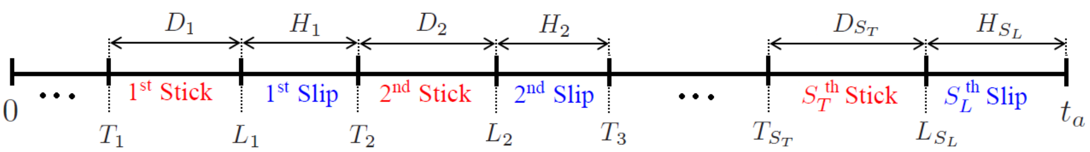
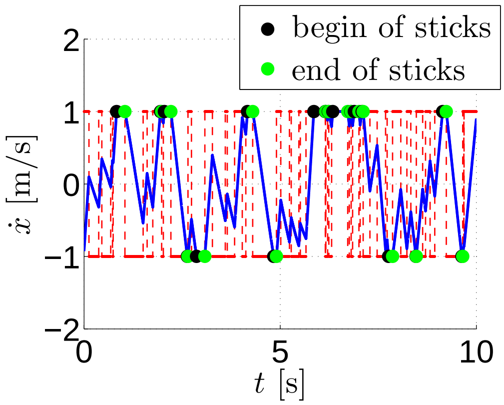


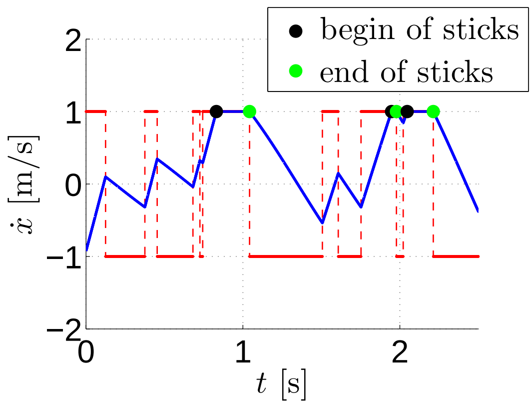
Figure 3 – Sketch of the sequence of sticks and slips in the system response for the case in which $S_T = S_L$.

of the stick-slip process were estimated by the Monte Carlo simulation method using $18,000$ independent realizations of random the system response (see for instance "Lima and Sampaio, 2012" and "Souza de Cursi and Sampaio, 2015"). For computation, duration t_a was chosen as 10 seconds and $\lambda = 5.0$. For the integration, it was used the function *ode45* of the *Matlab* software, which applies the Runge-Kutta 4th/5th-order method as time-integration scheme with a varying time-step algorithm. The maximal step size is equal to 10^{-4} seconds, and the relative and absolute tolerance are equal to 10^{-9}. The values of the parameters used in all simulations were 1.0 kg for the mass weight, 4.0 N/m for the spring stiffness, 9.8 N for the normal force, 0.51 for the constant friction coefficient and $v_0 = 1.0$ m/s for the modulus of the base speed. The initial conditions of the system were modeled as two independent random variables, X_0 and $\dot{X}_0$, uniformed distributed over $[-1,\ 1]$.

A realization of mass velocity is shown in Fig. 4(a) and a zoom over this graph is shown in Fig. 4(b). In these figures, the mass velocity is plotted in blue and base velocity is plotted in red. The vertical dashed lines indicates the instants at which occur the changes of sign in the base velocity. In Fig. 4(b) we can observe the first three sticks of the system response. The instants at which they start is indicated with a black ball, and the instants at which they end with a green ball. Observe that the first stick starts at $t = 0.83$ s and it is interrupted by the discontinuities on the base velocity at $t = 1.04$ s. Figure 5 shows the histogram of the number of changes of sign of the base velocity, i.e., the random variable Q with Poisson probability mass function. Fig. 6 shows the normalized histograms of the first six instants at which the changes of sign occur, i.e., random variables $Y_1, \cdots, Y_6$. Some statistics, as mean, variance, mode, skewness and kurtosis of $T_1, \cdots, T_6$, $D_1, \cdots, D_6$, $L_1, \cdots, L_6$ and $H_1, \cdots, H_6$ are shown in Table 1. An interesting result that can be obtained from this table is that every stick and slip have the same duration on average. On the other hand, the variance of the instants at which they start grows with the stick and slip number. To quantify this growth, we plotted the



(a)



(b)

Figure 4 – Realizations of the mass velocity.

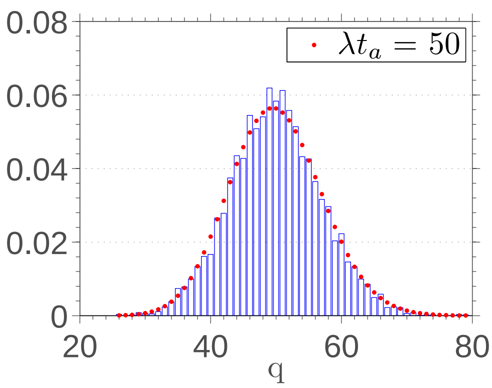


Figure 5 – Normalized histogram constructed with 18,000 samples of the number of changes of sign of the base velocity, Q.

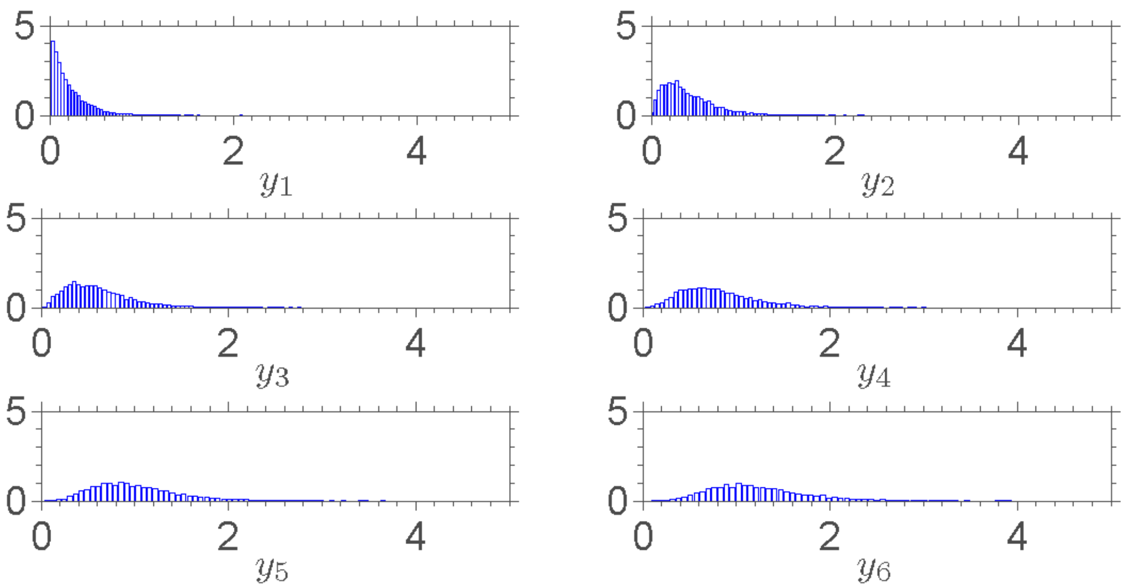


Figure 6 – Normalized histogram constructed with 18,000 samples of the first six instants at which the changes of sign occur, $Y_1, \cdots, Y_6$.

variances of these instants, which is shown in Figs. 7(a) and 7(b). Observing these graphs, we verify that the variance growth is linear with the stick and slip number. Knowing some moments of $T_1, \cdots, T_{S_T}$, $D_1, \cdots, D_{S_T}$, $L_1, \cdots, L_{S_L}$

	Mean	**Variance**	**Mode**	**Skewness**	**Kurtosis**
T_1	0.36	0.13	0.16	2.11	9.73
T_2	0.96	0.29	0.63	1.14	5.01
T_3	1.55	0.46	1.44	0.83	4.06
T_4	2.14	0.62	1.71	0.71	3.83
T_5	2.74	0.79	2.83	0.66	3.71
T_6	3.34	0.95	2.30	0.62	3.82
L_1	0.56	0.16	0.19	1.59	7.25
L_2	1.17	0.34	0.73	1.03	4.59
L_3	1.75	0.49	1.88	0.79	3.96
L_4	2.34	0.65	2.075	0.71	3.79
L_5	2.94	0.81	2.73	0.63	3.65
L_6	3.54	0.98	3.02	0.616	3.77
D_1	0.20	0.04	0.19	1.72	6.76
D_2	0.20	0.04	0.01	1.87	7.65
D_3	0.20	0.04	0.01	1.92	8.23
D_4	0.20	0.04	0.01	1.97	9.07
D_5	0.20	0.04	0.07	1.83	7.40
D_6	0.20	0.04	0.01	1.75	7.09
H_1	0.39	0.13	0.016	1.79	8.80
H_2	0.38	0.12	0.001	1.75	8.15
H_3	0.39	0.12	0.001	1.77	8.36
H_4	0.39	0.13	0.001	1.77	8.04
H_5	0.40	0.13	0.014	1.94	9.84
H_6	0.39	0.13	0.013	1.97	9.99

Table 1 – Statistics of $T_1, \cdots, T_6$, $D_1, \cdots, D_6$, $L_1, \cdots, L_6$ and $H_1, \cdots, H_6$.

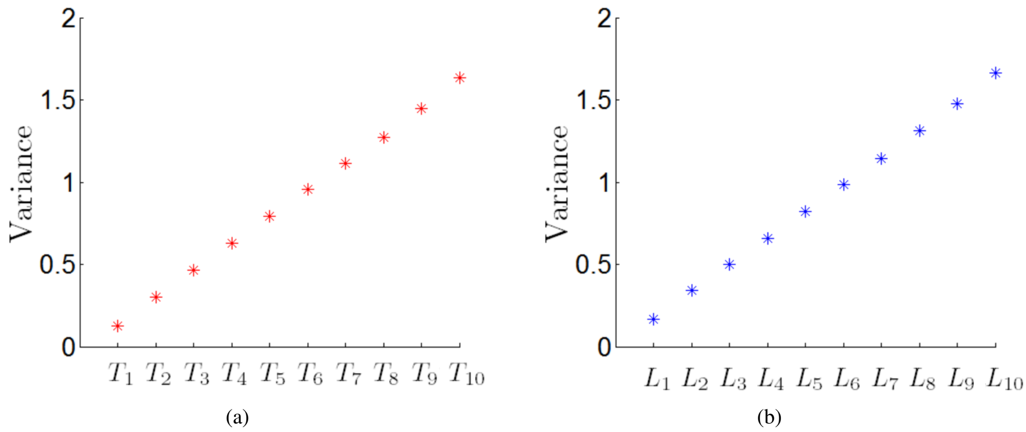


Figure 7 – Variance of the instants at which the(a) sticks and (b) slips start.

and $H_1, \cdots, H_{S_L}$ it is possible to determine on average the stochastic response of dry-friction oscillator. This average behavior give us a good overview of the stochastic stick-slip process, although it does not have enough information to characterize it from a probabilistic view point. To improve our predictions about the instants at which sticks and slips start and their duration, we propose to investigate their marginal and joint densities functions. Estimations of them were obtained through normalized histograms constructed with the response of Monte Carlo simulations. Figures 8(a) and 8(b) shows the normalized histograms of the number of time intervals in which stick and slip occur, i.e, the random variables S_T and S_L. The estimated mean and variance to S_T are respectively 16.71 and 7.94. To S_L these values are 16.54 and 7.89.

Figures 9 and 10 shows the normalized histograms of the first six instants at which sticks and slips begin, i.e., $T_1, \cdots, T_6$ and $L_1, \cdots, L_6$. Observing them we verify that $T_1, \cdots, T_6$ and $L_1, \cdots, L_6$ are not identically distributed

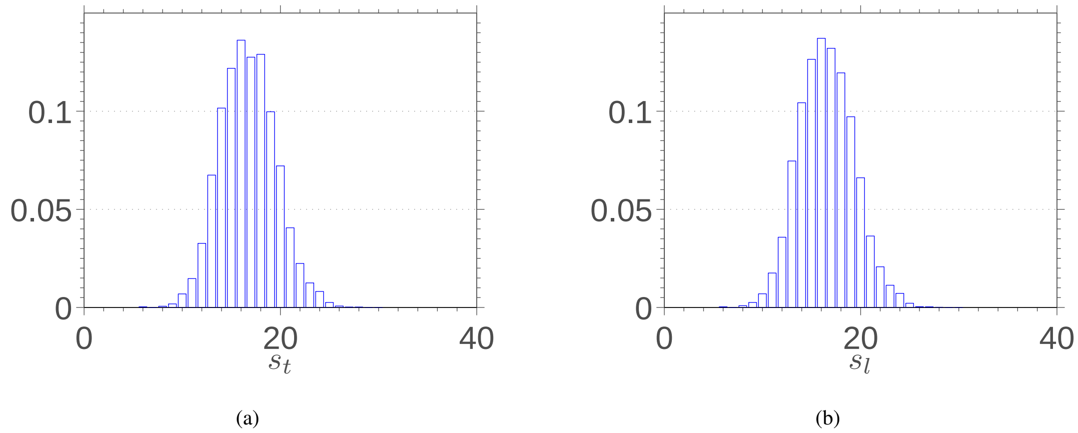


Figure 8 – Normalized histograms constructed with 18,000 samples of (a) number of time-intervals in which stick occur and (b)number of time-intervals in which slip occur.

random variables. As, $T_1 < \cdots < T_{S_T}$ and $L_1 < \cdots < L_{S_L}$, we have that $T_1, \cdots, T_{S_T}$ and $L_1, \cdots, L_{S_L}$ are not independent. To observe graphically the dependence between the two random variables T_1 and T_2, we propose to observe their normalized joint histogram, which is shown in Fig. 11(a), and to compare it with the graph that is obtained from the product of the normalized marginal histograms of T_1 and T_2, shown in Fig. 11(b). At a glance, we verify that the two graphs are different. Figure 12 shows the normalized histograms of the duration of the first six sticks, $D_1, \cdots, D_6$.

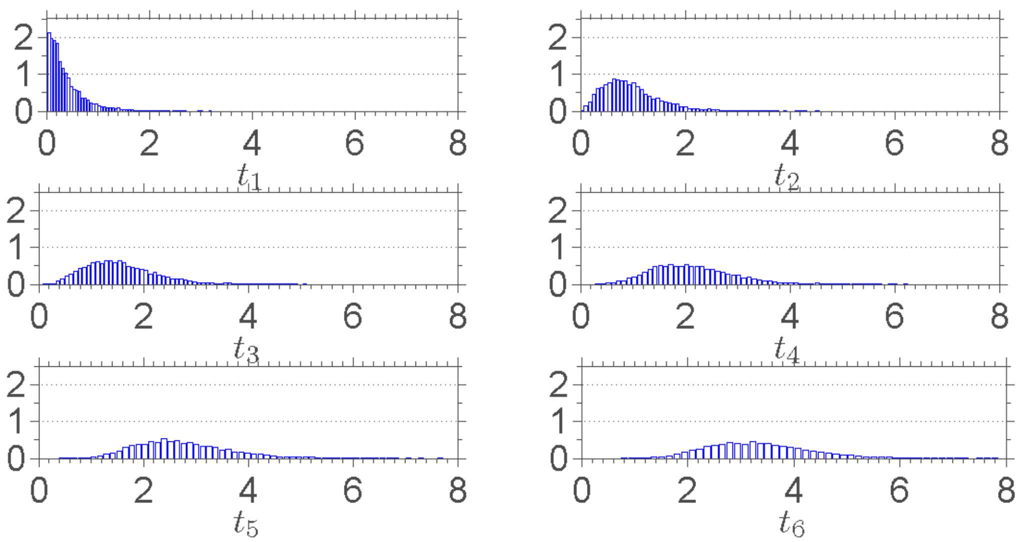


Figure 9 – Normalized histograms constructed with 18,000 samples of the first six instants at which the sticks begin, $T_1, \cdots, T_6$

The similarity between them suggests that $D_1, \cdots, D_{S_T}$ are identically distributed random variables. Given that, we may ask if they are independent. However, just based on these histograms, nothing can be said about it. To answer the question, it would be necessary to prove that the joint density function of $D_1, \cdots, D_{S_T}$ is equal to to product of their marginal density functions. But, as we do not known analytical expressions to neither of them, we are not able to answer the question. However, since we wanted to take a step forward into the quest, we computed numerically the product of the normalized histogram of D_1 and D_2 and compare it with the normalized joint histogram of the pair D_1, D_2. The two obtained surfaces are shown in Figs. 13(a) and 13(b). With a visual comparison, they seem to be similar, which suggests that D_1 and D_2 are independent. Figure 14 shows the histogram of the duration of the first six slips, $H_1, \cdots, H_6$ and, again the similarity between them suggests that $H_1, \cdots, H_{S_L}$ are identically distributed random variables.

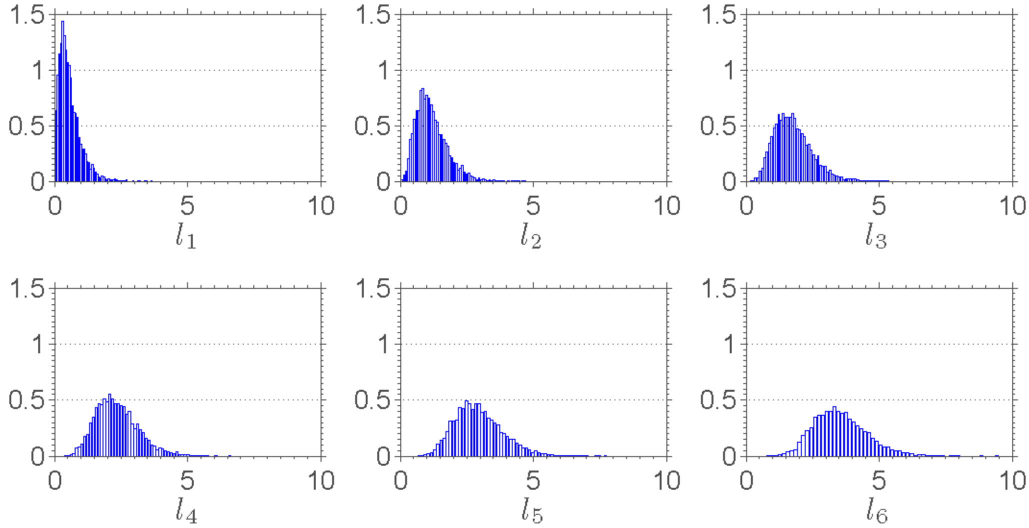


Figure 10 – Normalized histograms constructed with 18,000 samples of the first six instants at which the sticks begin, $L_1, \cdots, L_6$

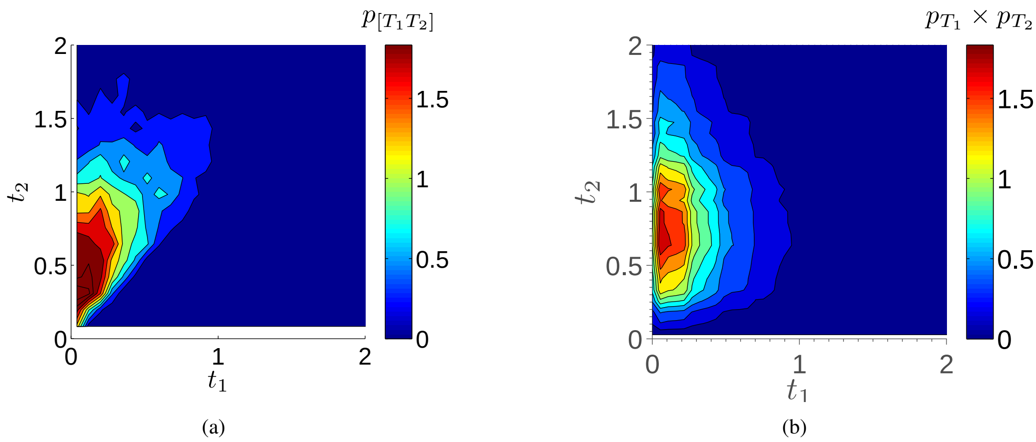


Figure 11 – (a) Contour map of the normalized histogram of the joint probability of (T_1**,** T_2**) constructed with 18,000 samples. (b) Contour map of the product of the normalized marginal histograms of** T_1 **and** T_2**.**

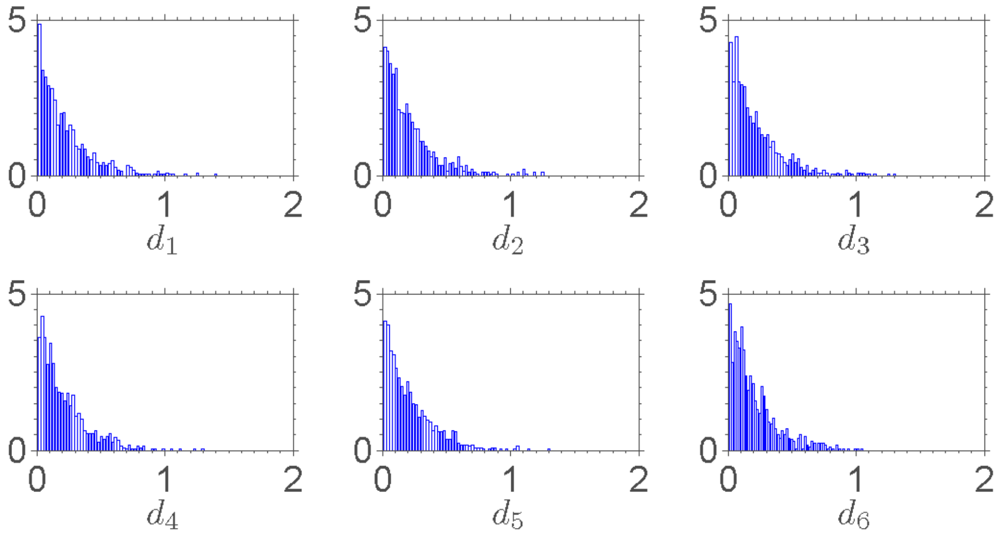
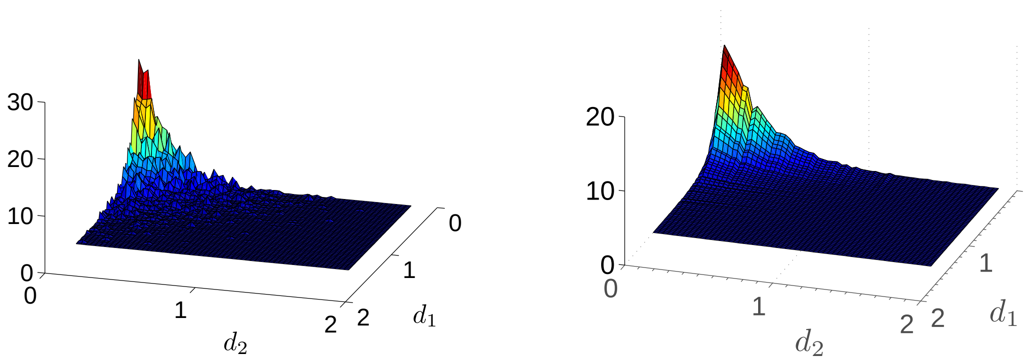


Figure 12 – Normalized histograms constructed with 18,000 samples of the duration of the first six sticks, i.e., random variables $D_1, \cdots, D_6$.



(a) (b)

Figure 13 – (a) Normalized histogram of the joint probability of (D_1, D_2) constructed with 18,000 samples. (b) Product of the normalized marginal histograms of D_1 and D_2.

5 CONCLUSIONS

In this paper the dynamics of a dry-friction oscillator driven by a stochastic base motion has been studied. We constructed a stochastic model to the base motion and analyzed the system response from a probabilistic view point. The variables of interest in the analysis were modeled as stochastic objects and statistics of them were estimated by the Monte Carlo simulation method. The simulations performed in the paper considered just one value to the friction coefficient, μ and to the parameter λ which represents the expected value of number of sign changes of the base velocity per unit of time. To make a more complete analysis of the process, we consider that it is important to verify the influence of λ and μ.

This is an ongoing work, and there are still many investigation to perform with this stick-slip oscillator to help to optimize drilling process. For instance, a refinement of the system model (making it closer to a real drillstring), a refinement of the stochastic modeling and a robust optimization to maximize the performance of the drilling process considering the uncertainties in the friction model.

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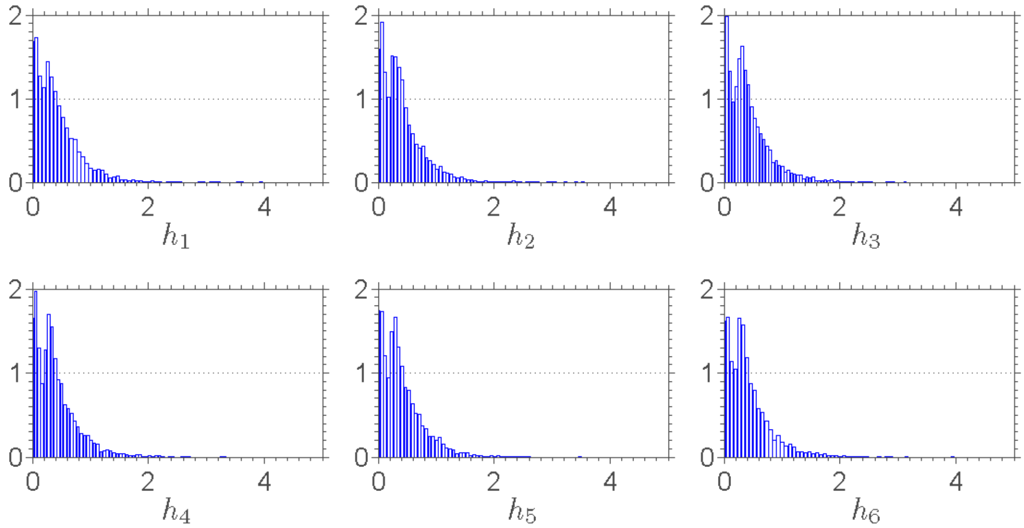

Figure 14 – Normalized histograms constructed with 18,000 samples of the duration of the first six slips, i.e., random variables $H_1, \cdots, H_6$.

7 REFERENCES

Awrejcewicz, J. and Olejnik, P., 2007, "Occurrence of stick-slip phenomenon", Journal of Theoretical and Applied Mechanics, Vol. 175, No. 1, p.p. 33-40.

Bengisu, M. and Akay, A., 1999, "Stick-slip oscillations: dynamics of friction and surface roughness", Journal of the Acoustical Society of America, Vol. 105, No. 1, p.p. 194-205.

Berger, E., 2002, "Friction modeling for dynamic system simulation", Applied Mechanics Reviews, Vol. 55, No. 6, p.p. 535-577.

Feeny, B., Guran, A., Hinrichs, N., Popp, K., 1998, "A historical review on dry friction and stick-slip phenomena", Applied Mechanics Reviews, Vol. 51, No. 5, p.p. 321-341.

Feng, Q., 2003, "A discrete model of a stochastic friction system", Computer Methods in Applied Mechanics and Engineering, Vol. 192, p.p. 2339-2354.

Fidlin, A., 2006, Nonlinear Oscillations in Mechanical Engineering, Springer, Berlin, 355 p.

Galvanetto, U., 1999, "Non-linear dynamics of multiple friction oscillators", Computer Methods in Applied Mechanics and Engineering, Vol. 178, No. 3-4, p.p. 291-306.

Jordan, D.W. and Smith, P., 2007, Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th Edition, Oxford University Press, New York, USA, 540 p.

Lima, R. and Sampaio, R., 2012, "Modelagem estocástica e geração de amostras de variáveis e vetores aleatórios", Notas de Matemtica Aplicada, SBMAC, *http://www.sbmac.org.br/arquivos/notas/livro_70.pdf*, Vol. 70, 144 p.

Lima, R. and Sampaio, R., 2014, "Stick-mode duration of random dry-friction oscillators". 8th European Nonlinear Dynamics Conference (ENOC 2014), Vienna, Austria. *https://www.academia.edu/16582286/Stick-Mode_Duration_of_Random_Dry-Friction_Oscillators*.

Lima, R. and Sampaio, R., 2015a, "Stick-mode duration of a dry-friction oscillator with an uncertain model", Journal of Sound and Vibration, Vol. 353, p.p. 259-271.

Lima, R. and Sampaio, R., 2015b, "A stochastic analysis of the trajectories of a dry-friction oscillator in a belt: how much sticked and slipping?", Conferência Brasileira de Dinâmica (DINCON 2015), Controle e Aplicações, 7 p.

Souza de Cursi, J.E. and Sampaio, R., 2015, "Uncertainty quantification and stochastic modeling with Matlab", ISTE Press, England, 456 p.

Vande Vandre, B., Van Campen, D. and De Kraker, A., 1999, "An approximate analysis of dry-friction-induced stick-slip vibrations by a smoothing procedure", Nonlinear Dynamics, Vol. 19, p.p. 157-169.

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