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State estimation based on stochastic polynomials and variational approximation

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Abstract: In many engineering applications, it becomes necessary to estimate the internal state of a system for the purposes of monitoring its health and/or achieving control of its dynamic behavior. Normally, the state estimation must be accomplished based on a few (sometimes indirect) measures of its state variables. These measures are in most cases affected by some level of noise/uncertainty. Typical methods to deal with such a problem are, for instance, least squares and/or Bayesian based approaches. In order to offer a different approach for the solution of state estimation problems, in this paper, we proposed a new method based on uncertainty quantification, more precisely, the representation of random variables using stochastic polynomials. The main idea of the proposed approach is to expand the state variables in terms of the random variables which represent the noise/uncertainty of the system, and then, apply a variational approximation to the iterative steps of a given numerical solution method (e.g. Euler). The proposed approach is first presented to discrete systems, and then, it is extended to nonlinear continuous systems discretized by numerical methods. Two examples are analyzed in order to demonstrate the effectiveness of the proposed method, including the estimation of the state variables of the Hodgkin and Huxley's model. The results of this analysis show that the proposed estate estimation method correctly estimates the values of the state variables of dynamical systems subjected to noise/uncertainty.

Keywords: uncertainty quantification, state estimation, polynomial chaos

1 INTRODUCTION

State estimation of discrete/continuous time systems is a fundamental problem in engineering sciences. For example, it is significant in engineering systems since accurate estimates allow: (i) better and more adapted actions in order to keep their desired behavior, (ii) to avoid undesired states, faults and damages, (iii) to furnish realistic prognostic of the future, and so on.

State estimation is often carried by Bayesian filters which look for the minimization of a statistical error. One of the most popular ones is Kalman's filter, which is limited to additive Gaussian noise and linear situations (see for instance, Kal). In order to overcome these limitations modifications of the original Kalman filter have been introduced, such as the extended Kalman filter or unscented Kalman filter (Evensen (2009)) as well as particle filters (see for instance, Estumano et al. (2013)). However, the application of the particle filters may face expensive computational cost when large number of samples are required for the estimation process, specially for high-dimensional problems and complex forward models (Madankan et al. (2013)).

An alternative approach that has been recently investigated to alleviate this computational burden is the use of polynomial chaos (PC) representations (Wiener (1938); Xiu and Karniadakis (2002)) for uncertainty quantification (UQ) in association with Bayesian filters to solve state estimation problems (Madankan et al. (2013)). One of the significant features of the UQ approach is that it allows the full determination of the probability distribution of a given random variable, which makes it possible the evaluation of any of its statistical moments and/or desired probabilities.

Within this context, this paper proposes an alternative approach for state estimation problems entirely based on the PC representation of random variables. The main idea of the proposed approach was presented in previous works of the authors (Lopez et al. (2011, 2013)), in which the random variables to be approximated are expanded as a function of the input random parameters of the problem. In the context of the state estimation problems, it means that the state variables are expanded in terms of the noise vector. Then, the solution of the state estimation problem becomes the determination of the deterministic coefficients of this expansion. For this purpose, we present an approach based on the solution of a sequence of variational equations. The main difference between the approach proposed in this paper, in comparison with the method presented in Madankan et al. (2013) and references therein, is that it is entirely based on the PC representation of random variables, *i.e.*, none Bayesian filter is required. It is important to mention that the main assumption of the proposed method is that there are no systematic errors, *i.e.*, $\mathbf{w}$ has a null mean.

This paper is organized as follows: Section 2 presents the general framework of the proposed approach, while Sections 3 and 4 detail the application of the new method to linear discrete systems and ordinary differential equations, respectively. In the numerical analysis section (Section 5), two examples are analyzed including a discrete linear system and the state estimation problem in the Hodgkin-Huxley's model. Finally, Section 6 presents the main conclusion drawn from this work.

2 THE PROPOSED APPROACH

In order to define the state estimation problem, consider a model for the evolution of the vector $\mathbf{x}$ in the form:

$$\mathbf{x}^{(p+1)} = \mathbf{f}\left(\mathbf{x}^{(p)}, \mathbf{w}^{(p)}\right) , \tag{1}$$

in which $^{(p)}$ denotes a time instant p in a dynamic problem, $\mathbf{f}$ is a iteration function, $\mathbf{x} \in R^n$ is comprised by the state variables to be dynamically estimated and $\mathbf{w} \in R^n$ is the state noise vector. Here, we assume that the evolution model given above is a Markovian process and

$$w_i^{(p)} = N(0, \alpha_{w_i}) , \tag{2}$$

$$E\left[w_i^{(p)} w_j^{(p)}\right] = 0 , \tag{3}$$

$$E\left[w_i^{(p)} w_j^{(q)}\right] = 0 \text{ for i, j = 1,..., n,} \tag{4}$$

where $N(...)$ stands for gaussian distribution and α_{w_i} is its standard deviation, E is the expected value operator, and p and q are two given time steps.

We are interested here in the estimation of the unnoisy values of $\mathbf{x}^{(p)}$, i.e. without the effect of $\mathbf{w}^{(p)}$. The basic idea of the proposed state estimation approach is to construct $\mathbf{x}$ as a random vector function of the state noise vector $\mathbf{w}$. Then, we estimate the unnoisy value of $\mathbf{x}$ by taking the mean value of this constructed random vector. In the following paragraphs, we detail the construction of $\mathbf{x}$ as a function of $\mathbf{w}$.

One of the classical approaches for uncertainty quantification consists in determining an expansion of $\mathbf{x}^{(p)}$ as a function of a convenient random vector $\xi^{(p)} = \left(\xi_1^{(p)}, ..., \xi_n^{(p)}\right)$ on a Hilbertian basis $\mathcal{F}$ - such as, for instance, polynomial chaos expansions (Wiener, 1938; Xiu and Karniadakis, 2002). In practice, a finite expansion is used in order to obtain a finite-dimensional approximation $\mathbf{px}^{(p)}$ - the projection of $\mathbf{x}^{(p)}$ onto a convenient finite dimensional space S. Our objective is to establish a quantitative connection between the uncertainty in the input - represented by the noise $\mathbf{w}^{(p)}$ - and the uncertainty in the response - represented by $\mathbf{x}^{(p)}$.

Here, we assume that $\mathbf{x}^{(p)}$ is a square summable random variable (e. g., $E\left(x_i^2\right) < +\infty$). Hence, we may consider a *total family* $\mathcal{F} = \{\varphi_k\}_{k\in\mathbb{N}}$ of the functional space $L^2(\Omega)$ and an approximation $\mathbf{px}^{(p)}$ belonging to the finite-dimensional subspace

$$S = \left[\left\{\varphi_1\left(\xi^{(p)}\right), ..., \varphi_{N_X}\left(\xi^{(p)}\right)\right\}\right]^n = \left\{\sum_{k=1}^{N_X} \mathbf{D}_k \varphi_k\left(\xi^{(p)}\right) : \ \mathbf{D}_k \in \mathbb{R}^n, \ 1 \le k \le N_X\right\}, \tag{5}$$

where $N_X \in \mathbb{N}^*$ is the number of terms that form the basis of S, and $\xi^{(p)}$ is a convenient random vector. Let us consider $\varphi\left(\xi^{(p)}\right) = \left(\varphi_1\left(\xi^{(p)}\right), ..., \varphi_{N_X}\left(\xi^{(p)}\right)\right)^t$ and denote by $\mathcal{C}(a, b)$ the set of the real $a \times b-$matrices. Then, we have:

$$\mathbf{y} \in \mathbf{S} \Longleftrightarrow \mathbf{y} = \mathbf{D}\varphi\left(\xi^{(p)}\right), \mathbf{D} = (D_{ij}) \in \mathcal{C}(n, N_X) . \tag{6}$$

Thus,

$$\mathbf{px}^{(p)} = \mathbf{C}^{(p)}\varphi\left(\xi^{(p)}\right) , \tag{7}$$

where $\mathbf{C}^{(p)} = \left(C_{ij}^{(p)}\right) \in \mathcal{C}(n, N_X)$ is to be determined. Once $\mathbf{C}^{(p)}$ is determined, we construct

$$\left(\mathbf{px}^{(p)}\right)_j = \sum_{k=1}^{N_X} C_{jk}^{(p)} \varphi_k\left(\xi^{(p)}\right) . \tag{8}$$

In order to construct the approximation, the coefficients of the projection $\mathbf{C}^{(p)}$ have to be determined. We propose here a variational approximation, i.e. $\mathbf{px}^{(p)}$ is determined by solving the variational equation (Lopez et al., 2013):

$$E\left[\mathbf{y}.\mathbf{px}^{(p+1)}\right] = E\left[\mathbf{y}.\mathbf{f}(\mathbf{px}^{(p)})\right], \tag{9}$$

for any $\mathbf{y}$ in the subspace S. Finally, we estimate the unnoisy value of $\mathbf{x}^{(p)}$ as

$$\mathbf{x}^{(p)}_{estimated} = E\left[\mathbf{px}^{(p)}\right] \ . \tag{10}$$

In the next sections, we detail the proposed approach for linear discrete systems and ordinary differential equations discretized by the Euler's method.

3 LINEAR DISCRETE SYSTEMS

Consider the discrete linear problem:

$$\mathbf{x}^{(p+1)} = \mathbf{A}\mathbf{x}^{(p)} + \mathbf{b}\mathbf{u}^{(p)} + \mathbf{w}^{(p)} \ . \tag{11}$$

where $\mathbf{A}$ is the evolution matrix, and $\mathbf{u}$ and $\mathbf{b}$ represent controlled inputs.

Since Eq. (11) is affected by uncertainty, the components of $\mathbf{x}^{(p)}$ and consequently $\mathbf{x}^{(p+1)}$ may be modeled as random variables. In order to accomplish that, we employ a finite representation of these vectors as a function of known random variables and deterministic coefficients as shown in Eq. (8).

It is natural to assume that $\mathbf{x}$ depends on the random vector $\mathbf{w}$ (see for instance Lopez et al. (2011)). Since the noise is considered to be independent from one step (p) to the other, we assume here that $\mathbf{X}^{(p)}$ may be written as function of $\mathbf{w}^{(q)}$, i.e. the noise of the q^{th} time step of the process. Thus, replacing $\xi^{(p)}$ by $\mathbf{w}^{(q)}$ in Eq. (8), we have

$$\left(\mathbf{px}^{(p)}\right)_j = \sum_{k=1}^{N_X} C^{(p)}_{jk}\varphi_k\left(\mathbf{w}^{(q)}\right), \tag{12}$$

and introducing it into (11), we have

$$\mathbf{C}^{(p+1)}\varphi\left(\mathbf{w}^{(q)}\right) = \mathbf{A}\mathbf{C}^{(p)}\varphi\left(\mathbf{w}^{(q)}\right) + \mathbf{b}\mathbf{u}^{(p)} + \mathbf{w}^{(p)} \ . \tag{13}$$

In order to determine the coefficients C_{ik}, we employ the variational approximation described in Eq. (9). Then, we have for the i^{th} state variable

$$E\left[\varphi^t\left(\mathbf{w}^{(q)}\right)\mathbf{c}_i^{(p+1)}\varphi\left(\mathbf{w}^{(q)}\right)\right] = E\left[\varphi^t\left(\mathbf{w}^{(q)}\right)\mathbf{A}\mathbf{c}_i^{(p)}\varphi\left(\mathbf{w}^{(q)}\right)\right] + E\left[\varphi^t\left(\mathbf{w}^{(q)}\right)b_i u_i^{(p)}\right] + E\left[\varphi^t\left(\mathbf{w}^{(q)}\right)w_i^{(p)}\right] \ ,$$

or in matrix form,

$$\mathbf{M}\mathbf{c}_i^{(p+1)} = \mathbf{M}\mathbf{c}_i^{(p)}\mathbf{A}^T + \mathbf{f_m}_i^{(p)} + \mathbf{w_m}_i^{(p)} \tag{14}$$

where

$$\begin{aligned}
\mathbf{c}_i^{(p+1)} &= \left(C^{(p+1)}_{i1}, ..., C^{(p+1)}_{iN_X}\right)^t \\
(\mathbf{M})_{ij} &= E\left[\varphi_i\left(\mathbf{w}^{(q)}\right)\varphi_j\left(\mathbf{w}^{(q)}\right)\right] \ , \\
\mathbf{f_m}_i^{(p)} &= b_i u_i^{(p)} E\left[\varphi\left(\mathbf{w}^{(q)}\right)\right] \text{ and} \\
\mathbf{w_m}_i^{(p)} &= E\left[w_i^{(p)}\varphi\left(\mathbf{w}^{(q)}\right)\right] \ .
\end{aligned}$$

Thus, the value of $\mathbf{px}^{(r)}$ may be easily evaluated iterating Eq. (14) from $p = 0$ to $p = r$. Now, in order to estimate the value of $\mathbf{x}^{(r)}$ without noise, we proposed in this paper to evaluate the mean value of $\mathbf{px}^{(r)}$ as mentioned in the previous section.

4 NONLINEAR SYSTEMS

Consider the nonlinear state estimation problem:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) + \mathbf{w}\,. \tag{15}$$

For the sake of simplicity, consider the Euler's method to iteratively solve Eq. (15). We may then write it as:

$$\mathbf{x}^{(p+1)} = \mathbf{x}^{(p)} + \mathbf{f}(\mathbf{x}^{(\mathbf{p})})dt + \mathbf{dw}, \tag{16}$$

where $\mathbf{dw}$ is a normal random vector with zero mean and known variance. Adopting the variational approach, we have

$$\varphi\left(\xi^{(p)}\right)^t \mathbf{D}^t \mathbf{p}\mathbf{x}^{(p+1)} = \varphi\left(\xi^{(p)}\right)^t \mathbf{D}^t \left(\mathbf{p}\mathbf{x}^{(p)} + \mathbf{f}\left(\mathbf{p}\mathbf{x}^{(p)}\right)dt + \mathbf{dw}\right), \forall\, \mathbf{D} \in \mathcal{C}(n, N_X)\,.$$

and using Eq. (8)

$$\sum_{k,m=1}^{N_X}\sum_{i=1}^{n} D_{im}\varphi_m\left(\xi\right) C_{ik}^{(p+1)}\varphi_k\left(\xi^{(p+1)}\right) =$$
$$\sum_{k,m=1}^{N_X}\sum_{i=1}^{n} D_{im}\varphi_m\left(\xi^{(p)}\right)\left(C_{ik}^{(p)}\varphi_k\left(\xi^{(p)}\right) + f_i\left(\mathbf{C}^{(p)}\varphi\left(\xi^{(p)}\right)\right)dt + dw_i\right)\,,$$

$$1 \le i \le n, 1 \le m \le N_X\,.$$

Thus,

$$\sum_{k,m=1}^{N_X}\sum_{i=1}^{n} D_{im}E\left[\varphi_m\left(\xi^{(p)}\right) C_{ik}^{(p+1)}\varphi_k\left(\xi^{(p+1)}\right)\right] =$$
$$\sum_{m=1}^{N_X}\sum_{i=1}^{n} D_{im}E\left[\varphi_m\left(\xi^{(p)}\right) C_{ik}^{(p)}\varphi_k\left(\xi^{(p)}\right) + \varphi_m\left(\xi^{(p)}\right) f_i\left(\mathbf{C}^{(p)}\varphi\left(\xi^{(p)}\right)\right)dt + \varphi_m\left(\xi^{(p)}\right)dw_i\right]\,,$$
$$1 \le i \le n, 1 \le m \le N_X\,.$$

By taking $D_{im} = \delta_{ir}\delta_{ms}$ in this equality, we have

$$\sum_{k=1}^{N_X} E\left[\varphi_s\left(\xi^{(p)}\right)\varphi_k\left(\xi^{(p+1)}\right)\right] C_{rk}^{(p+1)} = \sum_{k=1}^{N_X} E\left[\varphi_s\left(\xi^{(p)}\right)\varphi_k\left(\xi^{(p)}\right)\right] C_{rk}^{(p)} +$$
$$E\left[\varphi_s\left(\xi^{(p)}\right) f_r\left(\mathbf{C}^{(p)}\varphi\left(\xi^{(p)}\right)\right)\right] + E\left[\varphi_s\left(\xi\right)dw_r\right]\,,$$

$$1 \le r \le n, 1 \le s \le N_X\,. \tag{17}$$

These equations form a linear system for the unknowns $C_{rk}^{(p+1)}$. In order to solve Eq. (17), we need to choose the random vectors $\xi^{(p)}$ and $\xi^{(p+1)}$. As in the case of discrete linear system, it is a natural choice to consider that $\mathbf{x}^{(p)}$ and $\mathbf{x}^{(p+1)}$ be functions of $\mathbf{w}^{(p)}$ and $\mathbf{w}^{(p+1)}$, respectively. Hence, we assume here that $\xi^{(p)} = \mathbf{w}^{(p)}$ and $\xi^{(p+1)} = \mathbf{w}^{(p+1)}$. Introducing this in Eq. (17) and writing it in matrix form, the r^{th} state variable is evaluated by:

$$\mathbf{M_1}^{(p)}\mathbf{c}_r^{(p+1)} = \mathbf{M_2}^{(p)}\mathbf{c}_r^{(p)} + \mathbf{f_m}^{(p)}dt + \mathbf{w_m}^{(p)} \tag{18}$$

where

$$
\begin{aligned}
\mathbf{c}_r^{(p+1)} &= \left(C_{r1}^{(p+1)}, ..., C_{rN_X}^{(p+1)}\right)^t \\
\left(\mathbf{M_1}^{(p)}\right)_{sk} &= E\left[\varphi_s\left(\mathbf{w}^{(p)}\right)\varphi_k\left(\mathbf{w}^{(p+1)}\right)\right] \\
\mathbf{c}_r^{(p)} &= \left(C_{r1}^{(p)}, ..., C_{rN_X}^{(p)}\right)^t \\
\left(\mathbf{M_2}^{(p)}\right)_{sk} &= E\left[\varphi_s\left(\mathbf{w}^{(p)}\right)\varphi_k\left(\mathbf{w}^{(p)}\right)\right] \\
\left(\mathbf{f_m}^{(p)}\right)_{s} &= E\left[\varphi_s\left(\mathbf{w}^{(p)}\right)f_r\left(\mathbf{C}^{(p)}\varphi\left(\xi^{(p)}\right)\right)\right] \\
\left(\mathbf{w_m}^{(p)}\right)_{s} &= E\left[\varphi_s\left(\mathbf{w}^{(p)}\right)dw_r\right] .
\end{aligned}
$$

As described in Section 2, we iterate Eq. (18) from $p = 0$ to $p = t$, and then we employ Eq. (10) to estimate the unnoisy value of $\mathbf{x}^{(t)}$ at any desirable time step t.

5 NUMERICAL RESULTS

5.1 Example 1: A Linear Discrete System

Consider the following linear discrete system:

$$\mathbf{x}^{(p+1)} = \mathbf{A}\mathbf{x}^{(p)} + \mathbf{b}\mathbf{u}^{(p)} + \mathbf{w}^{(p)} . \tag{19}$$

in which

$$\mathbf{A} = \begin{bmatrix} 0.3 & 10^{-7} & -0.6 \\ 0 & 0.9 & 0 \\ -0.05 & 0 & 0.8 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and}$$

$\mathbf{w}$ follows the assumptions given in Eqs. (2)-(4). In this example, we aim at estimating the values of $\mathbf{x}$ from $p = 0$ to $p = 30$, knowing that $\mathbf{x}_0 = [0, 2, 0]$. We expand the state variables into a complete first order polynomial in terms of the noise variables, which leads to $N_X = 8$. Thus, the i^{th} state variable is approximated as

$$
\begin{aligned}
x_i \approx \left(\mathbf{p}\mathbf{x}^{(p)}\right)_i = {} & C_{i1} + C_{i2}w_1^{(q)} + C_{i3}w_2^{(q)} + C_{i4}w_1^{(q)}w_2^{(q)} + C_{i5}w_3^{(q)} + \\
& C_{i6}w_1^{(q)}w_3^{(q)} + C_{i7}w_2^{(q)}w_3^{(q)} + C_{i8}w_1^{(q)}w_2^{(q)}w_3^{(q)} ,
\end{aligned} \tag{20}
$$

in which q is a given time step between 0 and 30, as described in Section 3. Here, we arbitrarily choose $q = 7$. In order to estimate the unnoisy values of the state variables in this time frame, we iterate Eq. (14) from $p = 0$ to $p = 30$ and evaluate the expected value of $\mathbf{p}\mathbf{x}^{(p)}$ at each time step. The estimated values obtained using the proposed approach are in Figure 1.

In order to quantify the difference between the estimated and the exact values of the state variables, the RMS of the approximation given by the proposed approach is in Table 1. One may easily see from these results that the proposed approach was able to successfully estimate the values of the three state variable in the analyzed time range . Even the values of x_2 were reasonably approximated when compared to the perturbed values.

Table 1: RMS for the state variables of Example 1

	RMS_{x_1}	RMS_{x_2}	RMS_{x_3}	time (s)
Proposed approach	0.13×10^{-8}	0.74	2.86×10^{-8}	
Perturbed data	39.64	46.87	35.60	

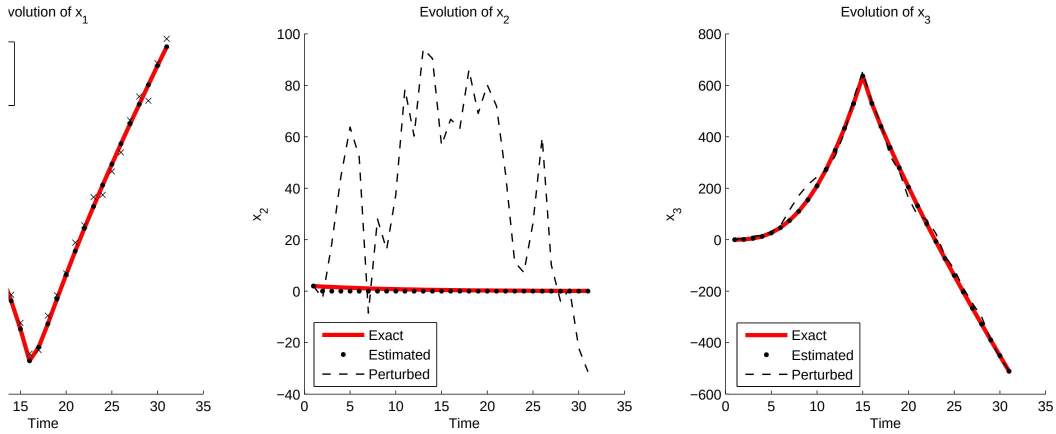

Figure 1: Estimation of the state variables of Example 1

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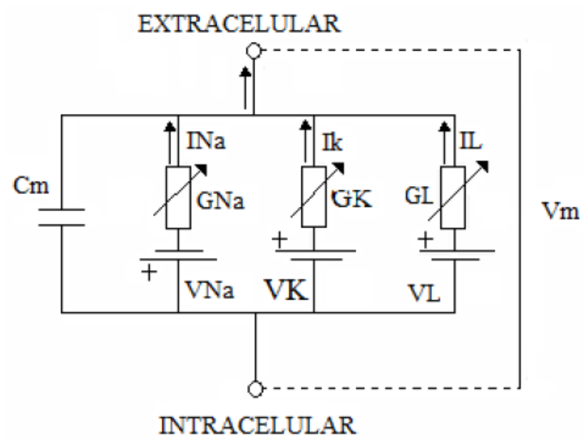


Figure 2: Eletric circuit for Hodgkin-Huxley's model : axon (Estumano et al. (2013)).

5.2 Example 2: Hodgkin-Huxley's model - Axon

In this example, we pursue the estimation of the state variables of the Hodgkin and Huxley's model. Such a model involves a non-linear system of four ordinary differential equations, whose coefficients are given in terms of functions of the applied potential. A brief description of this model is given in the following paragraphs and the interested reader is referred to the works by Hodgkin and Huxley (1952) and Estumano et al. (2013) for further details.

Hodgkin and Huxley, in their classical paper of 1952, examined the behavior of an axon under the effects of an imposed electric current across the cell membrane. The cell electric potential was assumed to be independent of the position within the cell, that is, the intracellular electric resistance was neglected. In their experiments, Hodgkin and Huxley observed that the conductance of some ions across the cell membrane, like sodium and potassium, varied with changes in the axon potential. The imposed electric current across the cell membrane was then modeled in terms of a capacitive current and ions' currents. Being the sodium and potassium ions recognized as the most important ones in this process, their currents were treated as separate from those of the other ions, which were quantified in a global manner and referred to as leakage current. Hodgkin and Huxley (1952) then proposed their model based on the electrical circuit depicted in Figure 2.

The current across the cell membrane is then given by:

$$I = I_{ions} + C_m \frac{dV_m}{dt}, \tag{21}$$

where C_m is the cell capacitance. The ions current is given by:

$$I_{ions} = I_{Na} + I_k + I_L. \tag{22}$$

The ionic currents are modeled by the conductances of the channels corresponding to each ion. Such conductances for the sodium and potassium ions were experimentally determined and written as Hodgkin and Huxley (1952)

$$G_{Na} = G_{Na}^{max} m^3 h \tag{23}$$
$$G_k = G_k^{max} n^4 \tag{24}$$

where m and n represent the open fraction, or probability of the channels being open, for sodium and potassium, respectively, while h is the probability of the channel being closed for the sodium ions. The variables m and n are also referred to as the activations of the ion transfer through the cell membrane, while h is referred to as the inactivation for the sodium ion transfer. In Eqs. (23) and (24), G_{Na}^{max} and G_k^{max} refer to the maximum sodium and potassium conductances, respectively. The electric currents resulting from the sodium and potassium ions flowing across the cell membrane are thus respectively given by

$$I_{Na} = G_{Na}^{max} m^3 h (V_m - V_{Na}) \tag{25}$$
$$G_k = G_k^{max} n^4 (V_m - V_k) \tag{26}$$

where V_{Na} and V_k give the equilibrium potential for the sodium and potassium ions, respectively. Similarly, the electric current resulting from the flow of the other ions is given by

$$I_L = G_L(V_m - V_L) \tag{27}$$

By substituting Eqs. (25)-(27) into Eqs. (21)-(22), we obtain

$$\frac{dV_m}{dt} = \frac{1}{C_m}\left(I - G_{Na}^{max} m^3 h (V_m - V_{Na}) + G_k^{max} n^4 (V_m - V_k) + G_L(V_m - V_L)\right). \tag{28}$$

Hodgkin and Huxley (1952) proposed he following ordinary differential equations to described the ion channels opening/closing dynamics

$$\frac{m}{dt} = \alpha_m(1-m) + \beta_m m, \tag{29}$$
$$\frac{h}{dt} = \alpha_h(1-h) + \beta_h h, \tag{30}$$
$$\frac{n}{dt} = \alpha_n(1-n) + \beta_n n. \tag{31}$$

Eqs. (28)-(31) comprise the non-linear system of ordinary differential equations of this example, in which the state variables are V_m, m, n and h. For notation convenience, we group the state variables in the vector $\mathbf{x} = [V_m, m, n, h]$. The coefficients α and β in each equation are given functions of V_m. While the coefficients α represent the inflow of ions towards the cell interior, the coefficients β represent the opposite effect. The following expressions were proposed by Hodgkin and Huxley (1952) for the axon studied in their experiment:

$$\alpha_m = \frac{0.1(25 - V_m)}{exp\left[0.1(25 - V_m) - 1\right]} \tag{32}$$
$$\beta_m = 4exp\left[-\frac{V_m}{18}\right] \tag{33}$$
$$\alpha_h = 0.07exp\left[-\frac{V_m}{20}\right] \tag{34}$$
$$\beta_h = \frac{1}{exp\left[0.1(30 - V_m) + 1\right]} \tag{35}$$
$$\alpha_n = 0.01\frac{10 - V_m}{exp\left[0.1(10 - V_m) - 1\right]} \tag{36}$$
$$\beta_n = 0.125exp\left[\frac{V_m}{80}\right] \tag{37}$$

Table 2: Parameters for Hodgkin-Huxley's model for an axon

Parameter	Value	Parameter	Value
$C_m(\mu F)$	1.00	$V_k(mV)$	-12.0
$G_{Na}^{max}(\mu S)$	120	$G_L^{max}(\mu S)$	0.30
$V_{Na}(mV)$	115	$V_L(mV)$	10.6
$G_k^{max}(\mu S)$	36.0	$I(mA)$	6.00

where V_m is given in milivolt. Other parameters appearing in the model were measured by Hodgkin and Huxley (1952) and are presented in Table 2.

Consider now that Eqs. (28)-(31) are perturbed by the random vector $\mathbf{w}$, such as:

$$\begin{aligned}
\frac{dV_m}{dt} &= \frac{1}{C_m}\left(I - G_{Na}^{max} m^3 h (V_m - V_{Na}) + G_k^{max} n^4 (V_m - V_k) + G_L(V_m - V_L)\right) + w_1,\\
\frac{m}{dt} &= \alpha_m(1-m) + \beta_m m + w_2,\\
\frac{h}{dt} &= \alpha_h(1-h) + \beta_h h + w_3,\\
\frac{n}{dt} &= \alpha_n(1-n) + \beta_n n + w_4.
\end{aligned}$$

The first step to estimate the values of the state variables using the proposed approach is to expand them in terms of the noise variables, as detailed in Eq. (12). For instance, using a complete first order polynomial, which leads in this example to $N_X = 16$, the j^{th} state variable is approximated by

$$\begin{aligned}
x_j \approx \left(\mathbf{p}\mathbf{x}^{(p)}\right)_j = {} & C_{j1} + C_{j2} w_1^{(p)} + C_{j3} w_2^{(p)} + C_{j4} w_1^{(p)} w_2^{(p)} + \\
& C_{j5} w_3^{(p)} + C_{j6} w_1^{(p)} w_3^{(p)} + C_{j7} w_2^{(p)} w_3^{(p)} + C_{j8} w_1^{(p)} w_2^{(p)} w_3^{(p)} + \\
& C_{j9} w_4^{(p)} + C_{j10} w_1^{(p)} w_4^{(p)} + C_{j11} w_2^{(p)} w_4^{(p)} + C_{j12} w_1^{(p)} w_2^{(p)} w_4^{(p)} + \\
& C_{j13} w_3^{(p)} w_4^{(p)} + C_{j14} w_1^{(p)} w_3^{(p)} w_4^{(p)} + C_{j15} w_2^{(p)} w_3^{(p)} w_4^{(p)} + C_{j16} w_1^{(p)} w_2^{(p)} w_3^{(p)} w_4^{(p)} .
\end{aligned}$$

Consider that we have information about the initial state of the system $\mathbf{x}_0 = [-5.00, 0.00, 0.33, 0.50]$. Here, we aim at estimating the state variables from 0 to 60 ms. The solution of the ordinary differential equation is obtained using a classic Euler's method discretizing the time range into 3000 steps. Thus, the next step is to iterate Eq. (18) from $p = 0$ to $p = 3000$. The expected values present in this equation were evaluated here using a limited sample, whose size was 25.

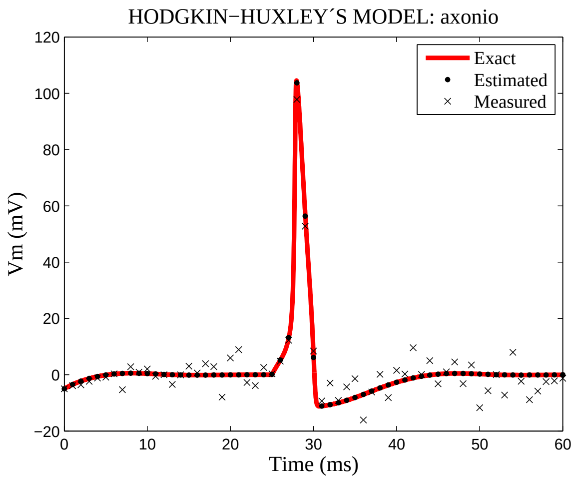
Figures 3a to 3d present the results of the estimation of the four state variables using the proposed approach. The RMS results of the proposed approach are presented in Table 3 and compared to two other state estimation methods named Sampling Importance Resampling (SIR) and Auxiliary Sampling Importance Resampling (ASIR) algorithms of the Particle Filter Method (Estumano et al., 2013). One may see from these results that the proposed approach provided a much lower RMS value for all the state variables.

Table 3: RMS for the state variables of the Hodgkin-Huxley's model: axon

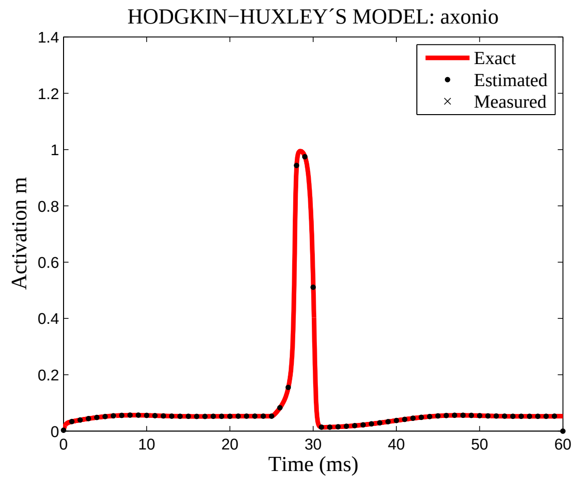
ns	RMS$_{Vm}$	RMS$_m$	RMS$_n$	RMS$_h$	time (min)
20	$1.60E-7$	$1.55E-9$	$6.17E-10$	$6.66E-10$	20
50	$9.44E-8$	$1.06E-9$	$3.18E-10$	$3.52E-10$	38
100	$8.01E-8$	$8.67E-10$	$2.55E-10$	$2.77E-10$	75
SIR (Estumano et al., 2013)	1.361	0.010	0.010	0.010	–
ASIR (Estumano et al., 2013)	1.145	0.012	0.012	0.007	–

6 CONCLUDING REMARKS

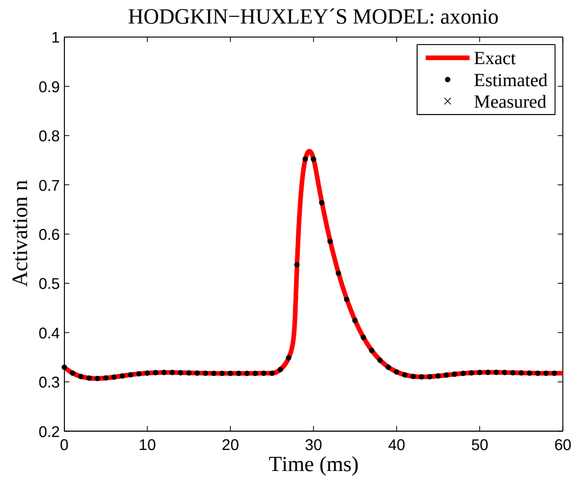
We presented in this paper a new approach for state estimation problems. This approach was based on the representation of random variables using stochastic polynomials. It consisted in expanding the state variables in terms of the noise



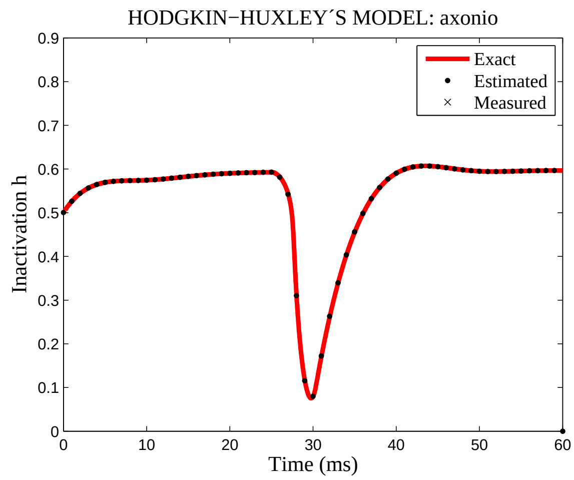
(a) Estimation of **Vm**.



(b) Estimation of **m**.



(c) Estimation of **n**.



(d) Estimation of **h**.

Figure 3: Variables estimations for Section 5.2.

variables and determining the deterministic coefficients of the expansion using a variational method. The new approach was detailed for problems involving discrete linear systems as well as ordinary differential equations discretized using the classical Euler's method.

In the numerical analysis section, two examples were analyzed including a discrete linear system and the state estimation problem in the Hodgkin-Huxley's model. In both examples, the proposed approach was able to estimate the values of the state variables with precision, i.e. with very low RMS values. In the last example, the comparison of the results obtained here to known techniques from the literature showed that the proposed approach was more accurate. We may consider these promising results and the authors believe that the proposed approach is worth of further development.

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