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Wave Propagation in Slowly Varying One-Dimensional Random Waveguides Using a Finite Element Approach

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Abstract: This work investigates structural wave propagation in one-dimensional waveguides with randomly varying material and geometric properties along the axis of propagation, specifically when the properties vary slowly enough such that there is negligible backscattering due to any changes in the properties of the medium. This variability plays a significant role in the so-called mid-frequency region, but wave-based methods are typically only applied to homogeneous and uniform waveguides. The WKB (after Wentzel, Kramers and Brillouin) approximation can be used to find a suitable generalisation of the wave solution in terms of the change of phase and amplitude of a wave propagating through a non-uniform waveguide, but it is typically restricted to analytical solutions of the equation of motion. In this paper a Wave and Finite Element (WFE) approach is proposed to extend the applicability of the WKB method to cases where no analytical solution is available. The wavenumber is expressed as a function of the position along the waveguide and a Gauss-Legendre quadrature scheme is used to the numerically integrate the phase. The WFE method is used to evaluate the wavenumbers at each integration point, and these are kept to a minimum to minimise computation cost while being able to capture the non-homogeneity to a given accuracy. The wave amplitude is calculated using conservation of power flow. The numerical example of a straight rod with a single propagating wave mode is considered. Random field properties are expressed in terms of a Karhunen-Loeve expansion. The forced response to a point excitation is calculated and results are compared to a standard Finite Element (FE) approach and to the WKB analytical solution. Results show good agreement and require only a few WFE evaluations, providing a suitable framework to account for spatially correlated randomness in waveguides.

Keywords: Wave propagation, Finite Elements, WKB approximation, Random Field, Karhunen-Loeve expansion

NOMENCLATURE

A = area of the cross-section, m^2
b = correlation length, m
$\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{h}$ = wave amplitude vectors
E = Young's modulus of elasticity
k_L = longitudinal wavenumber, m^{-1}
L = waveguide length, m
Y = input mobility, Nm/s

Greek Symbols

$\mathbf{\Lambda}(x_a, x_b)$ = propagation matrix from x_a to x_b
$\mathbf{\Gamma}$ = reflection matrix
ω = angular frequency, rad/s
ρ = mass density, kgm^3
ξ = zero mean, unity standard Gaussian random variable
σ = standard deviation

Superscripts

+ relative to positive going waves
- relative to negative going waves

Subscripts

L relative to left side of the waveguide
R relative to right side of the waveguide
0 relative to nominal value

INTRODUCTION

Wave-based methods commonly assume that waveguide properties are homogeneous in the direction of the travelling wave, limiting the application of such approaches. This assumption arises mainly because analytical solutions for non-homogeneous waveguides are only possible for very particular cases, for example acoustic horns, ducts, rods and beams - e.g. (Eisenberger, 1991, Guo, and Yang, 2012, Li, 2000) - particularly when the variation in the properties of the waveguide over distances to the wavelength are small. Moreover, randomly varying material and geometric properties along the axis of propagation play a significant role in the so-called mid-frequency region.

The WFE approach is a wave based method that is used to predict the wavenumbers and wave modes of a waveguide from a FE model, by post processing the mass and stiffness matrices, typically found using a FE package. This is particularly useful when no analytical solution is available and conventional FE models became excessively large. This method has been applied to a number of cases in structural dynamics including free and forced vibration (Ichchou, et al., 2007, Mace, et al., 2005, Manconi, and Mace, 2009, Mencik, 2014, Mencik, and Ichchou, 2005, Renno, and Mace, 2010, 2013; Waki, et al., 2009a,b). Even though it can be used to model non-uniform cross sections, this approach is however limited to homogeneous or piecewise constant waveguides in the direction of the travelling wave.

The classical WKB approximation is a method for finding suitable modifications of plane-wave solutions for non-homogeneous waveguides (Pierce, 1970). Named after Wentzel, Kramers and Brillouin, it was initially developed for solving the Schrödinger equation in quantum mechanics. The formulation assumes that the waveguide properties vary slowly enough such that there are no or negligible reflections due to these local changes, even if the net change is large, and can be extended to include spatially correlated random variability (Fabro, et al., 2015). It maintains the wave-like interpretation of non-uniform waveguides, but it is restricted to available analytical solutions.

In this work, a Wave and Finite Element (WFE) approach is proposed to extend the applicability of the WKB method to cases where no analytical solution is available. The wave proprieties are calculated using the WFE approach and they are expressed as a function of the position along the waveguide. A brief review of the method is presented. The phase change is calculated using a Gauss-Legendre quadrature scheme for numerical integration of the local wavenumber. The WFE method is used to evaluate the wavenumbers at each integration point, and these are kept to a minimum to reduce computation cost while being able to capture the non-homogeneity to a given accuracy. The wave amplitude change is calculated using conservation of power.

The numerical example of a straight rod with a single propagating wave mode is considered. Random field properties are expressed in terms of a Karhunen-Loeve (KL) expansion. The forced response to a point excitation is calculated and results are compared to a standard Finite Element (FE) approach and to the WKB analytical solution. Results show good agreement and require only a few WFE evaluations, providing a suitable framework to account for spatially correlated randomness in waveguides.

THE WKB APPROXIMATION

The WKB formulation has been applied in many fields of engineering, including ocean acoustics (Jensen, et al., 2011), acoustics (Arenas, and Crocker, 2001, Rienstra, 2003), structural dynamics (Fabro, et al., 2015, Firouz-Abadi, et al., 2007, Pierce, 1970). However, the WKB approximation breaks down if the properties change rapidly or when the travelling wave reaches a local cut-off section where the wave mode ceases to propagate. This transition, also known as a turning point, leads to an internal reflection, breaking down the main assumption in the theory, requiring a different approximation for certain frequency bands (e.g. (Nayfeh, 1973)).

Assuming a time harmonic solution, $u(x,t) = U(x)\, e^{-i\omega t}$, it is possible to define a local wavenumber $k(x)$. Thus, the *eikonal* function $S(x) = \ln \tilde{U}(x) + i\theta(x)$ is introduced, in order to find wave solutions of the kind (Jensen, et al., 2011, Whitham, 1974)

$$U(x) = e^{S(x)} = \tilde{U}(x) e^{\pm i\theta(x)}. \tag{1}$$

It is possible to define positive $\mathbf{b}^+ = \mathbf{\Lambda}^+(x_a, x_b)\mathbf{a}^+$ and negative going $\mathbf{b}^- = \mathbf{\Lambda}^-(x_a, x_b)\mathbf{a}^-$ propagation matrices for a wave travelling from the position x_a to x_b. Forced response can be considered as in Fig. 1, where the wave amplitudes are given at the excitation point by

$$\mathbf{c}^+ = \mathbf{e}^+ + \mathbf{b}^+ \text{ and } \mathbf{b}^- = \mathbf{e}^- + \mathbf{c}^-, \tag{2}$$

where $\mathbf{e}^+$ and $\mathbf{e}^-$ are the amplitude of the waves directly generated from the excitation that can be calculated from equilibrium and continuity of displacement conditions. Wave amplitudes at the boundaries are related by the reflection matrices as $\mathbf{a}^+ = \mathbf{\Gamma}_L \mathbf{a}^-$ and $\mathbf{d}^- = \mathbf{\Gamma}_R \mathbf{d}^+$. The traveling waves amplitudes are related by the propagating matrices as $\mathbf{b}^+ = \mathbf{\Lambda}^+(0, L_e)\mathbf{a}^+$, $\mathbf{d}^+ = \mathbf{\Lambda}^+(L_e, L)\mathbf{c}^+$, $\mathbf{a}^- = \mathbf{\Lambda}^-(L_e, 0)\mathbf{b}^-$, $\mathbf{c}^- = \mathbf{\Lambda}^-(L, L_e)\mathbf{d}^-$, $\mathbf{h}^+ = \mathbf{\Lambda}^+(L_e, L_r)\mathbf{c}^+$ and $\mathbf{h}^- = \mathbf{\Lambda}^-(L, L_r)\mathbf{d}^-$.

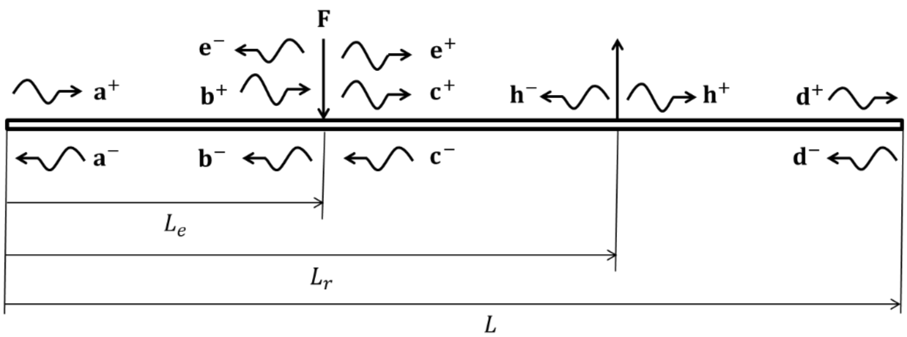


Figure 1. Point excitation and wave amplitudes on a waveguide with slowly varying properties.

These relations can be used to find

$$\mathbf{c}^{+} = [\mathbf{I} - \mathbf{\Lambda}^{+}(0, L_e)\mathbf{\Gamma}_L \mathbf{\Lambda}^{-}(0, L)\mathbf{\Gamma}_R \mathbf{\Lambda}^{+}(L_e, L)]^{-1}[\mathbf{e}^{+} + \mathbf{\Lambda}^{+}(0, L_e)\mathbf{\Gamma}_L \mathbf{\Lambda}^{-}(0, L_e)\mathbf{e}^{-}], \tag{3}$$

$$\mathbf{c}^{-} = \mathbf{\Lambda}^{-}(L, L_e)\mathbf{\Gamma}_R \mathbf{\Lambda}^{+}(L_e, L)\mathbf{c}^{+}, \tag{4}$$

from which the input mobility can be calculated. The same rationale can be used to calculate the response at any point in the waveguide from the wave amplitudes $\mathbf{h}^{+}$ and $\mathbf{h}^{-}$.

THE WAVE AND FINITE ELEMENT APPROXIMATION

In this section, a brief review of the WFE approach is presented for one-dimensional waveguides. A section of the waveguide of axial length Δ is cut from the structure and, assuming harmonic motion, its dynamic stiffness matrix $\widetilde{\mathbf{D}} = \mathbf{K} + i\omega\mathbf{C} - \omega^2\mathbf{M}$ can be obtained from a conventional FE analysis, such that $\widetilde{\mathbf{D}}\mathbf{q} = \mathbf{f}$, where $\mathbf{K}$, $\mathbf{C}$ and $\mathbf{M}$ are, respectively, the stiffness, damping and mass matrices, $\mathbf{q}$ is the vector of nodal degrees of freedom and $\mathbf{f}$ is the vector of nodal forces. The dynamic matrix $\widetilde{\mathbf{D}}$ can be condensed to eliminate the interior degrees of freedom, leading the matrix $\mathbf{D}$ that can be partitioned as

$$\begin{bmatrix} \mathbf{D}_{LL} & \mathbf{D}_{LR} \\ \mathbf{D}_{RL} & \mathbf{D}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{q}_L \\ \mathbf{q}_R \end{bmatrix} = \begin{bmatrix} \mathbf{f}_L \\ \mathbf{f}_R \end{bmatrix}, \tag{5}$$

associating the degrees of freedom and nodal forces on the left (L) and the right (R) cross-section (Mace, et al., 2005). For a wave freely propagating along the waveguide, a propagation constant relates displacements and forces from the left and right side of the section, i.e. $\mathbf{q}_R^s = \lambda \mathbf{q}_L^s$ and $\mathbf{f}_R^s = -\lambda \mathbf{f}_L^s$. Moreover, from continuity of displacements and equilibrium of forces between sections s and $(s+1)$ it follows that $\mathbf{q}_L^{s+1} = \mathbf{q}_R^s$ and $\mathbf{f}_L^{s+1} = -\mathbf{f}_R^s$. Then, a transfer matrix can be defined such that

$$\begin{bmatrix} \mathbf{q}_L^{s+1} \\ \mathbf{f}_L^{s+1} \end{bmatrix} = \mathbf{T} \begin{bmatrix} \mathbf{q}_L^{s} \\ \mathbf{f}_L^{s} \end{bmatrix}, \tag{6}$$

where

$$\mathbf{T} = \begin{bmatrix} -\mathbf{D}_{LR}^{-1}\mathbf{D}_{LL} & \mathbf{D}_{LR}^{-1} \\ -\mathbf{D}_{RL} + -\mathbf{D}_{RR}\mathbf{D}_{LR}^{-1}\mathbf{D}_{LL} & -\mathbf{D}_{RR}\mathbf{D}_{LR}^{-1} \end{bmatrix}. \tag{7}$$

The eigenvalues/eigenvectors of the transfer matrix are separated in two sets of n positive-going λ_j and $\boldsymbol{\phi}_j^{+}$ and n negative-going $1/\lambda_j$ and $\boldsymbol{\phi}_j^{-}$ wave types and the j^{th} eigenvalue is written as

$$\lambda_j = \exp(-ik_j\Delta). \tag{8}$$

The eigenvectors can be rearranged such that $\boldsymbol{\phi}^{+} = \begin{bmatrix} \boldsymbol{\phi}_q^{+} \\ \boldsymbol{\phi}_f^{+} \end{bmatrix}$ and $\boldsymbol{\phi}^{-} = \begin{bmatrix} \boldsymbol{\phi}_q^{-} \\ \boldsymbol{\phi}_f^{-} \end{bmatrix}$ and then are used for a linear transformation from the wave domain of the displacement and force to the physical domain

$$\mathbf{q} = \boldsymbol{\phi}_q^{+}\mathbf{a}^{+} + \boldsymbol{\phi}_q^{-}\mathbf{a}^{-} \text{ and } \mathbf{f} = \boldsymbol{\phi}_f^{+}\mathbf{a}^{+} + \boldsymbol{\phi}_f^{-}\mathbf{a}^{-}, \tag{9}$$

where $\mathbf{a}^{+}$ and $\mathbf{a}^{-}$ are respectively positive-going and negative-going travelling wave amplitudes. A number of parameters yielding information about the wave propagation characteristics can be calculated from this approach. In this work, it is particularly interesting to calculate the time average power transmitted through the cross-section, i.e.

$$P = -\frac{1}{2}\mathrm{Re}\{i\omega\mathbf{f}^H\mathbf{q}\} = \frac{\omega}{2}\mathrm{Im}\{\mathbf{f}^H\mathbf{q}\}, \tag{10}$$

where the superscript H stands for the Hermitian. Forced response to point excitation can be calculated using a well-conditioned formulation using the left eigenvectors of the transmission matrix $\mathbf{T}$ (Waki, et al., 2009b) and response to general excitation can be calculated following the procedure given by Renno and Mace (Renno, and Mace, 2010).

WAVE PROPAGATION WITH SLOWLY VARYING PROPERTIES

For the WKB approximation, it is necessary to calculate the phase change considering the locally defined wavenumber $k_j(x)$ as well as the amplitude change caused by the slowly varying waveguide properties. In this section, the WFE approach is used to estimate $k_j(x)$ at a number of points for calculating the phase change $\theta_j(x_a, x_b)$ from x_a to x_b. A numerical integration using a Gauss-Legendre quadrature scheme is applied, i.e.

$$\theta_j(x_a, x_b) = \int_{x_a}^{x_b} k_j(x)\, dx \approx \sum_{i=1}^{N_{gl}} W_i k_j(x_i), \tag{11}$$

where W_i are the weights and $k_j(x_i)$ is the j^{th} wavenumber calculated at the sampling point x_i defined from the Gauss-Legendre quadrature. The proprieties are evaluated at x_i from a given function describing the spatial variability and then assumed constant within the WFE cross-section. This is equivalent to a mid-point discretization for the spatial variability given by a random field, (Der Kiureghian, and Ke, 1988, Stefanou, 2009, Sudret, and Der Kiuereghian, 2000). The integration scheme gives the exact integral for a polynomial of a given order depending on the number of points N_{gl}. Therefore, this is equivalent to a polynomial fitting of the wavenumber over the waveguide between x_a and x_b. The number of points used by the quadrature must be kept to a minimum number of evaluations, to avoid excessive computational cost. No re-meshing of the FE model is necessary for each WFE evaluation.

The amplitude change can be calculated from the energy conserving property as a consequence of the WKB approximation (Pierce, 1970). Therefore, for a positive-going wave travelling from x_a, with amplitude a^+, to x_b, with amplitude b^+, as shown in Fig. 2, assuming no damping, the time average power transmitted through the cross-section, Eq. (10), at both positions must be equal, leading to

$$|a^+|^2 Re\{i\omega \boldsymbol{\Phi}_f^{+H}(x_a)\boldsymbol{\Phi}_q^{+}(x_a)\} = |b^+|^2 Re\{i\omega \boldsymbol{\Phi}_f^{+H}(x_b)\boldsymbol{\Phi}_q^{+}(x_b)\}. \tag{12}$$

This relation is written in the same way as in Eq. (17), giving

$$\gamma_j(x_a, x_b) = \log\left(\frac{|b^+|}{|a^+|}\right) = \frac{1}{2}\log\left(\frac{Re\{i\omega \boldsymbol{\Phi}_f^{H}(x_a)\boldsymbol{\Phi}_q(x_a)\}}{Re\{i\omega \boldsymbol{\Phi}_f^{H}(x_b)\boldsymbol{\Phi}_q(x_b)\}}\right). \tag{13}$$

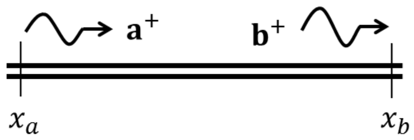


Figure 2. Positive-going wave travelling from x_a, with amplitude a^+, to x_b, with amplitude b^+, in an infinite waveguide.

Application example: straight rod

For the case of longitudinal vibration in a straight rod, only a propagating wave is considered, and the governing equation with spatially varying properties is given by

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right] + \rho A(x)\frac{\partial^2 u(x,t)}{\partial t^2} = p(x,t), \tag{14}$$

where $EA(x)$ is the spatially varying longitudinal stiffness and $\rho A(x)$ is the spatially varying mass per unit length, $p(x,t)$ is the excitation and $u(x,t)$ is the axial displacement. From the WKB approximation, it is possible to define a local wavenumber $k_L(x) = \omega / c_L(x)$, where $c_L(x) = \sqrt{EA(x)/\rho A(x)}$ is the local phase velocity at position x. Positive and negative going propagation matrices are given by

$$\boldsymbol{\Lambda}^+(x_a, x_b) = diag\left[\exp\left(-i\theta(x_a, x_b) + \gamma_j(x_a, x_b)\right)\right], \tag{15}$$

$$\boldsymbol{\Lambda}^-(x_a, x_b) = diag\left[\exp\left(-i\theta(x_a, x_b) - \gamma_j(x_a, x_b)\right)\right], \tag{16}$$

where the phase and amplitude change are respectively given by

$$\theta_L(x_1, x_2) = \int_{x_1}^{x_2} k_L(x)dx \quad \text{and} \quad \gamma_L(x_1, x_2) = \frac{1}{2}\ln\left[\frac{k_L(x_1)}{k_L(x_2)}\right], \tag{17}$$

where x_1 and x_2 are two points within the waveguide, and $\theta_L(0, L)$ and $\gamma_L(0, L)$ are given for a wave propagating from one boundary to the other. For a homogeneous waveguide, then $k_L = k_L(x)$ is a constant and the total phase change educes to $\theta_L(0, L) = k_L L$, as well as there being no change in the amplitude, i.e. $\gamma_L(0, L) = 0$.

For the sample example, a FE model for the rod cross-section is built using a single element with two nodes and one degree of freedom per node, with mass and stiffness matrices respectively given by (Petyt, 2010)

$$\mathbf{M}_e = \frac{\rho A\Delta}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } \mathbf{K}_e = \frac{EA}{\Delta}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{18}$$

assuming constant properties within the each element. Normalizing the displacement by Δ and the force by EA, it is possible to find the transfer matrix as (Duhamel, D. et al., 2003)

$$\mathbf{T} = \frac{1}{1 + \frac{(k_L\Delta)^2}{6}}\begin{bmatrix} 1 - \frac{(k_L\Delta)^2}{3} & -1 \\ (k_L\Delta)^2 - \frac{(k_L\Delta)^4}{12} & 1 - \frac{(k_L\Delta)^2}{3} \end{bmatrix}, \tag{19}$$

where $k_L = \sqrt{\rho\omega^2/E}$. The transfer matrix eigenvalues are given by

$$\lambda^{\pm} = \frac{1}{1 + \frac{(k_L\Delta)^2}{6}}\left[1 - \frac{(k_L\Delta)^2}{3} \pm i\sqrt{1 - \frac{(k_L\Delta)^2}{12}}\right], \tag{20}$$

For small values of $k_L\Delta$, they are approximately equal to $\exp(\pm\, ik_L\Delta)$ and the FE discretization gives accurate values for $k_L\Delta \ll \sqrt{12}$. The WFE wavenumber is given by $k_{WKB} = (i/\Delta)\log\lambda$, which are used to calculate the phase change evaluating the wavenumber at the positions defined by the Gauss-Legendre (GL), Eq. (11). The amplitude change is calculated from Eq. (13) with

$$\frac{|b^+|}{|a^+|} = \sqrt{\frac{k_L(x_a)}{k_L(x_b)}}\sqrt[4]{\frac{1 - [k_L(x_a)\Delta]^2/12}{1 - [k_L(x_b)\Delta]^2/12}}, \tag{21}$$

and using

$$\phi_q^+(x) = \frac{1}{\sqrt{2}} \text{ and } \phi_f^+(x) = -\frac{1}{\sqrt{2}ik_L\Delta\sqrt{1 - \frac{[k_L(x)\Delta]^2}{12}}}. \tag{22}$$

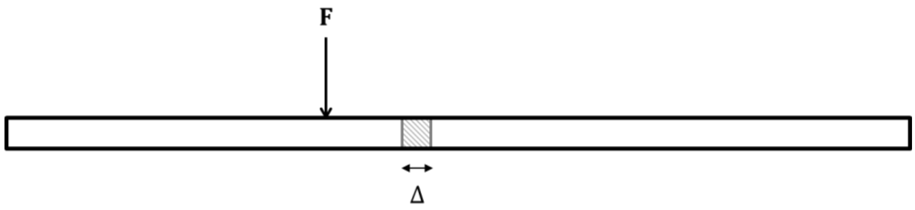
Under the same assumption, i.e. for $k_L\Delta \ll \sqrt{12}$, Eq. (21) is approximately equal to the exact solution obtained from the WKB approach, as expected from the standard WFE.

The forced response requires a number N_{gl} of WFE evaluations for the calculation of propagating matrices at the left and N_{gl} at the right side of the excitation point, one evaluation at the left and right boundaries and one evaluation at the excitation point itself, with a total of $N_{eval} = 2N_{gl} + 3$ WFE evaluations. Figure 3 presents these evaluations for $N_{gl} = 3$ compared to the homogeneous case, in which only one WFE evaluation is necessary.

RANDOM VARIABILITY

Random field theory can be used to model spatially distributed randomness using a probability measure. There are a number of methods available in the literature for generating random fields (Ghanem, and Spanos, 2012, Schueller, 1997, Stefanou, 2009, Sudret, and Der Kiureghian, 2000, Vanmarcke, 2010), including formulations using series expansions that are able to represent the field using deterministic spatial functions and random uncorrelated variables. The KL expansion is a special case where these deterministic spatial functions are orthogonal and derived from the covariance function.

a)



b)

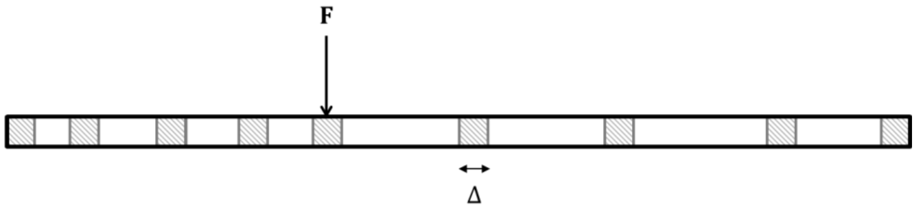


Figure 3. WFE evaluations for (a) homogeneous and (b) slowly varying waveguide for $N_{gl} = 3$.

A Gaussian homogeneous random field $H(x, p)$ with a finite, symmetric and positive definite covariance function $C_H(x_1, x_2)$, defined over a domain D, has a spectral decomposition in a generalized series as (Ghanem, and Spanos, 2012)

$$H(x) = H_0(x) + \sum_{j=1}^{\infty} \sqrt{\lambda_j}\, \xi_j f_j(x), \tag{23}$$

where ξ_j are Gaussian uncorrelated random variables, λ_j and $f_j(x)$ are eigenvalues and eigenfunctions. The eigenvalues and eigenfunctions can be ordered in descending order of eigenvalues and the KL expansion is then calculated with a finite number of terms N_{KL}, chosen by the accuracy of the series in representing the covariance function (Huang, et al., 2001). As a rule of thumb, N_{KL} can be chosen such that $\lambda_{N_{KL}}/\lambda_1 < 0.1$, and N_{KL} will depend on the correlation length of the random field.

In general, this problem can only be solved numerically by discretizing the covariance function. However, for some families of correlation functions and specific geometries, there exist analytical solutions. One such case is the one dimensional exponentially decaying autocorrelation function, $C(x_1, x_2) = e^{-|x_1 - x_2|/b}$, where b is the correlation length, in the interval $-L/2 \leq x \leq L/2$, where L is the length of the domain and where x_1 and x_2 are any two points within the interval. In this case, the KL expansion, for a zero-mean random field, can be written as

$$H(x) = \sum_{j=1}^{N_{kl}} \left[\alpha_j \xi_{1j} \sin(w_{1j} x) + \beta_j \xi_{2j} \cos(w_{2j} x)\right] \tag{24}$$

where ξ_{1j} and ξ_{2j} are Gaussian zero-mean, unity standard-deviation, independent random variables with the properties $\langle \xi_{1j} \rangle = \langle \xi_{1i} \rangle = 0, \langle \xi_{1i} \xi_{2j} \rangle = 0\,, \langle \xi_{1i} \xi_{1j} \rangle = \delta_{ij}$ where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$, and $\alpha_j = \sqrt{\lambda_{1j} / \left(\frac{L}{2} - \frac{\sin(w_{1j} L)}{2 w_{1j}}\right)}$, $\beta_j = \sqrt{\lambda_{2j} / \left(\frac{L}{2} + \frac{\sin(w_{2j} L)}{2 \mathrm{w}_{2\mathrm{j}}}\right)}$, $\lambda_{1\mathrm{j}} = 2c/(w_{1i}^2 + c^2)$, $\lambda_{2j} = 2c/(w_{2i}^2 + c^2)$, where $c = 1/b$ and w_{1i} and $\mathrm{w}_{2\mathrm{i}}$ are the i^{th} roots of the transcendental equations $c \tan w_1 + w_1 = 0$ and $w_2 \tan w_2 - c = 0$, respectively. This expansion is truncated to N_{KL} terms according the weight of the higher order eigenvalues in the series. Complete derivation can be found in the book by Ghanem and Spanos (Ghanem, and Spanos, 2012).

The KL expansion is then used to describe the Young's modulus as a random field in the numerical examples of the following section, given by

$$E(x) = E_0[1 + \sigma H(x)], \tag{25}$$

where E_0 is the nominal value for the Young's modulus and σ is the standard deviation, that can also be seen as a dispersion term quantifying the influence of $H(x)$ on the mean value E_0. The slowly varying condition can be achieved by choosing an appropriate value of the correlation length b. The larger the correlation length, the smoother the spatial variability. The Gaussian probability distribution implies that the Young's modulus could assume negative values, but the choice of the parameters makes it a very unlikely event. From a Monte Carlo (MC) sampling framework, the distribution can clipped to avoid values of Young`s modulus smaller than a given threshold. Even though an analytical solution of the KL expansion is used in this work, the proposed method is not restricted to it and any numerical solution for different correlation function or probability density function can be applied.

NUMERICAL EXAMPLE

In this section, two numerical examples are presented. The first considers a single sample from the random field to generate a spatially varying Young's modulus, i.e. a deterministic case. The second considers the statistics of the response calculated from a MC sampling scheme in which the approximation is discussed in terms the random field correlation length. For both, a numerical example of longitudinal vibration is presented for an aluminium rod with spatially varying Young's modulus, with $L = 5$ m total length, and rectangular cross-section 5 cm × 0.1 cm, free-free boundary conditions and point excitation at $L_e = 0.15L$.

The phase change $\theta(x_a, x_b)$ for the propagation matrices $\mathbf{\Lambda}^+(x_a, x_b)$, Eq. (15), and $\mathbf{\Lambda}^-(x_a, x_b)$, Eq. (16), was calculated from the wavenumbers found in the WFE analysis with cross-section length $\Delta = 0.01$ m, Eq. (20), evaluated at $N_{gl} = 3$ points defined by the Gauss-Legendre quadrature scheme over the waveguide and assuming constant properties within the element of the cross-section length. For calculation of the forced response a total $N_{eval} = 2N_{gl} + 3 = 9$ WFE evaluations were needed. This procedure was repeated for each MC sample. The analytical WKB solution and a standard FE model were used for comparison. This last was assembled with 120 elements and mid-point random field discretization.

Deterministic case

In this case, the Young`s modulus was generated from a single sample of a Karhunen-Loeve expansion shown in Figure 4 along with the excitation point and the points used for integration using the GL quadrature. Figure 5 shows the rod input mobility and Figure 6 shows the normalized value of $k_L\Delta$ at the excitation point and the ratio of the amplitude change $\gamma_L(0, L)$ calculated from the numerical approach (WFE and WKB) by the one obtained analytically (WKB only). It can be noted that the WFE approach is accurate for all the frequency range $\left(k\Delta \ll \sqrt{12}\right)$ and it agrees very well with the analytical result and the standard FE approach.

Stochastic case

In this case, the Young's modulus was considered to be a random field with $\sigma = 1$ and two different correlation lengths $b = L$ and $b = 0.25L$ were taken, according to Eqs. (24) and (25). The input mobility is calculated for the numerical WKB approach using $\Delta = 0.01$ m, which guarantees accurate results for the whole frequency range. MC simulation with 5,000 samples was used as the stochastic solver.

Figures 7 and 8 show the stochastic rod input mobility 95% confidence bounds and mean value for input mobility from the homogenous rod, i.e. constant Young's modulus E_0 along the rod. It can be noticed that results from the WKB approach using numerical evaluation agree very well with the results from the analytical WKB and standard FE. Also, it can be seen that the larger correlation length gives the larger confidence bounds, i.e. bigger variability, as expected

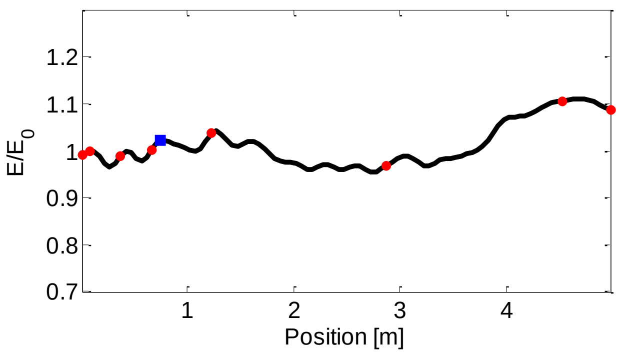


Figure 4. The Young's modulus as a function of the position used for the deterministic Also, the WFE evaluation points (red dot) and excitation point (blue square).

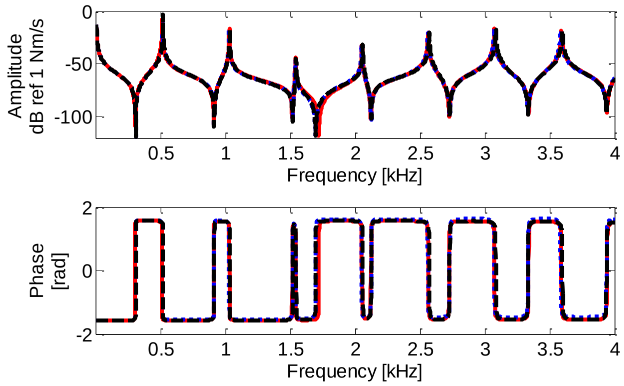


Figure 5. Input mobility at $x = 0.15L$ using FE (red), analytical WKB (blue dotted) and numerical WKB (black dashed) for $\Delta = 0.01$ m.

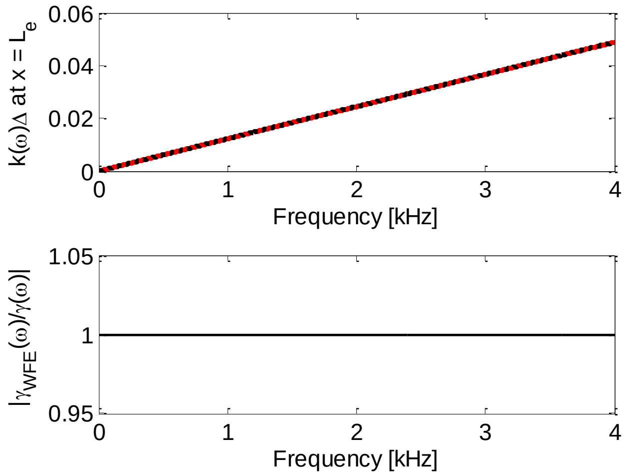


Figure 6. Analytical (red) and numerical (black) normalized wavenumber (upper) and amplitude change ratio from analytical and numerical WKB (lower) at the excitation point for $\Delta = 0.01$ m.

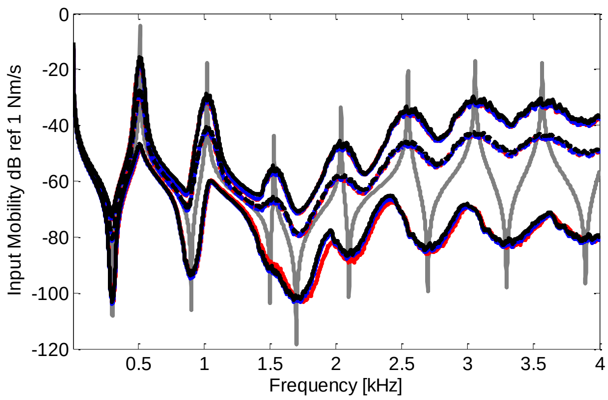


Figure 7. Input mobility 95% confidence bounds (full line) and mean value (dashed line) at $x = 0.15L$ using FE (red), analytical WKB (blue) and numerical WKB (black) with $\Delta = 0.01$ m using $\sigma = 0.1$ and $b = L$. Also, the input mobility from the homogenous rod (grey).

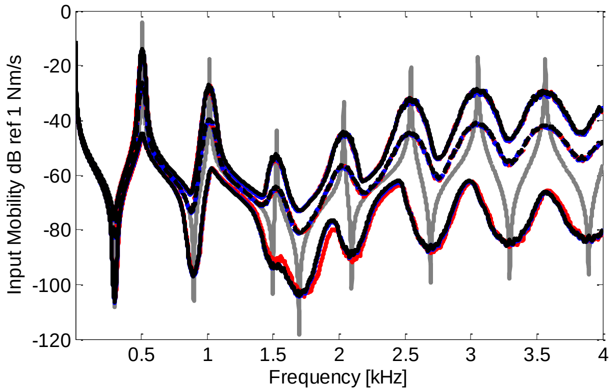


Figure 8. Input mobility 95% confidence bounds (full line) and mean value (dashed line) at $x = 0.15L$ using FE (red), analytical WKB (blue) and numerical WKB (black) with $\Delta = 0.01$ m using $\sigma = 0.1$ and $b = 0.25L$. Also, the input mobility from the homogenous rod (grey).

CONCLUDING REMARKS

A method is proposed to extend the applicability of the WKB approach to cases where no analytical solution exists by using a FE approximation. The phase change requires the numerical evaluation of the locally defined wavenumber, which are kept to a minimum, and is evaluated at localations defined by a Gauss-Legendre quadrature scheme. Also, the WKB solution implies conservation of the power which is used to calculate the amplitude change.

An example of a straight rod undergoing longitudinal vibration is presented in which the spatially correlated random variability of the Young's modulus is expressed by a specific analytical solution of the KL expansion for random fields. Even though an analytical KL expansion is used, the method can be extended straightforwardly to a numerical solution of the KL expansion and therefore to different correlation functions or probability density functions. Results are compared to analytical WKB and standard FE results, using a mid-point random field discretization, and agree very well within the validity of the WFE approximation, i.e. it depends on the FE discretization of the waveguide cross-section.

Further steps include extending the proposed approach to more complex waveguides, with different wave modes, exploring other random field types and the sensitivity to the random field discretization.

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REFERENCES

Arenas J.P., Crocker M.J., 2001, "A note on a wkb application to a duct of varying cross-section", Applied Mathematics Letters. Vol. 14, pp. 667–71

Der Kiureghian A., Ke J.-B., 1988, "The stochastic finite element method in structural reliability", Probabilistic Engineering Mechanics. Vol. 3, pp. 83–91

Duhamel, D., Mace, B.R., Brennan, M.J., 2003, "Finite element analysis of the vibrations of waveguides and periodic structures", *ISVR Technical Memorandum 922*, University of Southampton, UK

Eisenberger M., 1991, "Exact longitudinal vibration frequencies of a variable cross-section rod", Applied Acoustics. Vol. 34, pp. 123–30

Fabro A.T., Ferguson N.S., Jain T., Halkyard R., Mace B.R., 2015, "Wave propagation in one-dimensional waveguides with slowly varying random spatially correlated variability", Journal of Sound and Vibration. Vol. 343, pp. 20–48

Firouz-Abadi R.D., Haddadpour H., Novinzadeh A.B., 2007, "An asymptotic solution to transverse free vibrations of variable-section beams", Journal of Sound and Vibration. Vol. 304, pp. 530–40

Ghanem R., Spanos P.D., 2012, "Stochastic Finite Elements: A Spectral Approach", Minneola, N.Y.: Dover Publications. 240 pp. Revised edition ed.

Guo S., Yang S., 2012, "Wave motions in non-uniform one-dimensional waveguides", Journal of Vibration and Control. Vol. 18, pp. 92–100

Huang S.P., Quek S.T., Phoon K.K., 2001, "Convergence study of the truncated karhunen–loeve expansion for simulation of stochastic processes", International Journal for Numerical Methods in Engineering. Vol. 52, pp. 1029–43

Ichchou M.N., Akrout S., Mencik J.M., 2007, "Guided waves group and energy velocities via finite elements", Journal of Sound and Vibration. Vol. 305, pp. 931–44

Jensen F.B., Kuperman W.A., Porter M.B., Schimidt H., 2011, "Computational Ocean Acoustics", New York: Springer

Li Q.S., 2000, "Exact solutions for free longitudinal vibrations of non-uniform rods", Journal of Sound and Vibration. Vol. 234, pp. 1–19

Mace B.R., Duhamel D., Brennan M.J., Hinke L., 2005, "Finite element prediction of wave motion in structural waveguides", The Journal of the Acoustical Society of America. Vol. 117, No. 5, pp. 2835–43

Manconi E., Mace B.R., 2009, "Wave characterization of cylindrical and curved panels using a finite element method", The Journal of the Acoustical Society of America. Vol. 125, No. 1, pp. 154–63

Mencik J.-M., 2014, "New advances in the forced response computation of periodic structures using the wave finite element (wfe) method", Computational Mechanics. Vol. 54, No. 3, pp. 789–801

Mencik J.-M., Ichchou M.N., 2005, "Multi-mode propagation and diffusion in structures through finite elements", European Journal of Mechanics - A/Solids. Vol. 24, No. 5, pp. 877–98

Nayfeh A.H., 1973, "Perturbation methods", New York: Wiley

Petyt M., 2010, "Introduction to Finite Element vibration analysis", New York, USA: Cambridge University Press. 2nd ed.

Pierce A.D., 1970, "Physical interpretation of the wkb or eikonal approximation for waves and vibrations in inhomogeneous beams and plates", The Journal of the Acoustical Society of America. Vol. 48, pp. 275–84

Renno J.M., Mace B.R., 2010, "On the forced response of waveguides using the wave and finite element method", Journal of Sound and Vibration. Vol. 329, pp. 5474–88

Renno J.M., Mace B.R., 2013, "Calculation of reflection and transmission coefficients of joints using a hybrid finite element/wave and finite element approach", Journal of Sound and Vibration. Vol. 332, No. 9, pp. 2149–64

Rienstra S.W., 2003, "Sound propagation in slowly varying lined flow ducts of arbitrary cross-section", Journal of Fluid Mechanics. Vol. 495, pp. 157–73

Schueller G.I., 1997, "A state-of-the-art report on computational stochastic mechanics", Probabilistic Engineering Mechanics. Vol. 12, pp. 197–321

Stefanou G., 2009, "The stochastic finite element method: past, present and future", Computer Methods in Applied Mechanics and Engineering. Vol. 198, pp. 1031–51

Sudret B., Der Kiureghian A., 2000, "Stochastic finite element methods and reliability: a state-of-art report", *UCB/SEMM-2000/08*, University of California, Berkeley

Vanmarcke E., 2010, "Random Field: Analysis and Synthesis", Cambridge, MA: Word Scientific. 2nd Revised and Expanded ed.

Waki Y., Mace B.R., Brennan M.J., 2009a, "Numerical issues concerning the wave and finite element method for free and forced vibrations of waveguides", Journal of Sound and Vibration. Vol. 327, pp. 92–108

Waki Y., Mace B.R., Brennan M.J., 2009b, "Free and forced vibrations of a tyre using a wave/finite element approach", Journal of Sound and Vibration. Vol. 323, No. 3–5, pp. 737–56

Whitham G.B., 1974, "Linear and nonlinear waves", New York: John Wiley & Sons

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