Proceedings of the 3rd International Symposium on Uncertainty Quantification and Stochastic Modeling
D.A. Rade; A. M. G. Lima (Editors), ABCM, Maresias, SP, Brazil, February 15-19, 2016

Ranking Hyperelastic models for simple and pure shear at large deformation using the Bayesian Framework

TG Ritto[1] **and LCS Nunes**[2]

[1] Universidade Federal do Rio de Janeiro, Department of Mechanical Engineering, Rio de Janeiro, Brazil.

[2] Universidade Federal Fluminense, Department of Mechanical Engineering (TEM/PGMEC), Laboratory of Opto-Mechanics (LOM/LMTA), Rio de Janeiro, Brazil.

Abstract: Hyperelastic materials are extensively employed in a wide range of applications. Although there are well-established models for describing the mechanical behavior of the hyperelastic materials, relatively few papers have attempted to rank different models. This paper aims to identify parameters of some constitutive models for pure and simple shear of an incompressible isotropic hyperelastic material under large deformation, and also aims to propose a strategy to rank different models. The constitutive models considered in the present analysis are the following: Mooney-Rivlin, Yeoh, Ogden (1 and 2 terms), Lopez-Palmies (1 and 2 terms), and Gent. In the first part of the paper, the Bayesian framework is applied for the identification of the parameters of the models, where experimental data are used to update the prior probabilistic model of the unknown parameters. The Maximum a Posteriori Estimate is obtained, and the error between the model prediction and the experimental data is computed to rank the models. In the second part of the paper, the Bayesian framework is again employed, but now as an strategy for model selection. Instead of ranking the models using the error between the model prediction and the experimental result, more ingredients, such as the Ockham factor, are taken into account for the model selection. The results indicate that all models and experimental results for pure shear are in good agreement, but Mooney-Rivlin, Gent and Yeoh models were not able to well describe the available experimental data from simple shear. The best model for pure shear is Mooney-Rivlin and the best model for simple shear is Ogden (2 terms), considering the available experimental data and the criteria proposed in the present paper.

Keywords: hyperelastic models, pure and simple shear, ranking models, Bayesian model selection, parameter identification

INTRODUCTION

Nowadays, hyperelastic materials, such as rubber, are extensively employed in a wide range of applications. Many of these applications involve development of components with complex geometries under a variety of loading conditions. Sometimes uncertainties in large deformation processes must be taken into account (Acharjee and Zabaras, 2007). In general, numerical simulation is used to solve these types of problems (Swanson et al., 1985). In this way, a constitutive model for predicting the mechanical behavior of the material is often required. There are several models to predict the mechanical behavior of hyperelastic materials that can be found in the literature, see, for instance, Markmann and Verron (2006). The stress-strain response of these types of materials is derived from a strain-energy function. Here, for comparison purpose, four classical and well-known models were chosen: Mooney-Rivlin (Mooney, 1944; Rivlin, 1998); Yeoh (1993); Gent (1996) and Ogden (1972). First- and second-order Ogden were taken into account. Besides, a more recent model proposed by Lopez-Pamies (2010) (1 and 2 terms) is also considered. These models describe well the stress-strain relationship for a uniaxial tension test. However, few of them are capable of describing complex load conditions. For that reason, in the present paper, simple and pure shear tests were selected (Moreira and Nunes, 2013; Nunes and Moreira, 2013).

To rank the constitutive models, first we compute the error between the model prediction and the experimental data. Then, employing a Bayesian model selection strategy, we investigate the probability of each model, given the experimental data. With such strategy, not only the error between the model prediction and the experimental data is taken into account, but also the Ockham factor is considered, penalizing additional parameters of the models. In general, a good constitutive model requires few parameters to describe the experimental data accurately. Models with a large number of parameters can be used to fit complex response, but special care is required in order to avoid over-fitting.

This paper is organized as follows. Next section depicts the models analyzed. Section 3 presents the identification procedure of the model parameters within the Bayesian framework. Section 4 explains the Bayesian model selection strategy and Section 5 presents the results. Finally, the concluding remarks are made in the last Section.

SIMPLE AND PURE SHEAR MODELS

The concepts of simple and pure shear are well known in mechanics. Simple shear is related to a state of deformation, while pure shear denotes a state of stress that is characterized by tr $\sigma = 0$. The Cauchy stress tensor σ of an incompressible isotropic hyperelastic material may be expressed in terms of strain-energy function Ψ (Rivlin, 1948).

There are several strain-energy density functions that are used to describe the mechanical behaviour of hyperelastic materials. Here, for comparison purpose, Mooney-Rivlin, Yeoh, Ogden, Lopez-Pamies and Gent models are used.

1. Mooney-Rivlin model

One of the first strain energy function was proposed by Mooney and extended by Rivlin. The well-known Mooney-Rivlin model is given by (Rivlin, 1948):

$$\Psi_{MR} = C_{10}(I_1 - 3) + C_{01}(I_2 - 3)\,, \tag{1}$$

with the initial shear modulus equal to $\mu = 2(C_{10} + C_{01})$.

$$\sigma_{11} = \mu(\lambda^2 - \lambda^{-2})\,, \tag{2}$$

$$\sigma_{12} = \mu k\,. \tag{3}$$

2. Yeoh model

The model proposed by Yeoh, which is referred to as generalized neo-Hookean materials, is based on three terms that depends only on the first strain invariant. The Yeoh model is defined by (Yeoh, 1993):

$$\Psi_Y = C_1(I_1 - 3) + C_2(I_1 - 3)^2 + C_3(I_1 - 3)^3\,, \tag{4}$$

where C_1, C_2 and C_3 denote material parameters. The initial shear modulus is equal to $\mu = 2C_1$.

$$\sigma_{11} = 2(\lambda^2 - \lambda^{-2})[C_1 + 2C_2(\lambda^2 + \lambda^{-2} - 2) + 3C_3(\lambda^2 + \lambda^{-2} - 2)^2]\,, \tag{5}$$

$$\sigma_{12} = 2k[C_1 + 2C_2(\lambda^2 + \lambda^{-2} - 2) + 3C_3(\lambda^2 + \lambda^{-2} - 2)^2]\,. \tag{6}$$

3. Ogden model with 1 term

An alternative model for hyperelastic materials was postulated by Ogden, where the strain energy is a function of the principal stretch, instead of the first strain invariant. The Ogden model is expressed by (Ogden, 1972):

$$\Psi_O = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3)\,, \tag{7}$$

where the initial shear modulus is $\mu = \frac{1}{2}\sum_{p=1}^{N} \mu_p\alpha_p$ with $\mu_p\alpha_p > 0$. Considering only one term of series, the Cauchy stress components are

$$\sigma_{11} = \mu_1(\lambda^{\alpha_1} - \lambda^{-\alpha_1})\,, \tag{8}$$

$$\sigma_{12} = \mu_1 \frac{\lambda}{1 + \lambda^2}(\lambda^{\alpha_1} - \lambda^{-\alpha_1})\,, \tag{9}$$

with $\lambda = \dfrac{k + \sqrt{k^2 + 4}}{2}$ (> 1) and $\mu = \dfrac{\mu_1\alpha_1}{2}$.

4. Ogden model with 2 terms

Taking into consideration the first two terms of the series (Ogden, 1972):

$$\sigma_{11} = \mu_1(\lambda^{\alpha_1} - \lambda^{-\alpha_1}) + \mu_2(\lambda^{\alpha_2} - \lambda^{-\alpha_2})\,, \tag{10}$$

$$\sigma_{12} = \frac{\lambda}{1 + \lambda^2}[\mu_1(\lambda^{\alpha_1} - \lambda^{-\alpha_1}) + \mu_2(\lambda^{\alpha_2} - \lambda^{-\alpha_2})]\,, \tag{11}$$

with $\mu = \dfrac{\mu_1\alpha_1 + \mu_2\alpha_2}{2}$.

5. Lopez-Pamies model with 1 term

A simple model based on the first strain invariant for rubber elastic solids was proposed by Lopez-Pamies. The stored-energy function is given by (Lopez, 2010):

$$\Psi_{LP} = \sum_{r=1}^{M} \frac{3^{1-\alpha_r}}{2\alpha_r} \mu_r (I_1^{\alpha_r} - 3^{\alpha_r}) \,, \tag{12}$$

where M indicates the number of terms, while μ_r and α_r are material parameters being $\mu_r > 0$ and $\alpha_r > 1/2$. The initial shear modulus is defined as $\mu = \sum_{r=1}^{M} \mu_r$. Considering only one term of series,

$$\sigma_{11} = 3\mu_1 \frac{\lambda^4 - 1}{\lambda^4 + \lambda^2 + 1} \left(\frac{\lambda^2 + \lambda^{-2} + 1}{3} \right)^{\alpha_1} , \tag{13}$$

$$\sigma_{12} = 3\mu_1 \frac{k}{k^2 + 3} \left(\frac{k^2 + 3}{3} \right)^{\alpha_1} , \tag{14}$$

with $\mu = \mu_1$.

6. Lopez-Pamies model with 2 terms

Taking into consideration the first two terms of the series (Lopez, 2010):

$$\sigma_{11} = 3\frac{\lambda^4 - 1}{\lambda^4 + \lambda^2 + 1} \left[\mu_1 \left(\frac{\lambda^2 + \lambda^{-2} + 1}{3} \right)^{\alpha_1} + \mu_2 \left(\frac{\lambda^2 + \lambda^{-2} + 1}{3} \right)^{\alpha_2} \right] , \tag{15}$$

$$\sigma_{12} = 3\frac{k}{k^2 + 3} \left[\mu_1 \left(\frac{k^2 + 3}{3} \right)^{\alpha_1} + \mu_2 \left(\frac{k^2 + 3}{3} \right)^{\alpha_2} \right] , \tag{16}$$

with $\mu = \mu_1 + \mu_2$.

7. Gent model

Gent suggested a high order I1 model based on a linear logarithm argument and on the concept of limiting chain extensibility. The Gent model is given by the following expression (Gent, 1996):

$$\Psi_G = -\frac{\mu}{2} J_m \ln \left(1 - \frac{I_1 - 3}{J_m} \right) , \tag{17}$$

where μ is the initial shear modulus and J_m is the constant limiting value for $I_1 - 3$.

$$\sigma_{11} = \mu \frac{J_m}{J_m - I_1 + 3} \left(\lambda^2 - \lambda^{-2} \right) , \tag{18}$$

$$\sigma_{12} = \mu \frac{J_m k}{J_m - k^2} \,. \tag{19}$$

IDENTIFICATION PROCEDURE

The constitutive model parameters are modeled as random variables, and a prior probability density function (pdf) is assigned for them. Using the Maximum Entropy Principle (MaxEnt) (Jaynes, 1957a; Jaynes, 1957; Kapur, 1990), introducing given bounds, yields the Uniform random variable. If any other information is considered, the prior distribution would not be Uniform, and the resulting random variable would have less entropy (i.e., more information). It is recommended to begin with a prior with low information, such that the available experimental data have an important role in updating the prior.

Let M_j, for $j = \{1, 2, ..n\}$, be a model class with a set of parameters $\mathbf{p}_j$. To identify the set of parameters $\mathbf{p}_j$ from the model class M_j, we will use the Bayesian framework (Beck and Katafygiotis, 1998; Kaipio and Somersalo,2005). Thus, the parameters are modeled as random variables. Then, the prior probability density function is updated with experimental data D.

We assume an additive Gaussian noise to model the prediction error. This probabilistic model is taking into account both measurement and modeling errors. This last error is related to the fact that, even if the measurements could be done perfectly, the response of the computational model would not be expected to perfectly match the data. The additive Gaussian noise model is an usual assumption, and it is corroborated by the MaxEnt, if we use the following information for the noise: (1) the support is R^n (where n is the number of points measured), (2) the mean is zero and (3) the variance is finite. With this information, the MaxEnt yieds an independent Gaussian random vector. Hence, for a given $\mathbf{p}_j$:

$$S(\lambda) = \sigma(\lambda, \mathbf{p}_j) + \mathbf{N}(\lambda)\,, \tag{20}$$

where σ is the pure/simple shear stress, $\mathbf{N}(\lambda) \sim Normal(0,\text{var}[I])$ is a white noise vector with unknown variance var, that takes into account all errors, and $[I]$ is the identity matrix. The response $S(\lambda)$ is random because of the random noise, and for each set of parameters $\mathbf{p}_j$ there is a different expression for σ, which depends on the model class M_j, as seen in Section . We want to identify the parameters that are entries of vector $\boldsymbol{\phi}_j = \{\mathbf{p}_j,\text{var}\}$, which is modeled by the random vector $\boldsymbol{\Phi}_j$. Using the Bayes formula, we can compute the updated (or posterior) probability density function of $\boldsymbol{\Phi}_j$, given data D:

$$posterior(\boldsymbol{\phi}_j|M_j) = p(\boldsymbol{\phi}_j|D, M_j) \propto p(D|\boldsymbol{\phi}_j, M_j)p(\boldsymbol{\phi}_j|M_j)\,, \tag{21}$$

where '$\propto$' means 'proportional to' and $p(D|\boldsymbol{\phi}_j, M_j)$ is the likelihood function.

To approximate the posterior distribution, which is given up to a normalization constant, the Markov Chain Monte Carlo (MCMC) / Metropolis-Hastings algorithm is applied (Gamerman, 2002).

BAYESIAN MODEL SELECTION

We want to compute the probability of M_1 given data D, $P(M_1|D)$, and compare it with the probability of the other model classes, $P(M_j|D)$. The probability of model class M_j given data D can be written as (Bayes theorem):

$$P(M_j|D) = \frac{p(D|M_j)P(M_j)}{p(D)}\,, \tag{22}$$

where $p(D)$ is equal for all model classes, $P(M_j)$ is the probability of model class M_j ($j = 2, 3, ...$), and $\sum P(M_j) = 1$. If there is no preferable model class a priori, i.e., $P(M_j)$ is the same for all models, then, the only term that matters, and that must be ranked among the models, is the evidence (or marginal likelihood) $p(D|M_j)$, which can be expressed as

$$p(D|M_j) = \int_\Omega p(D|\boldsymbol{\phi}_j, M_j)p(\boldsymbol{\phi}_j|M_j)d\boldsymbol{\phi}_j\,, \tag{23}$$

where Ω is the hypercube defined by the range of the parameters, $p(D|\boldsymbol{\phi}_j, M_j)$ is the likelihood function and $p(\boldsymbol{\phi}_j|M_j)$ is the prior probability density function (pdf) of the parameters, as stated before. The integral given by Eq. (23) might be computed numerically, using Monte Carlo integration, for instance.

RESULTS

The experimental data were obtained from the work developed by Moreira and Nunes (2013). An adhesive based on silane modified polymer, characterized by hyperelastic behavior, was used to manufacture samples for simple and pure shear tests.

Identification results

The support of each prior random variable is shown in table 6. With previous information obtained from Moreira and Nunes (2013), we defined the range of μ by adding and subtracting 80% of its mean (wich is about 0.5); thus $\mu = 0.5 \pm 0.4$ for all models. And, since $C_1 = \mu/2$, we defined $C_1 = 0.25 \pm 0.2$. Also using previous information obtained from Moreira and Nunes (2013), we limited the range of α-Ogden from 0 to 2, while the range of α-Lopez-Pamies was chosen from 0.5 to 2. The range of the other parameters are more dificult to determine. Since μ_2 of Ogden and Lopez-Pamies models represent a second term of the series, we limited its value up to 0.5. The range of C_2 and C_3, which for a rubber-like material has value around 100, was limited to [80,120]. For the variance of the prediction error, we considered the following reasoning. The maximum distance among the measurements was about 0.16. If we consider $\pm\sigma_N$ (about 70% confidence region), we have $\sigma_N = 0.08$. Finally, to accommodate an important error in the modeling, we multiply this value by four, such that $\sigma_N = 0.32$ ($\sigma_N^2 \approx 0.10$), therefore, we limited the support of var to [0.01,0.10].

Figure 1 illustrates a typical Markov Chain. Two thousands samples (n_s) were good enough for the chain convergence. The acceptance rate was kept in the range of 30% to 50% for all models, and 600 burn-in samples were not included in the analysis.

For illustrative purposes, Fig. 2 shows the prior probability density function (pdf) together with the updated (posterior) pdf of two parameters. Figure 2(a) is related to parameter μ_1 of the Lopez-Pamies (1 term) model for pure shear. The posterior range is significantly smaller than the range of the prior pdf. Figure 2(b) is related to parameter μ_1 of the Ogden (2 terms) model for simple shear. The posterior range is only a little smaller than the range of the prior pdf. A peaked pdf, Fig. 2(a), means that the response is sensitive to this parameter. The opposite happens when the distribution is very spread, Fig. 2(b), i.e., the response is not so sensitive to this parameter.

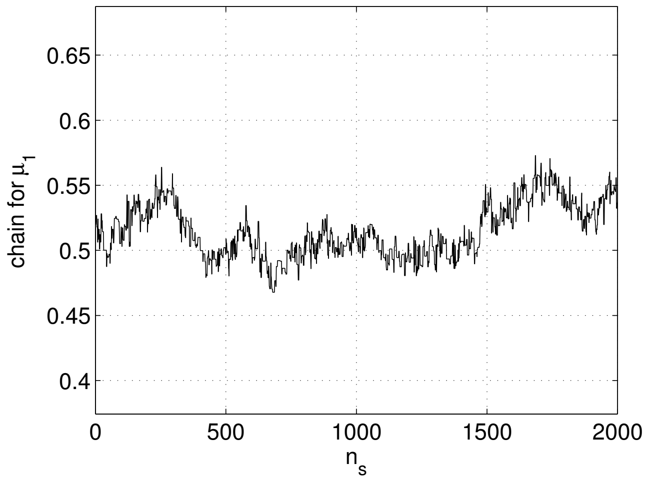


Figure 1 – Markov chain for μ_1 Lopez-Pamies (1 term) model.

Tables 7 and 8 show the identified values (MAP) and the standard deviation of the prior and the posterior pdf for each parameter. It can be observed that the standard deviation of the posterior pdf is smaller than the standard deviation of the prior pdf. The experimental data give additional information that help to localize the most probable values of the parameters, and to decrease the initial uncertainties (entropy) among them.

There are some parameters that are well correlated, and others that have no correlation whatsoever. For example, Figure 3(a) shows that the parameters μ and J_m of the Gent model for pure shear case are weakly correlated (results randomly distributed), and Fig. 3(b) shows that the parameters μ_1 and α_1, of the Ogden model (1 term) for simple shear are negatively correlated (aligned with negative slope). For the purpose of the present application, we are not so interested in the correlation information, but it might be of some value to better understand the connections among the parameters of the physical model.

Analyzing the curve fitting (see details in the next section), it was observed that all models for the pure shear case present good results. On the other hand, for the simple shear case, not all models exhibit a good agreement with experiments. For instance, take a look at Fig. 4, where experimental data and the 98% confidence region for simple shear Gent model are plotted. It is clear that Gent model is not able to predict the available experimental tests.

Further in this paper it will be concluded that Mooney-Rivlin model is the best one for the pure shear experimental tests available and Ogden-2terms is the best model for simple shear experimental tests available. For this reason, the prediction of these models, $\sigma(\lambda, \mathbf{p}_j)$, are shown in Figs. 5 and 6. The figures show the prediction of the two models using the identified parameters, together with the experimental data and the 98% confidence region.

Ranking by computing the error

In this section the models will be ranked taken into account curve fitting. The error is defined by the following equation

$$Error^{(M_j)} = \sum_{i=1}^{n_{exp}} ||\sigma^{M_j} - \sigma^{exp-i}||^2 \,, \tag{24}$$

where n_{exp} is the number of experiments, σ^{exp-i} is the i-th pure/simple shear experimental curve, and σ^{M_j} is the prediction of constitutive model M_j for pure/simple shear, obtained with the identified parameters.

Table 1 and 2 show the values of the errors for all the constitutive models for both tests (pure shear and simple

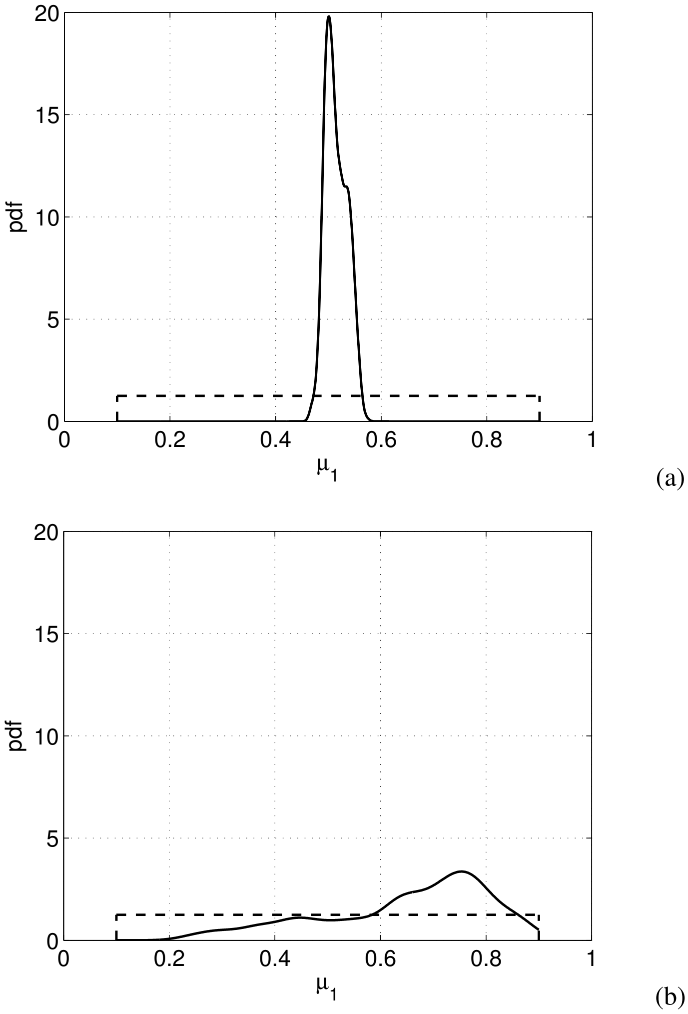


Figure 2 – Prior (dashed line) and posterior (solid line) probability density functions related to parameter μ_1, (a) Lopez-Pamies (1 term) model for pure shear and (b) Ogden (2 terms) for simple shear.

shear). It also shows the initial shear modulus (μ) for each model. All the constitutive models for pure shear yielded low errors, and presented a reasonable value for the initial shear modulus (μ), which is about 0.52 MPa; see Table 1. On the other hand, Table 2, shows that four constitutive models for simple shear (Lopez-Pamies, Ogden) are excellent (errors below 0.2), whether three (Mooney-Rivlin, Gent and Yeoh) presented large errors, i.e, they were not able to represent the available experimental data. In addition, the three models with large errors yielded unreasonable values for the initial shear modulus. It is not a surprise that the Mooney-Rivlin constitutive model would give poor results for the simple shear test, since the model is linear and the experimental curve is clearly nonlinear. Also, Yeoh and Fleming (1997) pointed out that the Yeoh's model is not suitable to describe the behavior of simple shear at small strain.

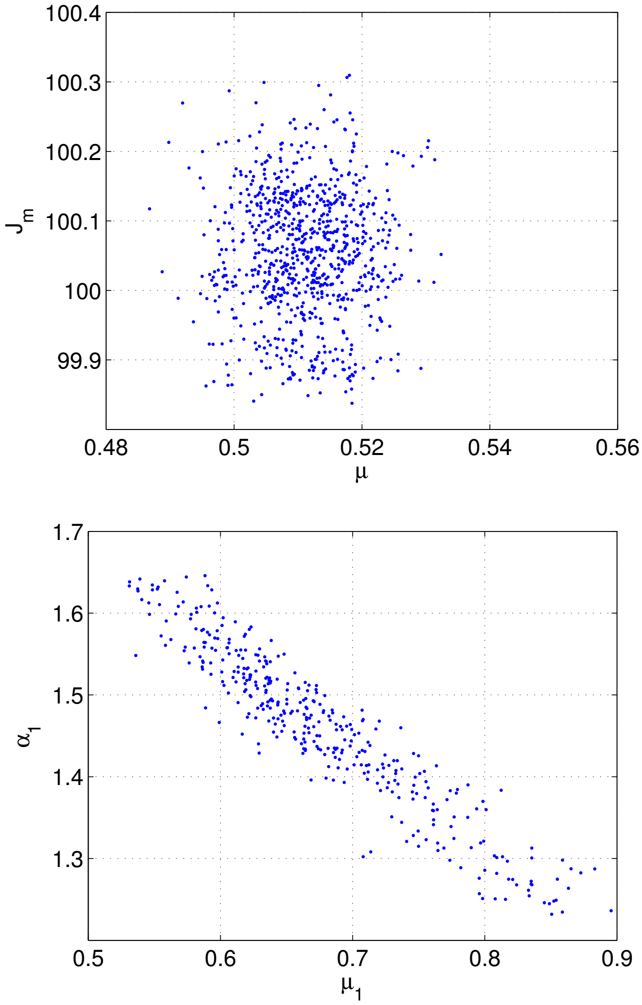
Finally, Table 3 shows the ranking obtained ordering the models from lower to higher errors. The best constitutive model for pure shear is Lopez-Pamies (2 terms) and the best constitutive model for simple shear is Ogden (2 terms) together with Lopez-Pamies (2 terms), considering the available data.

Ranking using the Bayesian framework

Table 4 shows the results obtained for $P(D|M)$ (see Eq. 23), which is computed by Monte Carlo simulations. Although the Monte Carlo method is not the most effective approach to compute this integral, the computational time for all models is so low, that this strategy could be used in the present application without compromising the results.

The models are ranked from higher to lower values of $P(D|M)$, which means that models with higher probabilities, given the experimental data, are better ranked; see Eq. 22. The results are less intuitive because not only the curve fitting is taken into account, but also prior and posterior probability density functions of the parameters play an important role.

Table 5 shows the ranking results of hyperelastic models for pure shear and simple shear cases. The results are quite different from the ones presented in the last section. The Mooney-Rivlin model presented the best result among the pure shear constitutive models. It seems that the other models were penalized for the extra parameters. The Ogden and Lopez-



(a)

(b)

Figure 3 – Correlation between (a) μ and J_m of Gent pure shear model and (b) μ_1 and α_1, Ogden model (1 term) for simple shear.

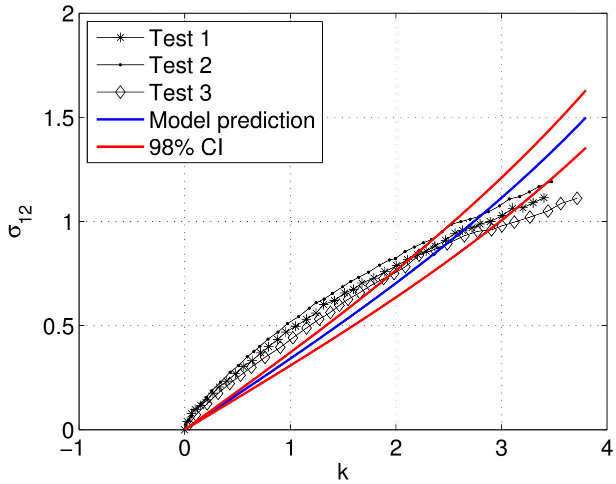
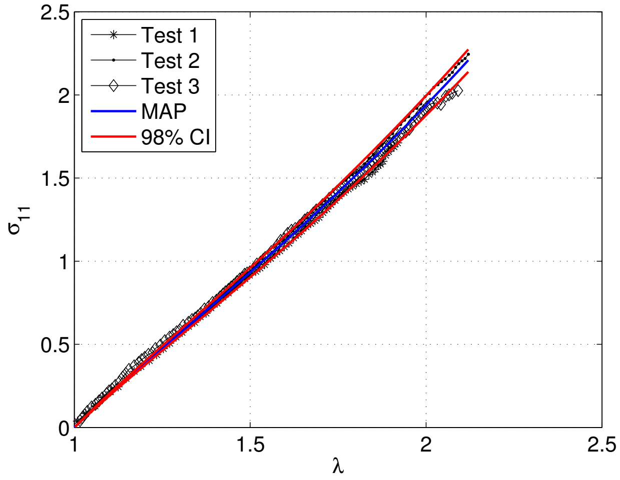
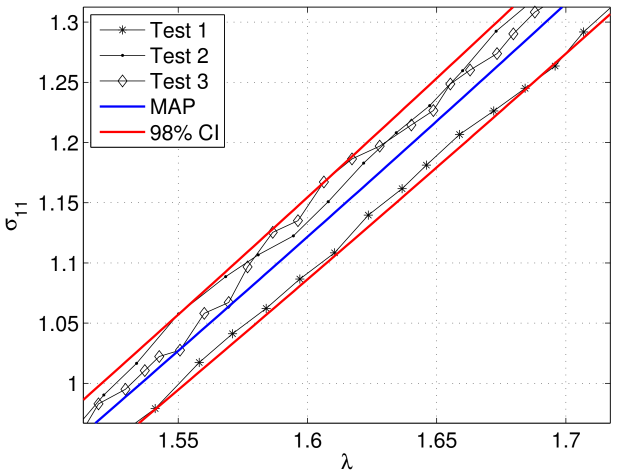


Figure 4 – Experimental data together with the model prediction (using the identified parameters) and the 98% confidence region of the Gent model for simple shear.

Pamies models for pure shear presented good results, then comes the Gent model, and later the Yeoh model, which was not able to describe the experimental data available. Concerning the simple shear constitutive models, the three models (Mooney-Rivlin, Gent and Yeoh) that presented high errors, also presented low $P(D|M)$, as expected. Three constitutive



(a)



(b)

Figure 5 – (a) Experimental data together with the model prediction (using the identified parameters) and the 98% confidence region of the Mooney-Rivlin model for pure shear and (b) zoom image.

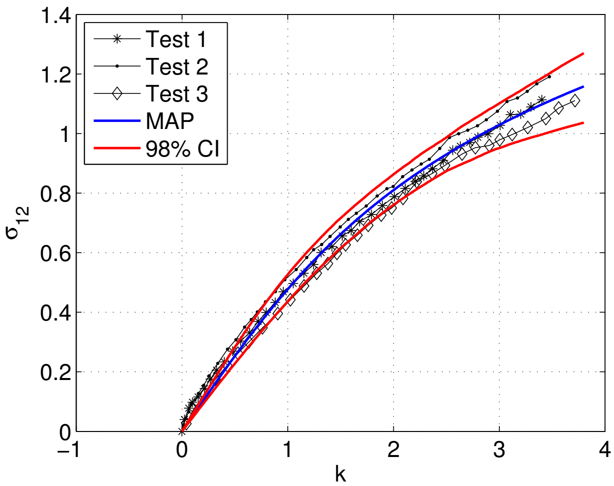


Figure 6 – Experimental data together with the model prediction (using the identified parameters) and the 98% confidence region of the 2-terms Ogden model for simple shear.

models for simple shear gave excellent results (Ogden and Lopez-Pamies-1term), and one (Lopez-Pamies-2terms) gave good results.

Figures 5 and 6 illustrate the predictions of the best models for pure and simple shear, i.e., Mooney-Rivlin and 2-terms

Model	Error	Initial Shear Modulus (μ, MPa)
Mooney-Rivlin	0.2377	0.517
Yeoh	0.2588	0.510
Ogden (1 term)	0.2315	0.520
Ogden (2 terms)	0.2314	0.523
Lopez-Pamies (1 term)	0.2434	0.526
Lopez-Pamies (2 terms)	0.2309	0.524
Gent	0.2915	0.510

Table 1 – Pure shear error results.

Model	Error	Initial Shear Modulus (μ, MPa)
Mooney-Rivlin	1.0275	0.370
Yeoh	1.5247	0.400
Ogden (1 term)	0.1621	0.517
Ogden (2 terms)	0.1579	0.517
Lopez-Pamies (1 term)	0.1610	0.520
Lopez-Pamies (2 terms)	0.1579	0.520
Gent	1.6007	0.338

Table 2 – Simple shear error results.

Ranking using Error	**Pure shear**	**Simple shear**
1st	Lopez-Pamies (2 terms)	Lopez-Pamies (2 terms) and Ogden (2 terms)
2nd	Ogden(2 term)	–
3rd	Ogden (1 term)	Lopez-Pamies (1 term)
4th	Mooney-Rivlin	Ogden (1 term)
5th	Lopez-Pamies (1 term)	Mooney-Rivlin
6th	Yeoh	Yeoh
7th	Gent	Gent

Table 3 – Ranking the models using the error measure.

Ogden models, respectively. The figures show also the experimental data and the 98% confidence limits.

Note that, to rank the models using the Bayesian criteria, curve fitting is not the only ingredient. The dispersion of the prior and posterior probability density functions of the parameters plays also an important role in model ranking.

Bayesian Model Selection	**Pure shear**	**Simple shear**
Model	$P(D\|M)$	$P(D\|M)$
Mooney-Rivlin	51.9×10^{-3}	0.8×10^{-3}
Yeoh	$< 1 \times 10^{-15}$	$< 1 \times 10^{-15}$
Ogden (1 term)	2.4×10^{-3}	12.4×10^{-3}
Ogden (2 terms)	2.0×10^{-3}	20.2×10^{-3}
Lopez-Pamies (1 term)	2.5×10^{-3}	10.1×10^{-3}
Lopez-Pamies (2 terms)	3.3×10^{-3}	1.4×10^{-3}
Gent	0.8×10^{-3}	7.0×10^{-7}

Table 4 – Probability of the data be explained by the different models for pure shear and simple shear.

Bayesian Model Selection	**Pure shear model**	**Simple shear model**
1st	Mooney-Rivlin	Ogden (2 terms)
2nd	Lopez-Pamies (2 terms)	Ogden (1 term)
3rd	Lopez-Pamies (1 term)	Lopez-Pamies (1 term)
4th	Ogden (1 terms)	Lopez-Pamies (2 terms)
5th	Ogden (2 term)	Mooney-Rivlin
6th	Gent	Gent
7th	Yeoh	Yeoh

Table 5 – Ranking the models using the Bayesian model selection strategy.

CONCLUDING REMARKS

For pure shear condition, all models demonstrated excellent agreement with the experimental data, while Mooney-Rivlin, Gent and Yeoh models were unsuitable for describing the available experimental simple shear. This discrepancy, as expected, can be attributed to linear stress-strain response of models. According to these results, the best models were Lopez-Pamies (2 terms) and Ogden model (2 terms) for both simple and pure shear. However, analyzing only curve fitting, it is not easy to select the best model for a given data.

It should be kept in mind that the best model must be able to describe complex behavior taking into account a small number of physical parameters. In this way, the proposed methodology based on Bayesian model selection strategy emerges as a powerful tool for ranking a set of models. With this approach, more information than just mean squared error is provided for estimating parameters. As a closing remark, taking into account the analyzed models, the best model was 2-terms Ogden model.

ACKNOWLEDGMENTS

The authors would like to thank the Brazilian agencies CNPq, CAPES and FAPERJ for the financial support.

REFERENCES

Acharjee S., and Zabaras N., 2007, A non-intrusive stochastic Galerkin approach for modeling uncertainty propagation in deformation processes, Computers and Structures, Vol.85(5-6), pp. 244–254.

Swanson S.R., Christensen L.W., and Ensign M., 1985, Large deformation finite element calculations for slightly compressible hyperelastic materials, Rubber Chem Technol , Vol. 21(1-2), pp. 81–88.

Markmann G., and Verron E., 2006, Comparison of hyperelastic models for rubber-like materials, Rubber Chemistry and Technology, Vol. 73(3), pp. 504–23.

Mooney M., 1944, Stress-strain curves of rubbers in simple shear, European Polymer Journal, Vol. 35, pp. 23–26.

Rivlin R.S., 1948, Large Elastic Deformations of Isotropic Materials. I. Fundamental Concepts, Phil. Trans. R. Soc. Lond. A , Vol. 240, pp. 459–490.

Yeoh O.H., 1993, Some forms of the strain energy function for rubber, Rubber Chem Technol , Vol. 66(5), pp. 754–71.

Gent A.N., 1996, A new constitutive relation for rubber, Rubber Chem Technol , Vol. 69(1), pp. 59–61.

Ogden R.W., 1972, Large deformation isotropic elasticity on the correlation of theory and experiment for incompressible rubberlike solids, Proc R Soc London A Math Phys Sci , Vol. 326, pp. 565–84.

Lopez-Pamies O., 2010, A new I1-based hyperelastic model for rubber elastic materials, C. R. Mecanique , Vol. 338, pp. 3–11.

Moreira D.C, and Nunes, L.C.S., 2013, Comparison of simple and pure shear for an incompressible isotropic hyperelastic material under large deformation, Polym. Test , Vol 32, pp. 240–248.

Nunes L.C.S, and Moreira D.C., 2013, Simple shear under large deformation: Experimental and theoretical analyses, Euro. Jour. Mech. A/Solids , Vol; 42, pp. 315–322.

Beck J.L., and Katafygiotis L.S., 1998, Updating models and their uncertainties. I: Bayesian statistical framework, Journal of Engineering Mechanics, Vol. 134(4), pp. 455–461.

Kaipio J., and Somersalo E., 2005, Statistical and Computational Inverse Problem, Ed. Springer.

Gamerman D., 2002, Markov Chain Monte Carlo: Stochastic Simulation for Bayesian Inference, Ed. Chapman and Hall/CRC, 2nd Edition.

Jaynes E., 1957, Information theory and statistical mechanics, The Physical Review , Vol. 106(4), pp. 1620–1630.

Jaynes E., 1957, Information theory and statistical mechanics II, The Physical Review , Vol. 108, pp. 171–190.

Kapur J.N., 1990, Maximum Entropy Models in Science and Engineering, Ed. Wiley-Blackwell.

Yeoh97 Yeoh O.H, and Fleming P.D., 1997, A New Attempt to Reconcile the Statistical and Phenomenological Theories of Rubber Elasticity, Journal of Polymer Science: Part B: Polymer Physics , Vol. 35, pp. 1919–1931.

RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.

TABLES WITH RESULTS

Model	Parameter	Range
Mooney-Rivlin	μ	[0.1,0.9]
Yeoh	C_1	[0.05,0.45]
	C_2	[0.0,0.5]
	C_3	[0.0,0.5]
Ogden (1 term)	μ_1	[0.1,0.9]
	α_1	[0,2]
Ogden (2 terms)	μ_1	[0.1,0.9]
	α_1	[0,2]
	μ_2	[0.0,0.5]
	α_2	[0,2]
Lopez-Pamies (1 term)	μ_1	[0.1,0.9]
	α_1	[0.5,2.0]
Lopez-Pamies (2 terms)	μ_1	[0.1,0.9]
	α_1	[0.5,2.0]
	μ_2	[0.0,0.5]
	α_2	[0.5,2.0]
Gent	μ	[0.1,0.9]
	J_m	[80,120]
Prediction error	var	[0.01,0.10]

Table 6 – Range of the parameters.

Model	Parameter	Id. Param.	Std$_{prior}$	Std$_{posterior}$
Mooney-Rivlin	μ	0.52	0.2309	0.0066
	var	0.099	0.0260	0.0159
Yeoh	C_1	0.26	0.1155	0.0039
	C_2	3×10^{-4}	0.1443	0.0012
	C_3	1×10^{-4}	0.1443	0.003
	var	0.047	0.0260	0.0032
Ogden (1 term)	μ_1	0.53	0.2309	0.0329
	α_1	1.98	0.5773	0.0783
	var	0.100	0.0260	0.0185
Ogden (2 terms)	μ_1	0.52	0.2309	0.0708
	α_1	1.97	0.5773	0.1189
	μ_2	0.17	0.1443	0.1124
	α_2	0.22	0.5773	0.1255
	var	0.094	0.0260	0.0147
Lopez-Pamies (1 term)	μ_1	0.53	0.2309	0.0206
	α_1	0.95	0.4330	0.0855
	var	0.100	0.0260	0.0170
Lopez-Pamies (2 terms)	μ_1	0.51	0.2309	0.1197
	α_1	0.98	0.4330	0.1970
	μ_2	0.01	0.1443	0.1213
	α_2	0.57	0.4330	0.1138
	var	0.096	0.0260	0.0157
Gent	μ	0.51	0.2309	0.0071
	J_m	100	11.547	0.0914
	var	0.099	0.0260	0.0156

Table 7 – Identified parameters (MAP) and standard deviation (Std) for the pure shear models.

Model	Parameter	Id. Param.	Std$_{prior}$	Std$_{posterior}$
Mooney-Rivlin	μ	0.37	0.2309	0.0138
	var	0.100	0.0260	0.0109
Yeoh	C_1	0.20	0.1155	0.0001
	C_2	6×10^{-6}	0.1443	0.0001
	C_3	1×10^{-6}	0.1443	0.0001
	var	0.050	0.0260	0.0001
Ogden (1 term)	μ_1	0.76	0.2309	0.0788
	α_1	1.36	0.5773	0.0974
	var	0.098	0.0260	0.0164
Ogden (2 terms)	μ_1	0.76	0.2309	0.1584
	α_1	1.36	0.5773	0.2450
	μ_2	0.001	0.1443	0.1344
	α_2	0.001	0.5773	0.6194
	var	0.005	0.0260	0.0241
Lopez-Pamies (1 term)	μ_1	0.52	0.2309	0.0506
	α_1	0.69	0.4330	0.0819
	var	0.098	0.0260	0.0166
Lopez-Pamies (2 terms)	μ_1	0.44	0.2309	0.1124
	α_1	0.62	0.4330	0.2224
	μ_2	0.08	0.1443	0.0996
	α_2	0.99	0.4330	0.1224
	var	0.099	0.0260	0.0165
Gent	μ	0.34	0.2309	0.0122
	J_m	100	11.547	0.1191
	var	0.100	0.0260	0.0079

Table 8 – Identified parameters (MAP) and standard deviation (Std) for the simple shear models.