

A NEW APPROACH FOR SOLVING THE NAVIER-STOKES EQUATIONS FOR COMPRESSIBLE FLOWS USING FOURIER COLLOCATION PSEUDO-SPECTRAL METHOD

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Abstract. The purpose of the present work is to expose a new methodology for solving the Navier-Stokes equations for compressible flows by using the pseudo-spectral Fourier collocation method, in the particular case of an isothermal and two-dimensional flow. Due to high values associated with the density gradient, spectral methods show several problems when solving this type of flow. Compressible flows are very important in several applications, such as turbomachinery and aerodynamics, and due to its importance, new methods for solving these equations are still being developed. Therefore, any contribution for this field is crucial for the development of new CFD methods.

Keywords: Navier-Stokes equations, Compressible flow, Fourier collocation pseudo-spectral, CFD.

1. INTRODUCTION

The study of fluids and its dynamics, known as Fluid Mechanics, has great importance in many science fields, such as physics, medicine and engineering. Due to its wide range of applications, it is currently very useful for industrial, research and development applications. Besides the development of numerical and computational methods for solving these problems, in particular the CFD (acronym which stands for Computational Fluid Dynamics), the simulation of compressible flows are often a challenge, due to the presence of shock waves and sharp density gradients (Ferziger et al., 2002). In particular, several limitations are found when solving these flows using high-order methods.

In the present work, the authors propose the use of the pseudo-spectral Fourier collocation method, for a particular case of an two-dimensional, isothermal, compressible flow of an ideal gas. This method presents high accuracy, high-order numerical convergence and a relatively low computational cost, when compared with other high-order methods (Canuto et al., 2006). Preliminary results are shown, and future perspectives include solving more complex cases, such as three-dimensional flows and energy transport conditions.

2. MATHEMATICAL MODEL

Consider a compressible flow of an ideal, isothermal gas, in a periodic domain. The mathematical model for this flow can be represented by the following equations (Ferziger and Peric, 2002):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = f(\mathbf{r}, t) \quad (1)$$

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla P + \nabla \cdot \left[\mu(\nabla \mathbf{v} + \nabla \mathbf{v}^T) - \frac{2}{3} \mu(\nabla \cdot \mathbf{v}) \bar{\bar{\delta}} \right] + \mathbf{g}(\mathbf{r}, t) \quad (2)$$

$$P = \rho R T \quad (3)$$

Equation (1) is the continuity equation, and Eq. (2) represents the balance of linear momentum in its compressible form. The terms f and $\mathbf{g}$ represent source terms. Equation (3) is the ideal gas law, which holds for the present case. Using the fact that this is an isothermal problem, for a continuous pressure and density field, from Eq. (3), the pressure gradient can be written as

$$\nabla P = R T \nabla \rho \quad (4)$$

The pressure gradient term, ∇P , appears in both Eq. (2) and Eq. (4). It is proposed to substitute the pressure gradient term in the Navier-Stokes equations by the proportional specific mass gradient term, $R T \nabla \rho$. Thus, one can rewrite these equations as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\tilde{\mathbf{v}}) = f(\mathbf{r}, t) \quad (5)$$

$$\frac{\partial \tilde{\mathbf{v}}}{\partial t} = -R T \nabla \rho + \overline{\text{RHS}}(\tilde{\mathbf{v}}, \mathbf{v}) + \mathbf{g}(\mathbf{r}, t) \quad (6)$$

where $\tilde{\mathbf{v}} = \rho \mathbf{v}$, and $\overline{\text{RHS}} = \nabla \cdot \left[\mu(\nabla \mathbf{v} + \nabla \mathbf{v}^T) - \frac{2}{3} \mu(\nabla \cdot \mathbf{v}) \bar{\delta} \right] - \nabla \cdot (\rho \mathbf{v} \mathbf{v})$. Using the definition of Fourier transform and its properties (Canuto et al., 2006), one obtains the continuity and Navier-Stokes equations transformed forward to the Fourier space ($\mathcal{F}$), as follows:

$$\frac{\partial \hat{\rho}}{\partial t} = -i \mathbf{k} \cdot \hat{\tilde{\mathbf{v}}} + \hat{f} \quad (7)$$

$$\frac{\partial \hat{\tilde{\mathbf{v}}}}{\partial t} = -R T i \mathbf{k} \hat{\rho} + \overline{\text{RHS}}(\hat{\tilde{\mathbf{v}}}, \mathbf{v}) + \hat{\mathbf{g}} \quad (8)$$

It is important to note that Eqs. (7) and (8) are coupled. Moreover, Eq. (8) does not have any pressure term. It can be considered a numerical advantage, since there is no need of a projection pressure-recovery technique during the post-processing, as proceeded by Canuto et al. (2006), Mariano et al. (2010) and Villela (2011) when simulating incompressible flows.

3. NUMERICAL METHOD

3.1. Discretization

In order to solve the problem, Eq. (7) and Eq. (8) can now be explicitly discretized. Using the Euler scheme for the time derivative, yields:

$$\hat{\rho}^{n+1} = \hat{\rho}^n - \Delta t (i \mathbf{k} \cdot \hat{\tilde{\mathbf{v}}}^n + \hat{f}^n) \quad (9)$$

$$\hat{\tilde{\mathbf{v}}}^{n+1} = \hat{\tilde{\mathbf{v}}}^n + \Delta t (-R T i \mathbf{k} \hat{\rho}^n + \overline{\text{RHS}}(\hat{\tilde{\mathbf{v}}}^n, \mathbf{v})^n + \hat{\mathbf{g}}^n) \quad (10)$$

From Eq. (9) and Eq. (10), it is now possible to find $\hat{\rho}^{n+1}$ and $\hat{\tilde{\mathbf{v}}}^{n+1}$. Thus, since $\tilde{\mathbf{v}} = \rho \mathbf{v} \Rightarrow \mathbf{v} = \tilde{\mathbf{v}}/\rho$, when $\rho \neq 0$ in its entire domain. Note that this relation is only valid for the physical space. However, it is possible to recover the velocity field $\mathbf{v}$ (in the physical space) at the instant $n + 1$ by using the inverse Fourier transform ($\mathcal{F}^{-1}$):

$$\mathbf{v}^{n+1} = \frac{\mathcal{F}^{-1}(\hat{\tilde{\mathbf{v}}}^{n+1})}{\mathcal{F}^{-1}(\hat{\rho}^{n+1})} \quad (11)$$

The iterative procedure can now march to the next time step. Alternatively, one can utilize a more accurate explicit scheme to evaluate the time derivative. In the present work, the authors tested Euler and the 2nd and 4th order Runge-Kutta schemes.

3.2. Error verification

For the sake of simplicity, as a preliminary test of this mathematical model, the authors decided to choose a two-dimensional case. The source terms were established aiming the use of the method of manufactured solutions (MMS) (Salari et al., 2000). Therefore, the source terms f and $\mathbf{g}$ are evaluated from the following manufactured solutions:

$$u = \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{2\pi y}{L}\right) \left(1 - e^{-\frac{\alpha t}{L^2}}\right) \quad (12)$$

$$v = \cos\left(\frac{2\pi x}{L}\right) \sin\left(\frac{2\pi y}{L}\right) \left(1 - e^{-\frac{\alpha t}{L^2}}\right) \quad (13)$$

$$\rho = \sin\left(\frac{2\pi x}{L}\right) \sin\left(\frac{2\pi y}{L}\right) \left(1 - e^{-\frac{\alpha t}{L^2}}\right) + C \quad (14)$$

where L and α are the length of the periodic domain and a diffusive-like coefficient, respectively. From the MMS perspective, these parameters are arbitrary. The positive constant C in Eq. (14) plays an important role, since it ensures that ρ never becomes zero in the whole periodic domain. This is essentially important when evaluating Eq. (11).

Hence, the results of Eqs. (9) and (10) can be compared with the analytical results of Eqs. (12), (13) and (14).

4. RESULTS

As a preliminary test, a simulation of 10 [s] was performed. The L2 norm between the numerical and analytical specific mass was calculated, and results for different temporal schemes are shown in Fig. (1). The red, green and blue lines represent the temporal advancement schemes Euler, 2nd and 4th order Runge-Kutta, respectively. It was observed that, using a 4th order Runge-Kutta scheme, the method reaches errors around 10^{-15} , i.e., closely to a machine precision level of error. Figure (2) shows the numerical result of the component u of the velocity field at time $t = 10$ [s].

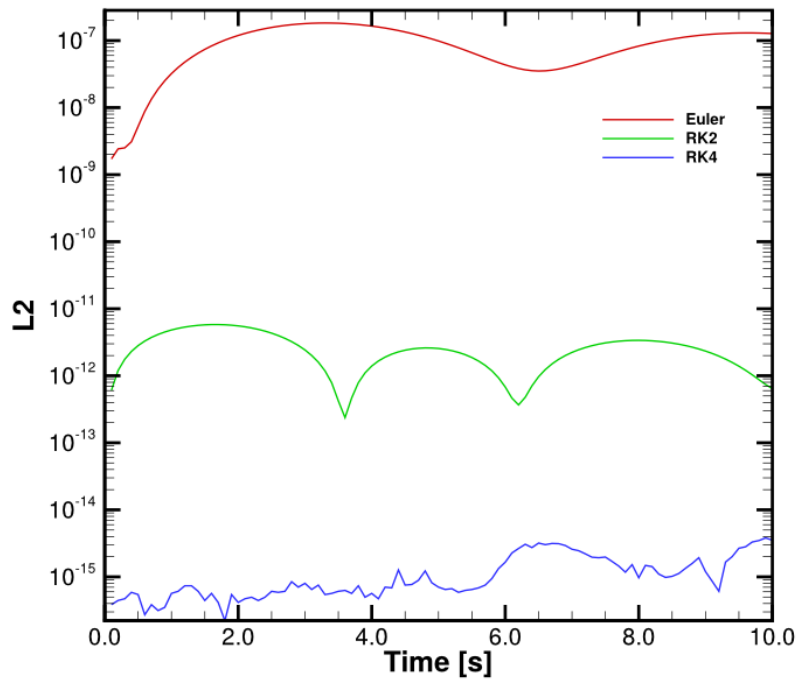


Figure 1. L2 norm of the density ρ for three different temporal advancement schemes.

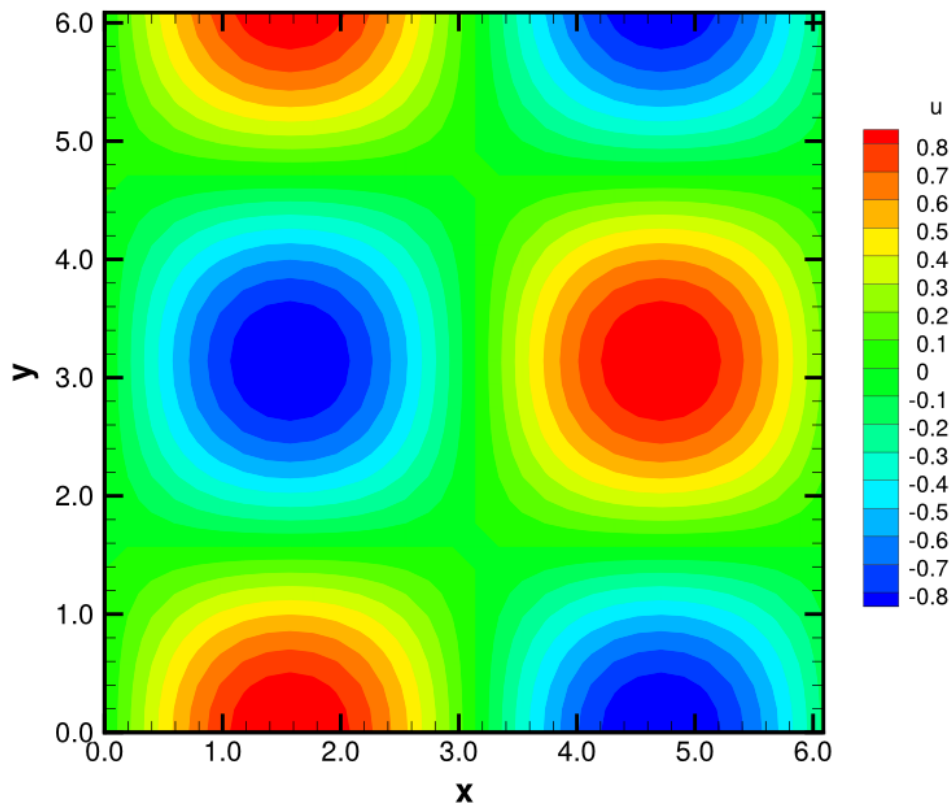


Figure 2. u -velocity field at $t = 10$ [s].

5. CONCLUSIONS

In the present work, two-dimensional numerical simulations of a compressible flow using the pseudo-spectral Fourier collocation method were presented. Preliminary results show that this method reaches errors on the order of 10^{-15} . Future efforts include extend the simulations for three-dimensional flows, and allow temperature gradients.

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