INPE – National Institute for Space Research
São José dos Campos – SP – Brazil – May 16-20, 2016

CONTROL OF ANOMALOUS TRANSPORT AND STICKINESS IN HAMILTONIAN SYSTEMS

Taline Suellen Krüger[1], *Paulo Paneque Galuzio*[1], *Thiago de Lima Prado*[1], *Ricardo Luiz Viana*[1], *José Danilo Szezech Jr.*[2], *Sergio Roberto Lopes*[1]

[1]Departamento de Física, Universidade Federal do Paraná, Curitiba, Brasil, email: lopes@fisica.ufpr.br
[2]Departamento de MatemÃątica, Universidade Estadual de Ponta Grossa, Ponta Grossa, Brasil

Abstract: We study how hyperbolic and nonhyperbolic regions in the neighborhood of a resonant island perform an important role allowing or forbidding stickiness phenomenon around islands in conservative systems. The vicinity of the island is composed of nonhyperbolic areas that almost prevent the trajectory to visit the island edge. For some parameters, there are tiny channels embedded in the nonhyperbolic area that are associated to hyperbolic fixed points localized in the neighborhood of the islands. Such channels allow the trajectory to be injected in the inner portion of the vicinity. When the trajectory crosses the barrier imposed by the nonhyperbolic regions, it spends a long time to abandon the surrounding of the island, since the barrier also prevents the trajectory to escape from the neighborhood of the island. In this scenario the nonhyperbolic structures are the responsible for the stickiness phenomena, and more than that, the strength of the sticky effect. We reveal that those properties of the phase space allow us to manipulate the existence of extreme events (and the transport associated to it) responsible for the non equilibrium fluctuation of the system.

keywords: stickiness phenomenon, anomalous transport, extreme events

1. INTRODUCTION

Recent works have shown that for a large class of systems, including those presenting mixed phase space, the transport can not be treated considering just ergodic theory or random phase approximation [1]. One class of those systems are low dimensional Hamiltonian systems presenting nonuniform phase space composed by regular and chaotic regions. The interface between these regions is not a smooth surface and the dynamics near the edge between chaotic and regular regions is complex and not well understood. The complexity comes mainly due to the presence of stickiness in the boundaries of islands [2]. Stickiness forces a trajectory injected into the boundary to stay near the boundary for long periods of time, leaving the system to present Poincaré recurrence time distributions displaying algebraically decay for long times rather than a exponential decay as expected for a normal transport system [1, 3–7]. In such a situation these systems are characterized as out of equilibrium.

We study how hyperbolic and non hyperbolic regions on the neighborhood of a resonant island perform an important role *to establish the presence and more than that, the strength of the sticky effect*. We show that the sticky effect is associated with the presence of injection channels related to the crossing of stable and unstable manifolds of hyperbolic fixed points in the vicinity of a island, allowing the trajectories to shift between sticky and nonsticky portions of the phase space. The effectiveness of such channels to capture trajectories to the sticky area is closely related to the degree of hyperbolicity of the close surrounding of hyperbolic fixed points located in the neighborhood of an island. Finally we make use of the presence of the hyperbolic channels to avoid (control) extreme recurrence events and the anomalous transport associated to it, resulting from a trajectory injection into the sticky area.

Here we study how hyperbolic and non hyperbolic regions on the neighborhood of a resonant island perform an important role *to establish the presence and more than that, the strength of the sticky effect*. We show that the sticky effect is associated with the presence of injection channels related to the crossing of stable and unstable manifolds of hyperbolic fixed points in the vicinity of a island, allowing the trajectories to shift between sticky and nonsticky portions of the phase space. The effectiveness of such channels to capture trajectories to the sticky area is closely related to the degree of hyperbolicity of the close surrounding of hyperbolic fixed points located in the neighborhood of an island. Finally we make use of the presence of the hyperbolic channels to avoid (control) extreme recurrence events and the anomalous transport associated to it, resulting from a trajectory injection into the sticky area.

2. THE MODEL

We characterize a hyperbolic region of the phase space $\mathcal{S}$ as an ensemble for which the tangent phase space splits continuously into stable (SM) and an unstable (UM) manifolds. SM and UM are invariant under the system dynamics: infinitesimal displacements in the stable (unstable) direction suffer exponentially decay as time goes forward (backward) [8]. It is required that the angles between the stable and unstable directions to be uniformly bounded away from zero. So, to quantify the degree of non hyperbolicity related to the phenomena we describe, consider an initial condition (p_0, x_0) and an unit vector $\boldsymbol{v}$, whose temporal evolution is given by $\boldsymbol{v}_{n+1} = \mathcal{J}(p_n, x_n)\boldsymbol{v}_n/||\mathcal{J}(p_n, x_n)\boldsymbol{v}_n||$, where $\mathcal{J}(p_n, x_n)$ is the Jacobian matrix of the map. For n large enough, $\boldsymbol{v}$ is parallel to the Lyapunov vector $\boldsymbol{u}(p, x)$ associated to the maximum Lyapunov exponent λ_u of the map orbit starting by (p_0, x_0). A backward iteration of the same orbit gives us a new vector $\boldsymbol{v}_n$ that is parallel to the direction $\boldsymbol{s}(p, x)$, the Lyapunov vector associated to the minimum Lyapunov exponent λ_s [9]. For regions where $\lambda_s < 0 < \lambda_u$ the vectors $\boldsymbol{u}(p, x)$ and $\boldsymbol{s}(p, x)$ are tangent to the UM and SM, respectively, of a point (p, x).

The (non)hyperbolic degree of a region $\mathcal{S}$ can be studied computing the local angles between the two manifolds

$$\theta(p, x) = \cos^{-1}(|\boldsymbol{u} \cdot \boldsymbol{s}|), \tag{1}$$

for $(p, x) \in \mathcal{S}$ [10]. $\theta(p, x) \sim 0$ denotes tangency between UM and SM at (p, x) [10].

Chaotic orbits of two dimensional mappings are often non hyperbolic since the SM and UM are tangent in infinitely many points. Here we consider a periodically kicked rotor subjected to a harmonic potential - the Chirikov-Taylor map [11], whose dynamics is two dimensional. The dynamics of a periodically kicked rotor is described in a periodic phase space $[-\pi, \pi) \times [-\pi, \pi)$, whose discrete-time variables p_n and x_n are respectively the momentum and the angular position of the rotor just after the nth kick, with the dynamics

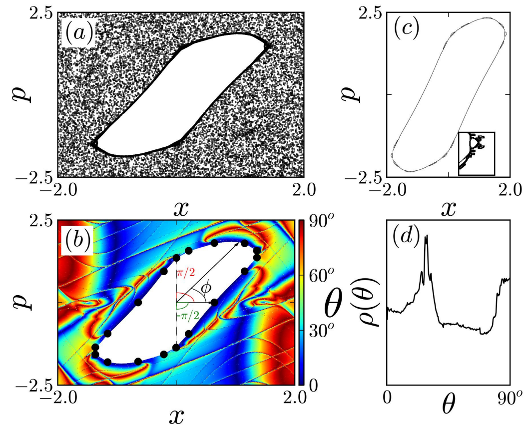

Figure 1 – (a) Phase space for the kicked rotor map, the denser areas near the island result from the sticky effect due to the time the trajectory remains near the edge of the island. (b) Phase space distribution of angle between unstable and stable manifold, dark (blue) tones mean strong non hyperbolicity. (c) An example of the edge detection algorithm used to compute the vicinity of the island (magnified in the inset). (d) PDF of the angle between stable and unstable manifolds, $\rho(\theta)$.

given by

$$\begin{aligned} p_{n+1} &= p_n + K\sin(x_n), \bmod 2\pi & (2)\\ x_{n+1} &= x_n + p_{n+1}, \bmod 2\pi & (3) \end{aligned}$$

where K is related to the kick strength.

Figure 1-a presents a portion of the phase space for a typical trajectory of the Chirikov-Taylor map for $K = 3.0$. To characterize the hyperbolicity of the surrounding areas of the island Fig. 1-b displays the local angle between stable and unstable manifolds (θ), Eq. (1), near the resonant island. The major part of the vicinities of the island is composed by strong nonhyperbolic areas, represented by dark blue areas in Fig. 1-b (tangencies between UM and SM). A small fraction of the vicinity is observed in red – yellow – green tones and corresponds to angles greater than 30^0 and responsible for the weak hyperbolic part of the vicinities. The role of both areas will be clear later on in the text. In Fig 1-b we also define the angle $\phi = \arctan(p/x)$, defined in the interval $[-\pi/2, +\pi/2]$, so each point in the vicinity of the island can be identified by a single number. The $9+9\,(\mathrm{symmetric})$ hyperbolic fixed points present in the neighborhood of the main island are identified as black bullets in Fig. 1-b. The vicinity of the island is shown in Fig 1-c that displays the result of our algorithm for edge island detection. Figure 1-d displays the probability distribution function (PDF) of the angles between SM and UM $\rho(\theta)$, so $\rho(\theta)d\theta$ represents the probability to find a angle between θ and $\theta + d\theta$ in the phase space ensemble displayed in Fig. 1-b. The large plateau for small θ angles reflects the strong nonhyperbolic character of the region.

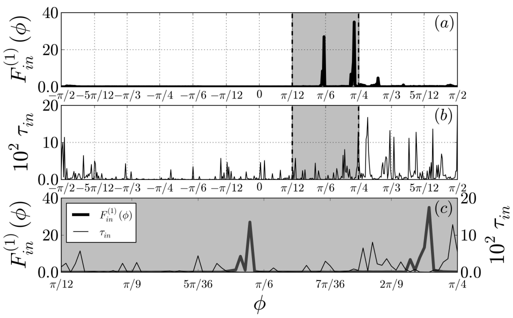

Figure 2 – (a) PDF of income angle in the vicinity of a island of the Chirikov-Taylor map. (b) Average time spent in the sticky area as a function of the injected angle.(c) Magnification of panel (a) and (b) near the maxima of $F_{in}^{(1)}(\phi)$

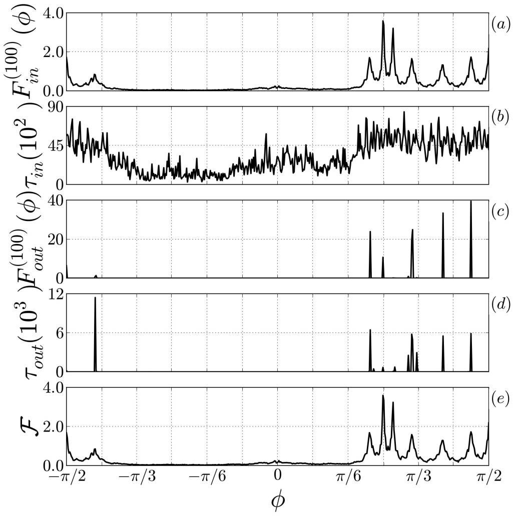

Figure 3 – (a) PDF of income angle into the vicinity of the main island of the Chirikov map. (b) Average spent time in the sticky area as a function of the injected angle. (c) PDF of outcome angle from the vicinity of the main island of the Chirikov map. (d) Average spent time in the sticky area as a function of the ejected angle. (e) Probability distribution of injected angle, subjected to the condition that the time of stickiness to be greater than 1000 iterations

3. RESULTS

The topological properties of the phase space in the vicinities of an island play an important role in the sticky mechanism. To explore the relation between topological characteristics of the phase space and the way trajectories visit an island vicinity and stick to it, we define the probability density function $F_{in}^{(1)}(\phi)$ for trajectories injected into the vicinities of the island considering the vicinity computed by our algorithm (Fig. 1-c). $F_{in}^{(1)}(\phi)d\phi$ is the probability that a typical chaotic trajectory to visit the vicinity of the island through a angle between ϕ and $\phi + d\phi$. Figure 2 displays $F_{in}^{(1)}(\phi)$, panel (a) as well as the average time the trajectory remains in the vicinity of the island when injected by a particular angle, panel (b). In panel (c) we magnify the gray region of panel (a) and (b). Although the major part of the trajectories are injected by just few angles inside the red-yellow-green tones regions around the island in Fig. 1-a (weak hiperbolic regions), these specific trajectories spend, in average, a short time mapping the sticky area. Trajectories are easily injected into the sticky areas by weak hyperbolic areas surrounding the island but almost all of them are also easily ejected from it. Those trajectories do not contribute for the phenomena of stickiness and do not make any substantial changes in the Poincaré recurrence time for the dynamics.

To distinguish sticky trajectories from those that just reach the island edge and leave it quickly, we compute the probability density function $F_{in}^{(100)}(\phi)$, of trajectories injected into the sticky area by a specific angle *considering that once a trajectory reaches the vicinity of the island, it remains mapping the vicinity for at least* 100 *iterations*. We identify such trajectories as sticky ones. The result is plotted in Fig. 3-a. Almost all sticky trajectories are injected in very specific intervals of angles. Each intervals are directly related to the angular location of the chain of periodic points (set as black bullets in Fig. 1-c). So trajectories are injected into the sticky area when they are tangent to the stable manifold of the 18^{th} order fixed points chain located in the vicinity of the main island. All trajectories that are not tangent enough to the stable manifold of the hyperbolic fixed point do not cross the tiny hyperbolic channel produced by the crossing between stable and unstable manifolds of the fixed point and are not injected into the stick area.

Considering those trajectories injected into the stickiness and remaining mapping the edge for at least 100 iterations, Fig. 3-b presents the average time they spend near the island (sticky trajectories) as a function of the injected angle. In Fig. 3-c we graph the PDF of sticky trajectories as a function of the ejected angle $F_{out}^{100}(\phi)$. It is clear the almost discrete nature of the distribution. All ejected trajectories follow the unstable manifold of the hyperbolic fixed points moving along a narrow channel departing from the fixed point. Fig. 3-d presents the average time the trajectories stay in the sticky region (at least 100 iterations inside the sticky area) as a function of the outcome angle. From Fig. 3-d we can conclude that sticky trajectories are ejected only by few angle intervals $\phi_{out} = \phi(\max(\tau_{out}))$. Therefore, we are able to calculate the probability $\mathcal{F}(\phi)d\phi$ that a given trajectory will enter the sticky region considering only trajectories that leave these region through the angle ϕ_{out}. The result is plotted in Fig. 3-e. The great similarity between $\mathcal{F}$ and $F_{in}^{(100)}$ suggests that the or previous conclusions are consistent. Therefore, we can argue that the local maximum of $F_{in}^{(100)}$ represent the sticky angles, *i. e.*, the angles that once a trajectory is injected from one of them, there is a great probability that this trajectory turn to be stuck to the island. These maxima correspond to the same region where are located hyperbolic points, confirming the hypotheses that these points provide

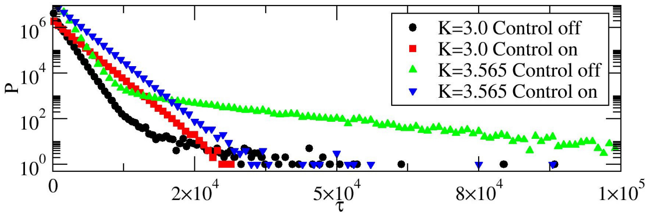

Figure 4 – (color online) Recurrence time for the Chirikov-Taylor map with and without control.

channel for a typical trajectory to enter in the sticky region. Figs. 3-a and 3-e show us that both figures are almost identical, a strong suggestion that all trajectories leave the sticky regions by the hyperbolic channels departing from the hyperbolic fixed points.

Such properties of the phase space allow us to manipulate the non equilibrium fluctuation of the system. To show that it is possible to control the non equilibrium fluctuations that arise due to the presence of stickiness, we track the position of the trajectory and once it maps a circle of radius 0.003 centered in one of the 18 fixed points we perturb the trajectory, so a possible crossing of the channel and consequent stick of the trajectory is avoided. Numerically, we perform a restart of the trajectory outside the injection channel. Results for the Poincaré recurrence time for the system with and without the control mechanism for two values of K, $K = 3.0$ and $K = 3.565$ (a large stickiness case) are plotted in Fig. 4. For large recurrence time, a strong fluctuation of the exponential law is observed, in fact the distribution has a power law decay as a result of the sticky phenomena in the recurrence time. For the controlled case, almost all fluctuation for long recurrence time is absent, corroborating the idea that all nonequilibrium fluctuation in the system is now absent since the stickiness is avoided. Additionally observe that the exponential rate for both K values is the same, supporting the idea that the behavior of the system is now completely diffusive independently of the K value. All results here are presented for two values of K but similar results are obtained for other values of the nonlinear parameter.

4. CONCLUSIONS

In conclusion, we describe a mechanism to suppress the effect of stickiness based on the knowledge of the nonhyperbolic structure of the edge of a island of a Hamiltonian system. The effectiveness of an island edge to stick trajectories is directly related to the degree of hyperbolicity of small areas surrounding fixed points around the island. Monitoring those areas of the phase space, it is possible to generate a complete diffusive processes without (almost) any influence of the large recurrence time due to the stickiness phenomena. Since very large sticky times make the system to present extreme events in the dynamics, it possible to affirm that, once under control, we can turn the out-of equilibrium system into a in equilibrium one.

ACKNOWLEDGMENTS

This work has financial support from CNPq and CAPES (Brazilian agencies). Computer simulations were performed at LCPAD/UFPR supported by FINEP (CT-INFRA).

References

[1] G.M. Zaslavsky, Phys. Rep., **371**, 461 (2002).

[2] J. D. Meiss and E. Ott, Phys. Rev. Lett. **55**, 2741 (1985).

[3] E. G. Altmann and T. Tel, Phys. Rev. Lett. **100**, 174101 (2008).

[4] J. M. Seoane, M. A.F. Sanjuán, Phys. Letts. A **372**, 110 (2008).

[5] E. G. Altmann and A. Endler, Phys. Rev. Lett. **105**, 244102 (2010).

[6] Y.-C. Lai and T. Tél *Transient Chaos Complex Dynamics on Finite-Time Scales* (Springer, New York. 2010).

[7] O. Alus, S. Fishman J. D. Meiss, arXiv:1410.7648v1 [nlin.CD], 2014.

[8] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer, New York, 2002).

[9] C. Grebogi *et al.*, Phys. Rev. Lett., **65**, 1527 (1990).

[10] F. Ginelli *et al.*, Phys. Rev. Lett., **99**, 130601 (2007).

[11] A. J. Lichtenberg, and M. A. Lieberman, M.A. *Regular and Chaotic Dynamics* (Springer, Berlin. 1992).