INPE – National Institute for Space Research
São José dos Campos – SP – Brazil – May 16-20, 2016

MHD EQUILIBRIA WITH GRAVITATIONAL FORCE IN SYMMETRIC SYSTEMS

R. L. Viana[1] *and* *Fabiane de Fatima Carvalho*[2]

[1]Departamento de Física, Curitiba, Brazil, rlv640@gmail.com
[2]Departamento de Física, Curitiba, Brazil, fabi.fc90@gmail.com

Abstract: A description of static equilibrium in symmetrical magnetohydrodynamics systems with one ignorable coordinate is presented. By add a gravitational potential in the general well know formulation via Grad-Shafranov equation, we investigate the influence of gravity in the magnetic field lines in some symmetric systems. We consider the isotherm and adiabatic cases.

keywords: magnetic field lines, gravitational effect.

1. INTRODUCTION

A simplified description of the static equilibrium in symmetric system with one ignorable coordinate can be made by the reduced set of ideal MHD equations. In our assumption there are tree external forces involved, the force of expansion due the pressure gradient, the Lorentz force and the gravitational force. The ideal MHD equations with the gravitational force in static equilibrium (where $\boldsymbol{v} = 0$) are:

$$\nabla p = \boldsymbol{J} \times \boldsymbol{B} - \rho \nabla \Phi, \tag{1}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}, \tag{2}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{3}$$

where p, $\boldsymbol{J}$, $\boldsymbol{B}$, Φ and ρ, are respectively the pressure, current density, magnetic field, gravitational potential and the mass density. An equation for static equilibrium can be deduced via Grad-Shafranov equation [1] [2] [3] including the gravitational field. We assume a spatially uniform gravitational potential. Since the sun is a natural fusion reactor this situation is particular useful to describe the static equilibrium in the solar surface, the solar corona [4].

- **Isotherm case**

The pressure satisfy the ideal gas equation:

$$p = \bar{R}T, \tag{4}$$

with $T = T(\Psi, \Phi)$, where Ψ is a transversal magnetic flux function. Differentiating about Φ and regard T as a constant yield the following expression for pressure:

$$p(\Psi, \Phi) = p_0(\Psi) e^{-(\Phi - \Phi_0)/\bar{R}T}. \tag{5}$$

Therefore the condition for static equilibrium in isotherm case must satisfy the following equation:

$$\Delta^* \Psi = -\mu_0 I D - \frac{1}{2}\mu_0^2 \frac{dI^2}{d\Psi} - \mu_0 g_{33} \frac{dp_0}{d\Psi} e^{-(\Phi - \Phi_0)/\bar{R}T}, \tag{6}$$

where $\Delta^* \Psi$ is the generalized Shafranov operator defined for a general curvilinear system,

$$\Delta^* \Psi \equiv \frac{g_{33}}{\sqrt{g}} \left[\frac{\partial}{\partial x^1} \frac{\sqrt{g}}{g_{33}} \left(g^{11} \frac{\partial \Psi}{\partial x^1} + g^{12} \frac{\partial \Psi}{\partial x^2} \right) + \right.$$
$$\left. + \frac{\partial}{\partial x^2} \frac{\sqrt{g}}{g_{33}} \left(g^{21} \frac{\partial \Psi}{\partial x^1} + g^{22} \frac{\partial \Psi}{\partial x^2} \right) \right]$$

and D is given by:

$$D = \frac{g_{33}}{\sqrt{g}}\left[\frac{\partial}{\partial x^2}\left(\frac{g_{31}}{g_33}\right) - \frac{\partial}{\partial x^1}\left(\frac{g_{32}}{g_33}\right)\right], \tag{7}$$

in orthogonal systems $D = 0$. Equation (6) can be solved by specifying the profiles of current, pressure and gravitational potential.

- **Adiabatic case**

The pressure now obeys the adiabatic condition,

$$p = K\rho^{\gamma} \tag{8}$$

with $\gamma = 5/3$. In this case the entropy is a constant, therefore $K = K(S)$ is also a constant since depends on the entropy. The static equilibrium in adiabatic case respect the following equation:

$$\Delta^*\Psi = -\mu_0 ID - \frac{1}{2}\mu_0^2\frac{dI^2}{d\Psi} - \mu_0 g_{33}\frac{dp_0}{d\Psi}\left(\frac{p(\Psi)}{K}\right)^{\frac{1}{\eta}-1} \times$$
$$\times\left[\left(\frac{p_0}{K}\right)^{1/\eta} - \frac{(\Phi-\Phi_0)}{\eta K}\right]^{\eta-1}, \tag{9}$$

the constant η is defined as:

$$\eta \equiv \frac{\gamma-1}{\gamma} = \frac{5}{2} \tag{10}$$

2. SOLUTION IN CYLINDRICAL COORDINATES

The contravariant coordinates in the cylindrical system are:

$$\begin{aligned} x^1 &= r, \\ x^2 &= \theta, \\ x^3 &= z, \end{aligned} \tag{11}$$

being z the ignorable coordinate. The metric tensor in this coordinate system is given by:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12}$$

Since the system is orthogonal $D = 0$. The gravitational field arise due a uniform linear mass density $\lambda = \frac{dm}{dz}$. By the Gauss law for gravity we have,

$$\nabla \cdot \boldsymbol{g} = -4\pi G\rho, \tag{13}$$

where $G = 6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2}$ is the gravitational constant and ρ is the mass bulk density. With $\boldsymbol{g} = -\nabla\Phi$, we get the Poisson's equation,

$$\nabla^2\Phi = 4\pi G\rho \tag{14}$$

which gives the following solution for the gravitational field,

$$\boldsymbol{g}(r) = \frac{-2G\lambda}{r}\hat{\boldsymbol{e}}_{\boldsymbol{r}} \tag{15}$$

2.1. Solution in isotherm case

For the isotherm case we regard the following profiles for pressure p and current I:

$$p_0 = A(\Psi - \Psi_0),$$

where A is a constant,

$$I^2 = I_0^2 = constant.$$

Thus eq.(6) becomes,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{\partial\Psi}{\partial r}\right) = -\mu_0 A\left(\frac{r_0}{r}\right)^{\xi}, \tag{16}$$

where $\xi = \frac{2G\lambda}{RT}$, we obtain the following solution,

$$\Psi(r) = \Psi_0 - A\frac{\mu_0 r_0^{\xi}}{2-\xi}(r^{2-\xi} - r_0^{2-\xi}) \tag{17}$$

2.2. Solution in adiabatic case

The profile current I is the same for isotherm case and for pressure p is:

$$p_0(\Psi) = K\Psi^{\eta}$$

Eq.(6) for adiabatic case is,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{\partial\Psi}{\partial r}\right) = -\mu_0 K\eta\Psi^{\eta-1}, \tag{18}$$

with the solution,

$$\Psi(r) = \left(\frac{-\mu_0 K\eta}{\alpha^2}\right)^{\frac{1}{2-\eta}} r^{\frac{2}{2-\eta}} \ln\left(\frac{r}{r_0}\right), \tag{19}$$

where $\alpha = \frac{2}{2-\eta}$. By relating the flux function Ψ with the components of the magnetic field we get the magnetic field lines equations for both cases which are for isotherm and adiabatic cases respectively:

$$\theta(r,z) = \frac{A}{(2-\xi)I_0}\left(\frac{r}{r_0}\right)^{\xi} z \tag{20}$$

$$\theta(r,z) = -\left[\frac{2}{(2-\eta)}\left(\frac{\mu_0 K\eta}{\alpha^2}\right)^{\frac{1}{2-\eta}} r^{\frac{\eta}{2-\eta}} + \frac{1}{\eta K}\frac{2G\lambda}{r_0}\right] z \tag{21}$$

In order to illustrate the magnetic field lines in isotherm case, how can seen in fig.1, we consider the exponent $\xi = 1$. The magnetic field lines for adiabatic case are shown fig.2. In fusion reactors with azimuthal symmetry the gravitational effect is negligible therefore the condition of static equilibrium in these systems produces magnetic field lines of helical path. However in figs.1,2 we see how gravity can distort the magnetic field lines if the gravitational effect is take into account in isotherm and adiabatic cases.

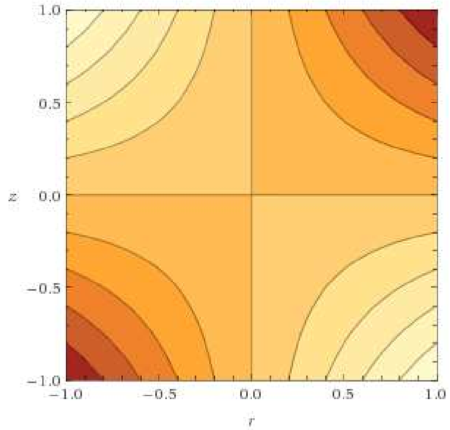

Figure 1 − Isotherm magnetic field lines for $\theta = constant$.

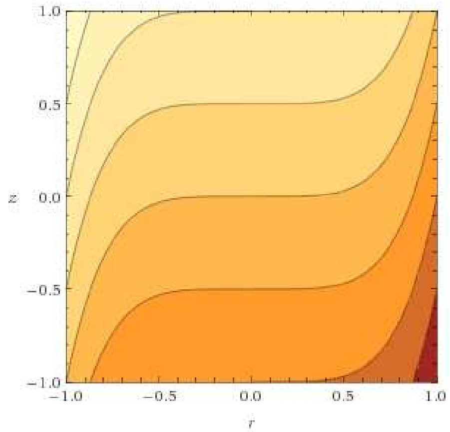

Figure 2 − Adiabatic magnetic field lines for $\theta = constant$.

3. CONCLUSION

The gravitational effect in the condition of static equilibrium in symmetrical magnetohydrodynamics systems with one ignorable coordinate like cylindrical system causes distortion of magnetic field lines.

ACKNOWLEDGMENTS

Thanks for the financial support of CAPES to the doctoral.

References

[1] Grad, H; Rubin, H., Hydromagnetic Equilibria and Force-free Fields. Journal of Nuclear Energy, Vol.7, n.8, pp.190-197, 1958.

[2] Shafranov, V.D., Plasma Equilibrium in a Magnetic Field, Reviews of Plasma Physics, Vol.2, pp.103, 1966.

[3] Kucink, M; Caldas, I.L. MHD equilibrium equation in symmetric systems, 2011.

[4] Shui Wang, S.T. Wu, A.H. Wang. Two-dimensional MHD equilibria in the solar corona. Chinese Astronomy and Astrophysics. Vol. 15, Issue 4, pp.470-479, 1991.