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LYAPUNOV SPECTRUM OF CHAOTIC MAPS WITH A COUPLING MEDIATED BY A DIFFUSING SUBSTANCE

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Abstract: We investigate the synchronization properties of networks of chaotic maps coupled through the influence of a rapidly diffusing substance, what induces a nonlocal coupling. The Lyapunov spectrum of the coupled map lattice is analytically obtained for chaotic maps with constant slope.

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1. INTRODUCTION

Nonlocal couplings consider the interaction of each site with all its neighbors, but take into account the lattice distance between them. Kuramoto has shown that such couplings occur among point oscillators mediated through a diffusing chemical substance [1]. In other words, each oscillator secrets a mediator substance with a rate proportional to the oscillator dynamics. This substance diffuses in the intersite medium and its local concentration also influences the dynamics of the oscillators. In the case of a very fast diffusion the coupling strength decreases with the lattice distance in an exponential way. This diffusion-mediated coupling model has been used to describe bursting synchronization in neuronal networks [2] and circadian rhythms in the suprachiasmatic nucleus [3].

The Lyapunov spectrum is a useful tool in the investigation of many dynamical properties of a coupled map lattice, since it yields information about the stability of the completely synchronized state. There are some techniques for the numerical obtention of Lyapunov exponents of a coupled map lattice. In some cases it is even possible to obtain analytically this spectrum, as we did in previous works dealing with power-law couplings [4, 5]. In this work we obtain the Lyapunov spectrum of coupled chaotic map lattices with non-local couplings obtained through the mediation of a chemical substance diffusing rapidly in the space. The calculation can be done analytically for maps with constant slope. This gives information about the regions of synchronization for such a coupled map lattice.

2. COUPLED MAP LATTICE

We consider a one-dimensional system whose state variable is denoted simply by $x_n^{(i)}$, and whose dynamics is governed by a map $x \mapsto f(x)$. We suppose that the dynamics in each site is affected by the local concentration of a chemical, denoted as $A(\vec{r}, n)$, through a coupling function g:

$$x_{n+1}^{(i)} = f(x_n^{(i)}) + g(A(\vec{r}_i, n)), \qquad (1)$$

where $\vec{r}$ denotes the position of the pointlike system in a d-dimensional Euclidean space and n is a discrete time. The chemical concentration satisfies a diffusion equation

$$\epsilon \frac{\partial A}{\partial n} = -\eta A + D\nabla^2 A + \sum_{j=1}^{N} h(x_n^{(j)})\delta(\vec{r} - \vec{r}_j), \qquad (2)$$

where $\epsilon \ll 1$ is a small parameter representing the fact that diffusion occurs in a timescale faster than the intrinsic period of individual sites, η is a phenomenological damping parameter (representing the chemical degradation of the mediating substance), and D is a diffusion coefficient.

The diffusion equation above has a source term h which depends on the state of individual sites located at $\vec{r}_i$, since each site is supposed to release a chemical with a rate depending on the current value of its own state variable. Assuming that the diffusion is very fast we may set $\epsilon \dot{A} = 0$ such that the concentration relaxes to a stationary value

$$A(\vec{r}_i) = \sum_{j=1}^{N} \sigma(\vec{r}_i - \vec{r}_j) h(x_n^{(j)}), \tag{3}$$

where σ is a Green function, which is the solution of $(\eta - D\nabla^2)\sigma = \delta(\vec{r}_i)$. If we consider a one-dimensional lattice then $G\sigma(r_{ij}) = \exp(-\gamma r_{ij})$, where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$, and $\gamma = \sqrt{\eta/D}$ is an inverse coupling length, and the constant C is determined from the normalization condition

$$\int dr \sigma(r) = 1. \tag{4}$$

There results

$$x_{n+1}^{(i)} = f(x_n^{(i)}) + \sum_{j=1}^{N} \sigma(\vec{r}_i - \vec{r}_j) g(h(x_n^{(j)})). \tag{5}$$

where we assume that g is a linear function of the state variable. Let us choose the following form for the latter

$$g(h(x_n^{(j)})) = \begin{cases} \varepsilon f(x_n^{(j)}), & \text{if } j \neq i, \\ -\varepsilon f(x_n^{(i)}), & \text{if } j = i, \end{cases} \tag{6}$$

which corresponds to the so-called future coupling, and where $\varepsilon > 0$ is a coupling strength.

Considering a one-dimensional lattice we have $x_n^{(i)} = x(z = i\Delta)$, with $i = 1, 2, \ldots N$ and $\Delta = 1$ for simplicity. Hence the non-locally coupled map lattice reads

$$x_{n+1}^{(i)} = (1-\varepsilon) f\left(x_n^{(i)}\right) + \varepsilon C \sum_{j=1}^{N} e^{-\gamma \Delta r_{ij}} f\left(x_n^{(j)}\right). \tag{7}$$

where the constant C is given by (4) as

$$C = \left[2 \sum_{j=1}^{N} e^{-\gamma \Delta r_{ij}} \right]^{-1}. \tag{8}$$

In numerical simulations of the spatio-temporal dynamics of the coupled map lattice above we consider a random initial condition pattern $x_0^{(i)}$ and periodic boundary conditions, such that

$$r_{ij} = \min_{\ell} |i - j + \ell N|. \tag{9}$$

Changing the summation index in the coupling term of (7) in a more symmetric form:

$$x_{n+1}^{(i)} = (1-\varepsilon) f\left(x_n^{(i)}\right) + \frac{\varepsilon}{\kappa(\gamma)} \sum_{s=1}^{N'} e^{-\gamma s} \left[f\left(x_n^{(i-s)}\right) + f\left(x_n^{(i+s)}\right) \right], \tag{10}$$

where the normalization factor has been written in the form

$$\kappa(\gamma) = 2 \sum_{s=1}^{N'} e^{-\gamma s}, \tag{11}$$

where $N' = (N-1)/2$ for N odd.

3. LYAPUNOV EXPONENTS

Let us write the coupled map lattice in the following general form

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n) \tag{12}$$

where $\mathbf{x}_n$ is the state vector. The corresponding tangent vector $\boldsymbol{\xi}_n$ satisfies the tangent map

$$\boldsymbol{\xi}_{n+1} = \mathbf{DF}(\mathbf{x}_n) \boldsymbol{\xi}_n, \tag{13}$$

where $\mathbf{DF}(\mathbf{x}_n) \equiv \mathbf{T}_n$ is the jacobian matrix of (12), with components

$$T_n^{(ij)} = \frac{\partial x_{n+1}^{(i)}}{\partial x_n^{(j)}}. \tag{14}$$

Starting from an initial value of $\boldsymbol{\xi}_0$ one iterates Eq. (13) n times to obtain

$$\boldsymbol{\xi}_n = \boldsymbol{\tau}_n \boldsymbol{\xi}_0, \tag{15}$$

where the ordered product of jacobian matrices is

$$\boldsymbol{\tau}_n = \mathbf{T}_{n-1} \mathbf{T}_{n-2} \ldots \mathbf{T}_1 \mathbf{T}_0. \tag{16}$$

Let the matrix

$$\hat{\boldsymbol{\Lambda}} = \lim_{n \to \infty} \left(\boldsymbol{\tau}_n^T \boldsymbol{\tau}_n \right)^{1/2n} \tag{17}$$

have eigenvalues $\{\Lambda_k\}_{k=1}^{N}$. The Lyapunov exponents of the coupled map lattice (12) are obtained as

$$\lambda_k = \ln \Lambda_k, \qquad (k = 1, 2, \ldots N). \tag{18}$$

If the Lyapunov exponents are ordered, the corresponding spectrum is denoted by $\{\tilde{\lambda}_k\}_{k=1}^{N}$.

For the coupled map lattice (10) the elements of the jacobian matrix are

$$T_n^{(ik)} = (1-\varepsilon) f'\left(x_n^{(i)}\right) \delta_{ik} + \frac{\varepsilon}{\kappa(\gamma)} \exp\left(-\gamma r_{ik}\right) f'\left(x_n^{(k)}\right) (1 - \delta_{ik}), \tag{19}$$

Hence the jacobian matrix (19) is

$$\mathbf{T}_n = \hat{\mathbf{B}} \mathbf{D}_n = \left[(1-\varepsilon)\mathbf{1} + \frac{\varepsilon}{\kappa(\gamma)} \mathbf{B} \right] \mathbf{D}_n, \tag{20}$$

where

$$D_n^{(ik)} = f'(x_n^{(i)})\delta_{ik}, \tag{21}$$

$$B_{ik} = e^{-\gamma r_{ik}}(1-\delta_{ik}), \tag{22}$$

$$\hat{B}^{(ik)} = (1-\varepsilon)\delta_{ik} + \frac{\varepsilon}{\kappa(\gamma)}B_{ik} \tag{23}$$

4. LYAPUNOV SPECTRUM OF BERNOULLI MAPS

Let x be a variable defined in the unit interval $[0,1)$. The so-called Bernoulli map is

$$x_{n+1} = f(x_n) = \beta x, \qquad (\mod 1), \tag{24}$$

which, for $\beta > 1$, is strongly chaotic (transitive). In fact, its Lyapunov exponent is $\lambda = \ln\beta > 0$.

Since $f'(x) = \beta$ there follows that

$$\mathbf{D}_n = \beta\mathbf{I}, \qquad \mathbf{T}_n = \beta\hat{\mathbf{B}}, \qquad \boldsymbol{\tau}_n = \beta^n\hat{\mathbf{B}}^n, \qquad \hat{\boldsymbol{\Lambda}} = \beta\hat{\mathbf{B}}.$$

If $\{b_k\}_{k=1}^N$ are the eigenvalues of $\mathbf{B}$, then for the matrix $\hat{\mathbf{B}}$ the corresponding eigenvalues are given by

$$\hat{b}_k = (1-\varepsilon) + \frac{\varepsilon}{\kappa(\gamma)}b_k, \qquad (k = 1, 2, \ldots N), \tag{25}$$

such that the eigenvalues of $\hat{\boldsymbol{\Lambda}}$ are $\Lambda_k = \beta\hat{b}_k$. Due to the periodic boundary conditions the matrix $\mathbf{B}$ is circulant and its diagonal elements are identically zero since such terms were ruled out when computing the coupling part of the jacobian matrix. A standard matrix calculation using Fourier transform yields

$$b_k = 2\sum_{m=1}^{N'} e^{-\gamma m}\cos\left(\frac{2\pi km}{N}\right), \qquad (k = 1, 2, \ldots N) \tag{26}$$

in terms of them the Lyapunov exponents of the coupled Bernoulli map lattice read

$$\lambda_k = \ln\beta + \ln\left|(1-\varepsilon) + \frac{\varepsilon}{\kappa(\gamma)}b_k\right|. \tag{27}$$

We have performed numerical simulations of the coupled map lattice (10) for the case $\gamma = 1.0$ and $N = 101$ coupled Bernoulli maps with $\beta = 1.9$. In Fig. 1 we plot the Lyapunov spectrum for different values of the coupling constant ε. It is clearly seen that the maximum Lyapunov exponent is $\ln\beta = 0.6418$, corresponding to the uncoupled maps (for this reason it is independent of ε). The spectrum of Lyapunov exponents is monotonic and its minimum decreases as ε increases. For $\varepsilon \lesssim 0.3$ all the exponents are positive, whereas part of them become negative as ε increases further.

5. DISCUSSION

The features observed in numerical simulations can be interpreted by considering the two limiting cases of the above expression. If γ goes to zero then $\kappa(0) = 2N' = N-1$ and we have a global (all-to-all) type of coupling

$$x_{n+1}^{(i)} = (1-\varepsilon)f\left(x_n^{(i)}\right) + \frac{\varepsilon}{N-1}\sum_{s=1, s\neq i}^{N} f\left(x_n^{(s)}\right). \tag{28}$$

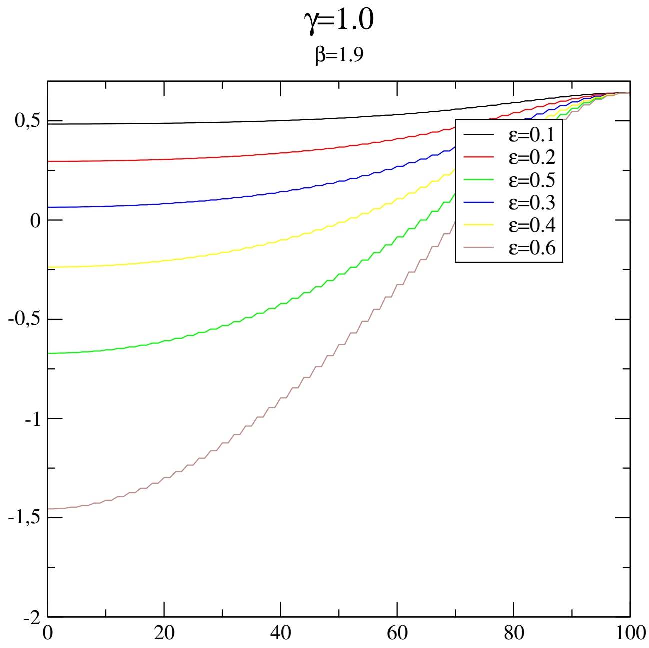

Figure 1 – Lyapunov spectrum of a one-dimensional lattice of $N = 101$ Bernoulli maps with $\beta = 1.9$ and chemical coupling with $\gamma = 1.0$ for various values of the coupling constant.

For $\gamma = 0$ Eq. (27) gives for the Lyapunov exponents

$$\lambda_k = \ln\beta + \ln\left|(1-\varepsilon) + \frac{2\varepsilon}{N-1}\sum_{m=1}^{N'}\cos\left(\frac{2\pi km}{N}\right)\right|. \tag{29}$$

Using Lagrange's identity for the sum of cosines there follows that

$$\lambda_k = \ln\beta + \ln\left|(1-\varepsilon) + \frac{2\varepsilon}{N-1}\left[-1 + \frac{\sin(k\pi)}{\sin\left(\frac{k\pi}{N}\right)}\right]\right|. \tag{30}$$

If $k \neq N$ we have the same exponents, a $(N-1)$-fold symmetry:

$$\lambda_k = \ln\beta + \ln\left|1 - \varepsilon\frac{N}{N-1}\right|, \tag{31}$$

whereas, if $k = N$, the exponent is simply $\lambda_N = \ln\beta$ [6].

Now, in the limit of large γ, the exponential factor in (10) decays very rapidly with the lattice distance ℓ, such that only the term with $\ell = 1$ contributes significantly to the summations, yielding $\kappa(\infty) = 2e^{-\gamma}$ and the coupling term takes into account only the nearest neighbors of a given site

$$x_{n+1}^{(i)} = (1-\varepsilon)f\left(x_n^{(i)}\right) + \frac{\varepsilon}{2}\left[f\left(x_n^{(i-1)}\right) + f\left(x_n^{(i+1)}\right)\right], \tag{32}$$

which is the well-known local (diffusive or Laplacian) coupling. Hence, depending on the value of the inverse coupling length we can pass continuously from a global to a local coupling type.

The $\gamma \to \infty$ case brings about the locally coupled lattice, for which the spectrum (27) yields

$$\lambda_k = \ln \beta + \ln \left| 1 - 2\varepsilon \sin^2 \left(\frac{\pi k}{N} \right) \right| . \quad (33)$$

in agreement with a previous result by Kaneko [7].

6. CONCLUSIONS

Analytical results for coupled map lattices are particularly rare since they represent dynamical systems with a large number of degrees of freedom. On the other hand, such results, when available, are important tools in the investigation of dynamical aspects of them, as the complete synchronization of chaotic orbits. In this work we considered a coupled map lattice with non-local coupling in the sense that a given map is influenced by all its neighbors in the network, the relative contribution being proportional to an exponentially decaying function of the spatial distance along the lattice. This coupling prescription can be physically interpreted as the result of a coupling mediated by a chemical substance which diffuses in the medium in which the pointlike systems are distributed. Such a coupling has many potentially important applications such as neuronal networks.

In this work we obtained analytically the Lyapunov spectrum of a non-locally coupled chaotic map lattice, where the uncoupled maps have constant slope (Bernoulli maps). The coupling prescription depends on a parameter which can be varied such that the coupling goes from global (all-to-all) to local (nearest-neighbor). The Lyapunov spectrum we obtained agrees with those previously known limiting cases. Our results can be used to investigate the transversal stability of the completely synchronized state of coupled chaotic map lattices.

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