INPE – National Institute for Space Research
São José dos Campos – SP – Brazil – May 16-20, 2016

EFFECT OF TURBULENT DIFFUSION ON PASSIVE SCALAR TRANSPORT INDUCED BY AN ISOLATED VORTEX

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Abstract: On the basis of an ellipsoidal vortex model and a Monte-Carlo-type diffusion simulation, we examine the flux and ensuing distribution of passive fluid particles through the boundary of an idealized geophysical vortex. Numerical calculations show that the diffusion significantly affects scalar spreading in the horizontal plane in the perturbed case while in the steady state the diffusion only induces linear growth according to a Gaussian distribution.

keywords: geophysical nonlinear dynamics, stochastic models, chaos and global nonlinear dynamics, diffusion, chaotic advection.

1. INTRODUCTION

Isolated vortices evolving in a sheared environment are often modeled by ellipsoidal vortices [1]. These vortices have been intensely studied over the last 30 years. However, the recent review [2] on the ellipsoidal vortex model evidences that there is still sizeable interest among researchers.

The model is derived such that fluid particles inside the vortex always move regularly with their trajectories coinciding with the stream-lines of the fixed vortex. However, for the fluid particles outside the vortex, that holds true only if the vortex does not change its form, thus generating a stationary velocity field. When moving periodically the vortex perturbs neighboring fluid inducing fluid particle chaotic advection.

Depending on the initial sizes of the ellipsoid's semi-axes, and its alignment relatively to the external strain, the ellipsoid is known to be able to oscillate, rotate slightly changing the sizes of its semi-axes, and infinitely elongate [3].

A significant restriction of the model is that there is no flux of fluid particles through the vortex's boundary. The boundary figures as a barrier ensuring a non-zero vorticity gradient and as a result precluding fluid particles to cross the boundary due to chaotic advection [4]. However, it is of interest how the interior of the vortex, which is often consisted of fluid with sufficiently different features such as salinity or temperature, can permeate through the boundary and spread in the exterior flow. A way to somehow overcome this restriction was introduced in the paper [5], where a low-scale diffusion process was implemented thus allowing fluid particles to jump through the boundary. In the paper, only the horizontal diffusion was considered. Here we extend the results of the previous one in the way of including into consideration the vertical component of diffusion.

2. PROBLEM FORMULATION

Let us consider the advection-diffusion equation

$$\left(\frac{\partial}{\partial t}+\mathbf{U}(\mathbf{r},t)\frac{\partial}{\partial \mathbf{r}}\right)q(\mathbf{r},t)=\left(\kappa\frac{\partial^2}{\partial x^2}+\kappa\frac{\partial^2}{\partial y^2}+\kappa_z\frac{\partial^2}{\partial z^2}\right)q(\mathbf{r},t),$$

$$q(\mathbf{r},0)=q_0(\mathbf{r}),$$

where the velocity field $\mathbf{U}(\mathbf{r},t)$ is the combined background deformation flow and flow induced by ellipsoidal vortex [1,3], $q(\mathbf{r},t)$ is the impurity concentration, κ is the horizontal diffusivity and κ_z is the vertical diffusivity. Now, we introduce a scalar field $\tilde{q}(\mathbf{r})$ complying with the equation,

$$\left(\frac{\partial}{\partial t}+\mathbf{U}(\mathbf{r},t)\frac{\partial}{\partial \mathbf{r}}\right)\tilde{q}(\mathbf{r},t)=-\boldsymbol{\alpha}(t)\frac{\partial\tilde{q}(\mathbf{r},t)}{\partial \mathbf{r}},\ \ \tilde{q}(\mathbf{r},0)=q_0(\mathbf{r}),$$

where $\boldsymbol{\alpha}(t)$ is a delta-correlated Gaussian process such as

$\langle \boldsymbol{\alpha}(t) \rangle = \mathbf{0}$, $\langle \alpha_i(t)\alpha_j(t') \rangle = 2\kappa\delta_{ij}\delta(t-t')$, i,j=1,2, $\langle \alpha_3(t)\alpha_j(t') \rangle = 2\kappa_z\delta_{3j}\delta(t-t')$, j=1,2,3 and δ_{ij} is the Kronecker delta, $\delta(t)$ is the Dirac function and t, t' are two instants in time. Taking advantage of the introduced function, one arrives at the averaged solution [5,6]

$$q(\mathbf{r},t) = \langle \tilde{q}(\mathbf{r},t) \rangle_{\boldsymbol{\alpha}}. \tag{1}$$

The random process $\boldsymbol{\alpha}(t)$ featuring in (1) is the modeled diffusion process depending on the time scale and diffusivity.

The equations governing a fluid particle trajectory subjected to the ellipsoid motion and diffusion impact ensue from (1)

$$\begin{aligned} u &= ex - \gamma y + \tilde{u}\cos\theta - \tilde{v}\sin\theta + \alpha_x, \\ v &= \gamma x - ey + \tilde{u}\sin\theta + \tilde{v}\cos\theta + \alpha_y, \\ w &= \alpha_z, \ \frac{d\tilde{q}(\mathbf{r}(t),t)}{dt} = 0, \end{aligned} \tag{2}$$

where the α_x, α_y terms correspond to the horizontal diffusivity κ, α_z complies with the vertical diffusivity κ_z, e, γ define the shear and rotational component of background flow and $\tilde{u}, \tilde{v}$ are the components of flow induced by an ellipsoid patch [1,3,5].

3. STEADY AND UNSTEADY VORTEX MOTION

It was shown in works [1,3] that the ellipsoid can move changing its form due to a linear shear. Hereafter we use the following values $e = 0.1, \gamma = 0$ and dimensionless ellipsoid vertical semi-axis $\tilde{c} = 1$. The values of horizontal ellipsoid semi-axes such as $\frac{a(0)}{b(0)} = 1.0551$ and its initial orientation $\theta(0) = \pi/4$ correspond to the case when the ellipsoid staying still thus creating a stationary velocity field in (2) for the $\kappa = \kappa_z = 0$. Figure 1 depicts the phase portrait of background and vortex induced flow

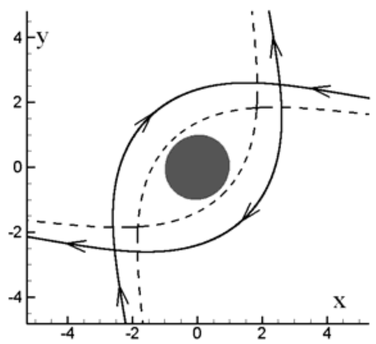
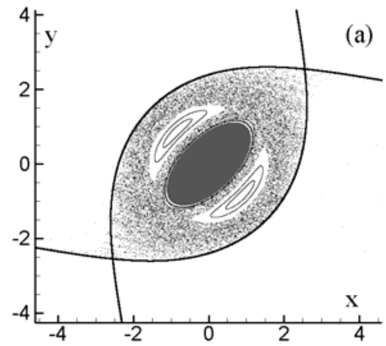


Figure 1 – Stream-line pattern. The filled central region is the stationary ellipsoid horizontal section at $z = 0$. The bold line is a separatrix at $z = 0$. The dashed line indicates the separatrix at depth $z = 1.5$.

If we take slightly different horizontal semi-axis the vortex starts moving with varying its horizontal axes and orientation periodically near steady values [3]. Thus, the vortex creates the unsteady velocity field. This variation induces a perturbation to fluid particle trajectories leading to chaotic mixing of passive markers.



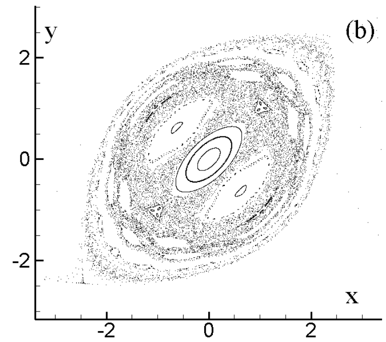


Figure 2 – Poincare sections of the perturbed state as the ellipsoid rotates varying its horizontal semi-axes. (a) $z = 0$, (b) $z = \tilde{c} = 1$ - the limiting point of the ellipsoid in the vertical direction.

4. MARKER DISPERSION DUE TO DIFFUSION

Now we take into account the diffusion. We distribute inside the ellipsoid some $6{\bullet}10^3$ scalars, such as there are 7 layers inside the ellipsoid with evenly distributed scalars in every layer. Then, we follow scalar distributions for 30 ellipsoid revolutions as both horizontal and vertical components of diffusion are at work. We carry out the same routine for $6{\bullet}10^3$ realizations of $\boldsymbol{\alpha}(t)$ stochastic process in order to smooth resulting scalar distributions due to assumed averaging.

4.1. Steady ellipsoid

Initially, we consider the steady state, as on figure 1. The ellipsoid semi-axes remain unchanged hence, there is no perturbation to exterior fluid particles. Then, when the diffusion is imposed, fluid particles are now able to redirect their paths from one stream-line to some other. Accordingly this leads to particle dispersion completely consistent with a Gaussian distribution. Figure 3 demonstrate the latter observation. The initial concentration of passive scalars within the ellipsoid is equal to the number of realizations. The figures indicate that because of the vertical component of diffusion the maximal value of the scalar concentration varies in space.

4.2. Unsteady ellipsoid motion

The ellipsoid changes its form and orientation in time at the perturbed state. Because of the varying lengths of the horizontal semi-axes, the ellipsoid stirs the neighboring fluid by periodically perturbing it thus causing exponential divergence of initially close fluid trajectories. This state is illustrated in figure 4. As the ellipsoid rotates, the phase space periodically shrinks and elongates, so to retain the self-similarity it is necessary to consider the dynamics only in the moments when the ellipsoid takes on its original form and orientation. The ellipsoid completes a full rotation and returns to its initial state in a $T_r = 1.89165$ time interval. Hence, 40-time interval equals near $21 \cdot T_r$. Figures 4 illustrate the concentration field after this time interval.

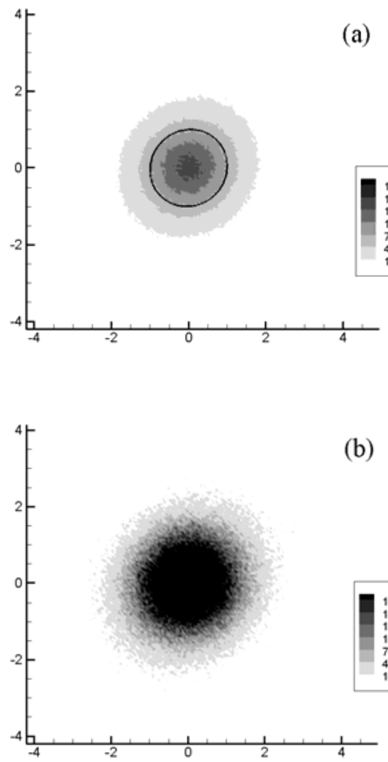


Figure 3 –Passive scalar distributions at the surface layer $z = 0$, after 40 dimensionless time intervals for steady state as on figure 1 from the initial concentration of $6 \cdot 10^3$ in each node as $\kappa = 0.01$. (a) $\kappa_z = 2\kappa$ the bold curve marks the steady ellipsoid boundary, (b) $\kappa_z = 0$.

4.3. Scalar probability density and dispersion

Now, we analyze in detail the concentration fields. We single out nested ellipsoidal rings of equal volume with the same ellipticity (1.0551 in the steady state, 2 in the perturbed state), and calculate the number of scalars that get into some of the ellipsoidal rings. As a result, we construct a distribution identifying the scalar concentration change in space $p = N_{\Delta V} / N$, where $N_{\Delta V}$ is the number of scalars inside an ellipsoidal ring of volume ΔV, N is the total number of scalars encompassed in total volume V. Initially, we had $p = 0.2$ inside the ellipsoid since we chose ΔV equal to $1/5$ of the initial volume, and $p = 0$ outside the ellipsoid.

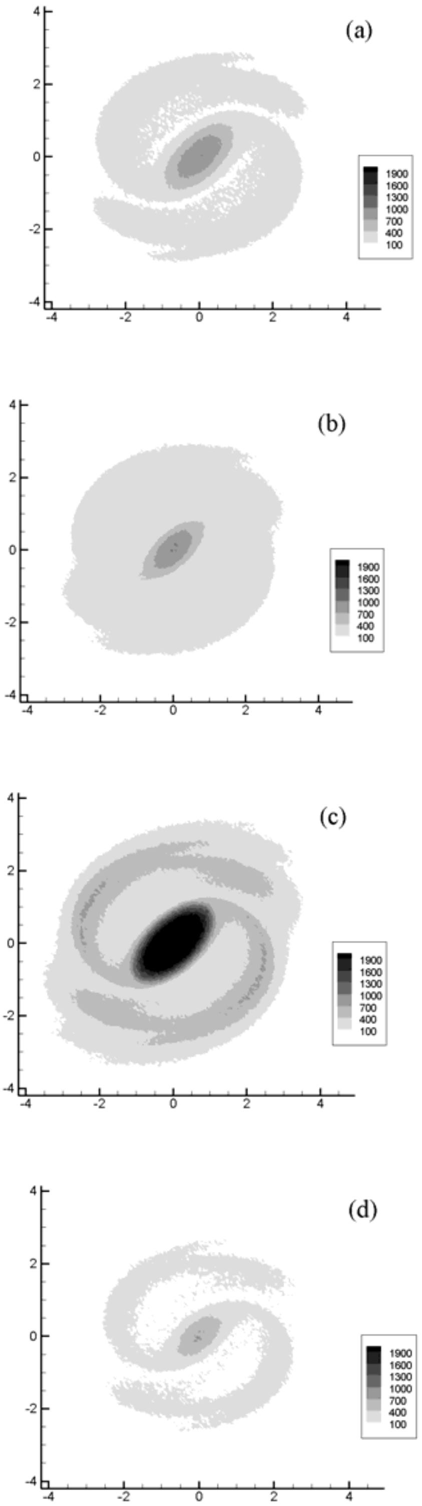


Figure 4 – Passive scalar distributions after 40 dimensionless time intervals for the perturbed state as $a(0)/b(0) = 2$, $\theta(0) = \pi/4$ from the initial concentration of $6 \cdot 10^3$ in each node as $\kappa = 0.01$. (a) $\kappa_z = 2\kappa$ - the surface layer, the bold curve marks the steady ellipsoid boundary, (b) $\kappa_z = 2\kappa$, $z = 1$, (c),(d) – the same as in (a),(b), respectively, except for that there is no vertical diffusion $\kappa_z = 0$.

Figure 5 illustrates the obtained density curves. Figures 5a,b correspond to the steady state with the vertical component of diffusion and without it, respectively. Accordingly, figures 5c,d correspond to the perturbed state with the vertical component of diffusion and without it, respectively. Each figure comprises four curves corresponding to 10, 20, 30 and 40-time intervals. Steady-state figures are clearly indicative of a Gaussian distribution outside the ellipsoid, whereas the perturbed ones demonstrate slightly different shapes.

5. CONCLUSION

In this paper, we analyzed the passive scalar dynamics in an ellipsoid vortex model evolving in a linear shear flow subjected to diffusion. Dealing with horizontal and vertical components of diffusion, we presented evidence that the vertical component of diffusion being several orders smaller than the horizontal one may be as much important and influential to the scalar transport and consequential general stability of vortex structures in such sheared media.

Comparing the steady and perturbed states, the ones where the ellipsoid stays still or periodically rotating, we demonstrated that the vertical component of diffusion sizably affects the scalar distribution in the perturbed state. That is because, in the perturbed state, the diffusion vertical component can force a scalar to move between the ellipsoid's horizontal sections which differ in area with depth.

ACKNOWLEDGMENTS

The reported study was partially supported by the RFBR research projects: 14-05-00017, 15-05-00103, 15-35-20105; and by the Ministry of Education and Science of Russian Federation, project: MK-3084.2015.1.

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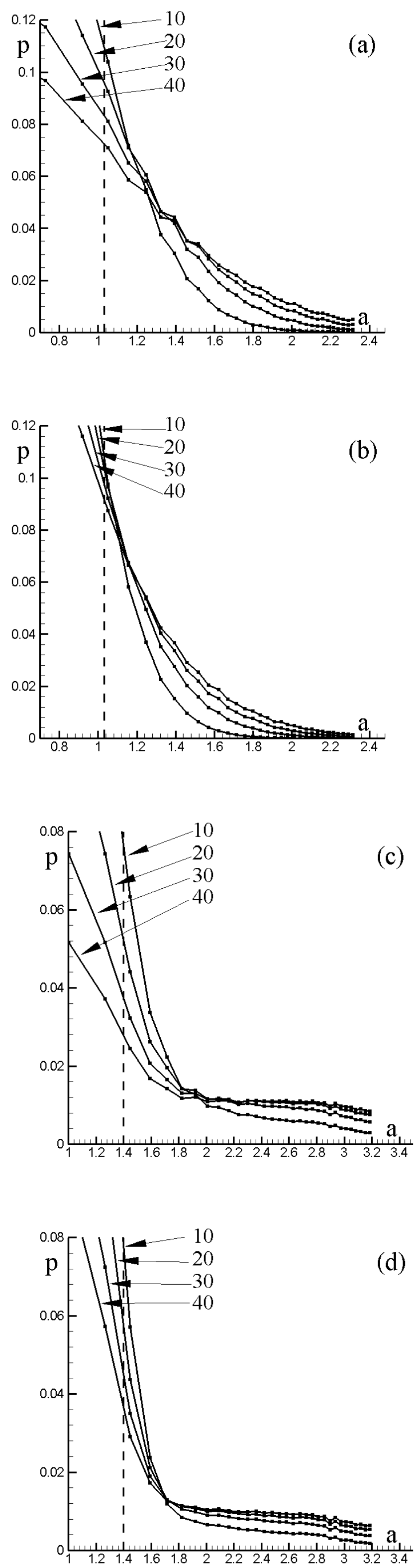

Figure 5 –Density p as a function of the ellipsoid horizontal major semi-axis a. The vertical dashed line marks the boundary of the ellipsoid. Four curves in each figures correspond to given dimensionless time intervals. (a) the steady state with the horizontal and vertical diffusion components; (b) the steady state with only the horizontal diffusion component; (c) the perturbed state with the horizontal and vertical diffusion components; (d) the perturbed state with only the horizontal diffusion component