INPE – National Institute for Space Research
São José dos Campos – SP – Brazil – May 16-20, 2016

BUILDING PHASE SYNCHRONIZATION EQUIVALENCE BETWEEN COUPLED BURSTING NEURONS AND PHASE OSCILLATORS

Fabiano A.S. Ferrari[1] *and Ricardo L. Viana*[2]

[1]Institute of Engeneering, Science and Technology, Federal University of Jequitinhonha's and Mucuri's Valleys, Janúba, Brazil, fabiano.ferrari@ufvjm.edu.br
[2]Physics Department, Federal University of Paraná, Curitiba, Brazil, viana@fisica.ufpr.br

Abstract: Bursting neurons are characterized by an alternation of active and inactive state, during the active state they exhibit train of spikes and quiescent behavior in the inactive state. When bursting neurons are coupled the beginning of their active state can synchronize depending on the coupling strength. If the coupling strength is increased above a critical coupling value, their active state show a transition from a partial synchronization, where just few neurons are synchronized, to a complete synchronization, in which all the neurons begin their active state in the same time. This transition has the same type observed in coupled phase oscillators. Adjusting the frequency distribution of the oscillators properly, it is possible to represent bursting neurons as coupled phase oscillators to describe phase synchronization.

keywords: bursting neurons, phase synchronization, Nonlinear Systems and Neural Dynamics,Synchronization in Nonlinear Systems, Nonlinear Dynamics and Complex Systems

1. INTRODUCTION

Bursting activity is observed in many neuronal firing patterns, it consist of an alternation between active and inactive state. During the active state bursting neurons show train of spikes and in the inactive state they are silent [1, 2]. When bursting neurons are coupled, according with their connection matrix and the coupling strength, they can exhibit phase synchronization, this results are observed at experiments in vitro and numerical simulations [3–5].

To explain the emergence of phase synchronization a model of coupled phase oscillators was proposed by Kuramoto [6]. In the Kuramoto model, phase linear oscillators are coupled through a nonlinear term. To determine phase synchronization is defined an order parameter, called Kuramoto order parameter. When the Kuramoto order parameter is one means the oscillators are sinchronized in phase and when the oscillators are desynchronized the order parameter is zero. This simple model, for some cases, provides an analytical expression for the characteristic transition towards phase synchronization [7].

The observed phase synchronization profile for coupled bursting neurons are similar to those observed in Kuramoto oscillators. We Compare the phase synchronization in bursting neurons with the analytical solutions for the Kuramoto model. We find a way to generate a network of Kuramoto oscillators with the same phase synchronization transition than a specific network of coupled Rulkov neurons [8]. In this paper we present a tutorial of how we can make this happen.

2. RULKOV MODEL

The Rulkov model is a bidimensional map described by [9]:

$$x_{n+1} = \alpha/(1+x_n^2) + y_n, \quad (1)$$
$$y_{n+1} = y_n - \eta(x_n - \sigma), \quad (2)$$

where $\eta = 0.001$ and $\sigma = -1$ are fixed parameters and α is the bifurcation parameter [9]. Different values of α result in different dynamical behaviours. One example of the bursting oscillations produced by this model is shown in the Figure 1.

For the Rulkov model a burst starts when the y-variable reaches its maximum value. Considering the maximum value of the y-variable we can define a phase φ, such that [10]

$$\varphi_n = 2\pi k + 2\pi \frac{(n - n_k)}{(n_{k+1} - n_k)}, \tag{3}$$

where n_k is the initial time of the k-th burst, $n_k = y$(k-th maximum point). The correspondence between this phase transformation and the burst oscillations is shown in the Figure 1.

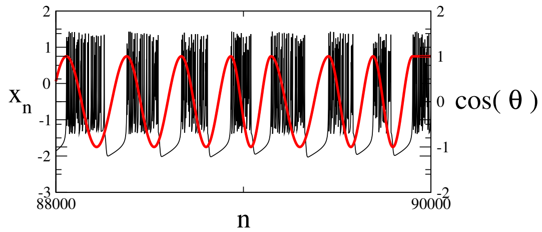

Figure 1 – Example of bursting oscillations and its associated phase produced by the Rulkov model. Black color represents the time serie of the x variable (left side in the graph) and red color the cosine of the phase time serie (right side in the graph). This time serie was produced after 88000 transient iterations, here $\alpha = 4.2$.

3. GENERALIZED KURAMOTO MODEL

The original version of the Kuramoto model describes globally coupled phase oscillators [6]

$$\frac{d\theta_i}{dt} = \omega_i + \frac{\varepsilon}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \tag{4}$$

where $i = 1, ..., N$, ω_i is the natural frequency and ε is the coupling strength. The natural frequency of oscillation is randomly chosen from an unimodal and symmetric probability distribution function $g(\omega)$. A more powerrful version of the Kuramoto model is called: Generalized Kuramoto model [11]. This version allow us to study phase synchronization for different network topologies

$$\frac{d\theta_i}{dt} = \omega_i + \varepsilon \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i), \tag{5}$$

where A_{ij} is the connection matrix.

When the natural frequencies ω obey a symmetric and unimodal frequency distribution $g(\omega)$ is possible to determine the critical value ε_C for the onset of frequency synchronization [12]

$$\varepsilon_c = \frac{2}{\pi g(0)} \frac{\langle k \rangle}{\langle k^2 \rangle}, \tag{6}$$

where $\langle k \rangle$ and $\langle k^2 \rangle$ are the mean and square mean of the connections number per site, respectively.

4. NETWORKS

To verify if the phase synchronization in the Kuramoto and Rulkov model are really equivalent we will analyse three different types of network: Erdös-Rényi, Small-World and Scale-Free.

4.1. Erdös-Rényi Network

This network consists of a random network type. It is generated from N initially uncoupled nodes, then N_K links are randomly chosen with an uniform probability p are added in the network [13]. This probability must to satisfy the relation

$$p = \frac{N_K}{N(N-1)}. \tag{7}$$

In the example of Figure 2 (a) is used a Erdös-Rényi network with $N_k = 5000$ links.

4.2. Small-World Network

Small-world networks are a type of networks where every node has few connections but the distance (in the number of sites) between any two sites in the network is small [14]. There are several methods to construct a small-worl network, here we will consider the Newman-Watts procedure [15]. In this procedure, we start with a regular network of size N, every site is connected with K first neighbours. According with a p uniform probability distribution pNK new connections are added in the network. In the example of Figure 2 (b) is used a small-world network with $K = 20$ and $p = 0.1$.

4.3. Scale-Free Network

To build this type of network we start with an initial Erdös-Rényi network of size N_0, then we insert a new site performing two connections, one random connection and another with the most connected site in the initial network. This procedure is repeated until the network reachs the desirable size N [16]. In the example of Figure 2 (c) is used a scale-free network with $N_0 = 23$ and $N_{0k} = 23$.

5. PHASE SYNCHRONIZATION

The transition from the desynchronized to synchronized state can be analysed through the Kuramoto order parameter. The temporal order parameter is

$$r_n = \frac{1}{N} \Big| \sum_{i=1}^{N} e^{i\varphi_n} \Big|, \tag{8}$$

and to analyse phase synchronization we use the mean order parameter

$$\overline{R} = \frac{1}{n'} \sum_{n=1}^{n'} r_n, \tag{9}$$

where n' is a suficient large number of iterations.

6. COMPARISON BETWEEN KURAMOTO AND RULKOV MODELS

Considering coupled Rulkov neurons, their equations are

$$x_{n+1}^{(i)} = \alpha^{(i)}/(1+(x_n^{(i)})^2) + y_n^{(i)} + \varepsilon \sum_{j=1}^{N} A_{ij} x_n^{(j)} \quad (10)$$

$$y_{n+1}^{(i)} = y_n^{(i)} - \eta(x_n^{(i)} - \sigma), \quad (11)$$

in which A_{ij} is the connection matrix, ε is the coupling strength, η and σ are the same of the uncoupled case. If $\alpha \in [4.1 : 4.3]$, for every α we will have a different bursting frequency $\tilde{\omega}$ [8]. For this reason, if we randomly determine $\alpha^{(i)}$ with a given probability density function, then the associated bursting frequency will naturally have the same distribution.

When the natural frequencies in both models, Rulkov and Kuramoto, obey the same probability distribution function then we can build an equivalent network of Kuramoto oscillators. To show this property we will consider that the frequencies obey an uniform distribution, in this case

$$g(\omega) = \begin{cases} \frac{1}{2a} & \text{for} \quad -a \leq \omega \leq a \\ 0 & \text{otherwise} \end{cases}, \quad (12)$$

where a is the range, $\omega \in [-a : a]$. For this $g(\omega)$ the critical value for the Kuramoto network will be

$$\varepsilon_c = \frac{4a}{\pi} \frac{\langle k \rangle}{\langle k^2 \rangle} \quad (13)$$

this equation can be re-writted as

$$a = \frac{\pi \varepsilon_c}{4} \frac{\langle k^2 \rangle}{\langle k \rangle}. \quad (14)$$

The equation (14) allow us to build a network of Kuramoto oscillators with the same onset for phase synchronization than a specific network of Rulkov neurons with a known ε_c. To generate this equivalent network we just need to determine the a value. Consequently, they will not have just the same ε_c they also will have the same phase synchronization. Examples using the three mentioned networks are shown in the Figure 2. For this example, the Kuramoto oscillators were integrated with classical Runge-Kutta method with step $h = 0.01$, the order parameter is an average of 10^4 iterations per site after 16×10^4 transient iterations and 20 distinct random initial conditions. For the Rulkov neurons, the order parameter is an average of 10^4 iterations per site after 8×10^4 transient iterations and 50 distinct random initial conditions.

7. CONCLUSION

When neuronal bursting models have a parameter control for their phase frequency and we adjust this parameter control to produce frequencies with a specific frequency distribution. Then it is possible to build an equivalent network of Kuramoto oscillators exhibiting, under certain conditions, the same phase synchronization.

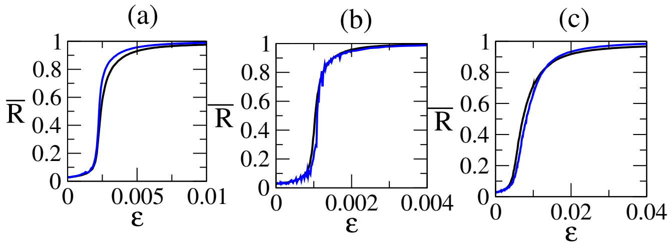

Figure 2 – Comparison in the order parameter for the three different network types: (a) Erdös-Rényi, (b) Small-World and (c) Scale-free. Black color indicates the result for coupled Rulkov neurons and blue color for Kuramoto oscillators. The network size is $N = 1000$.

ACKNOWLEDGMENTS

This work was made possible through financial support from the Brazilian Research Agencies, CAPES and CNPq.

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