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SPECTRAL PROPERTIES OF THE ORBITS OF THE HÉNON MAP

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Abstract: Recently, many works have explored the spectral characterization of one-dimensional discrete-time chaotic signals, considering them as sample functions of a stochastic process defined by the map. In this paper, we numerically access the autocorrelation sequence and power spectral density of orbits generated by a two-dimensional chaotic map, namely the Hénon map. We study how they vary with the map parameters. These results may be relevant for chaos-based communication systems that uses maps to codify messages.

keywords: Time Series Analysis, Discrete Dynamical Systems, Analysis and Control of Nonlinear Dynamical Systems with Practical Applications, Chaotic communication, Spectral Analysis.

1. INTRODUCTION

A chaotic signal is deterministic, aperiodic, bounded and has sensitive dependence on initial conditions [1]. In the last 30 years, many possible applications of chaos in Telecommunication Engineering have been proposed [2–4].

The study of spectral characteristics of chaotic signals is a relevant issue when it comes to using them in practical communications. Because every real world communication channel is bandlimited, to characterize and control the Power Spectral Density (PSD) of the generated chaotic signals is of paramount importance [4, 5]. Recently, many works have thoroughly analysed the PSD of signals generated by one-dimensional maps [6–9]. However, we are not aware of any systematic study of the PSD of signals generated by higher dimensional maps.

In this extend abstract, we report our initial studies involving the autocorrelation sequence (ACS) and PSD of signals generated by the two-dimensional Hénon map. We numerically analyse how they vary with the map parameters.

2. THE HÉNON MAP

The Hénon map $f_H : \mathbb{R}^2 \to \mathbb{R}^2$, is defined as [10]

$$\boldsymbol{s}(n+1) = f_H\left(\boldsymbol{s}(n)\right) = \begin{bmatrix} s_2(n) + 1 - a s_1^2(n) \\ b s_1(n) \end{bmatrix} \tag{1}$$

where $\{a, b\} \in \mathbb{R}$ are parameters and $\boldsymbol{s}(n) = \left[\, s_1(n) \; s_2(n) \,\right]^T$.

Fig. 1(a) and (b) show $\boldsymbol{s}(n)$ with initial condition $\boldsymbol{s}(0) = \left[\, 0 \; 0 \,\right]^T$ for $a = 1.4$ and $b = 0.3$. The atractor, obtained disregarding the first 1000 iterations is presented Fig 1(c). The highest Lyapunov exponent of this attractor can be obtained numerically resulting $h_H = 0.42$ [1, p. 201], confirming its chaotic nature.

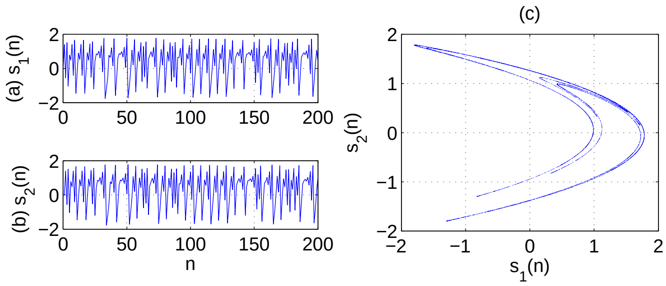

Figure 1 – Hénon map for $a = 1.4$ e $b = 0.3$: (a) and (b) orbit $s(n)$ with initial condition $\boldsymbol{s}(0) = [0\ 0]^T$; (c) attractor.

The bifurcation diagram of $s_1(n)$ and the highest Lyapunov exponent for $b = 0.3$ e $0 < a \leq 1.4$ are presented in Fig. 2. Figure 3 shows orbits obtained for (a) $a = 0.6$, (b) $a = 1.069$, (c) $a = 1.2$, (d) $a = 1.232$ and (e) $a = 1.4$. This values of a are signalized by dashed lines in Figure 2. In cases (a) and (d) we have periodic orbits and in (b), (c) and (e) chaotic ones.

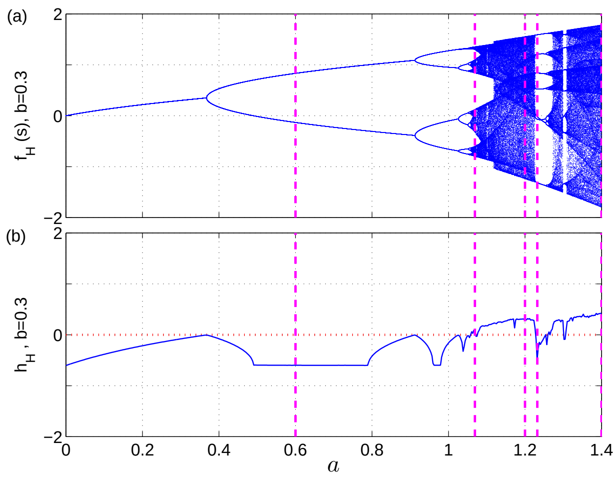

Figure 2 – (a) Bifurcation diagram for $s_1(n)$ and (b) highest Lyapunov exponent for the Hénon map.

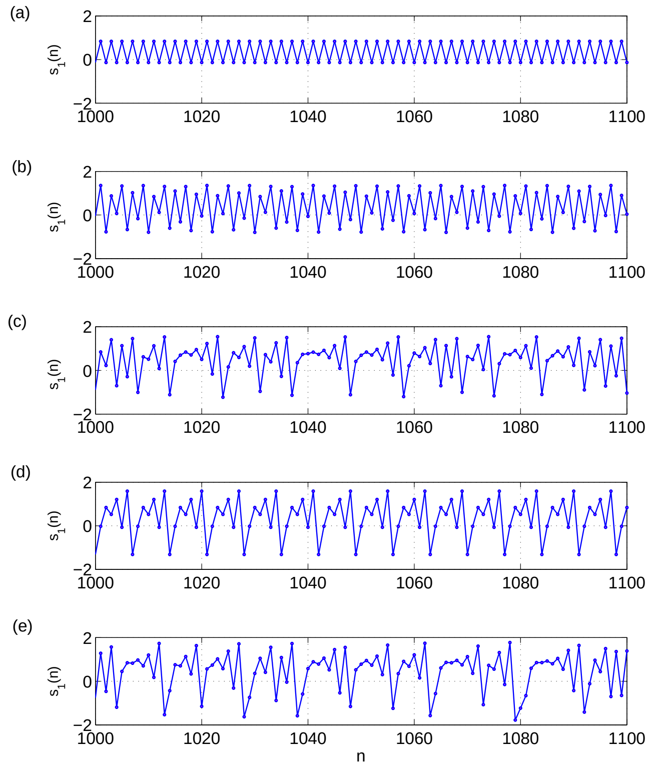

Figure 3 – Orbits of Hénon map for fixed $b = 0.3$ and (a) $a = 0.6$, (b) $a = 1.069$, (c) $a = 1.2$, (d) $a = 1.232$ and (e) $a = 1.4$.

3. AUTOCORRELATION SEQUENCES AND POWER SPECTRAL DENSITY

Chaotic signals generated by a map can be considered as sample function of an ergodic stochastic process [11]. The ACS $R(k)$ for an integer k is defined by

$$R(k) \equiv E\left[s(n)s(n+k)\right], \quad k \geq 0, \quad \forall n \geq 0. \tag{2}$$

The expected value $E[\cdot]$ is taken over all initial conditions that generate chaotic signals. For negative values of k, we conveniently consider

$$R(k) \equiv R(-k), \quad k < 0. \tag{3}$$

The PSD $P(\omega)$ is the discrete-time Fourier transform of $R(k)$, considering k as the time variable [5]. It displays how power is allocated in the discrete-time frequency $0 \leq \omega < \pi$.

Fig. 4 shows the ACS of $s_1(n)$ for $b = 0.3$ and the values of a considered in Fig. 3. Their estimated PSD using 6×10^6 samples are also shown.

In the periodic cases of (a) $a = 0.6$ (period $N = 2$) and (d) $a = 1.232$ ($N = 7$), it can be clearly seen the periodic nature of the ACS and the peaks at frequency $2\pi/N$ and its harmonics. In the chaotic cases (c) $a = 1.2$ and (e) $a = 1.4$, the ACS present a non-impulsive slowly decaying and oscillatory pattern that reflects in a continuous peakless PSD. The power is concentrated predominantly in higher frequencies.

In the case (b) $a = 1.069$, we are still next to the emergence of chaos in the bifurcation diagram of Figure 3. Although positive, the highest Lyapunov number is close to zero. This is reflected in a very slowly decaying ACS and peaks in the DSP, specially at $2\pi/4$ and its harmonics. This way, we have a narrowband chaotic signal.

The general behavior of the PSD for $1.2 < a \leq 1.4$ and $b = 0.3$ is plotted in Figure 5. The onset of chaos and periodic windows can be clearly noticed. It is also possible to note the power concentration in the high frequencies of the chaotic attractors.

Other simulations are presented in animated form at `http://www.lcs.poli.usp.br/~marcio/NSCHenon`.

4. CONCLUSION

This preliminary work analyzes through computational simulations the ACS and PSD of orbits generated by the Hénon map. The simulations suggest that the map can be used to generate narrowband and highpass chaotic signals, what can be relevant for chaos-based communication schemes.

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References

[1] K. Alligood, T. Sauer, and J. Yorke, *Chaos: An Introduction to Dynamical Systems*. Textbooks in Mathematical Sciences, Springer, 1997.

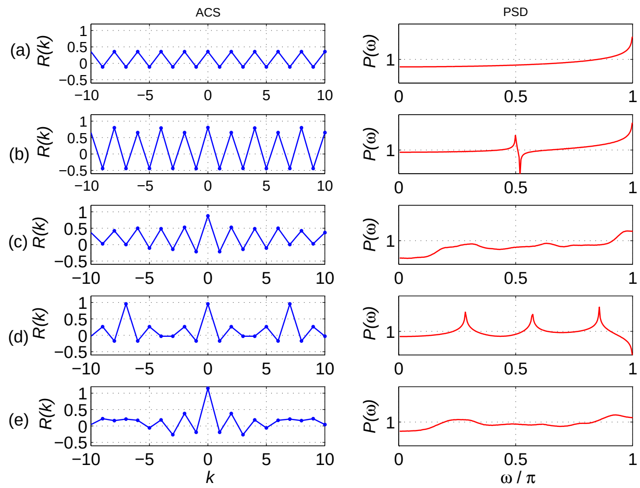

Figure 4 – ACS and PSD of orbits of the Hénon map for $b = 0.3$ **and (a)** $a = 0.6$**, (b)** $a = 1.069$**, (c)** $a = 1.2$**, (d)** $a = 1.232$ **and (e)** $a = 1.4$**.**

[2] F. C. M. Lau and C. K. Tse, *Chaos-based digital communication systems*. Berlin: Springer, 2003.

[3] M. P. Kennedy, G. Setti, and R. Rovatti, eds., *Chaotic Electronics in Telecommunications*. Boca Raton, FL, USA: CRC Press, Inc., 2000.

[4] M. Eisencraft, R. R. F. Attux, and R. Suyama, eds., *Chaotic Signals in Digital Communications*. CRC Press, Inc., 2013.

[5] B. Lathi and Z. Ding, *Modern Digital and Analog Communication Systems*. The Oxford Series in Electrical and Computer Engineering, Oxford University Press, Incorporated, 2009.

[6] H. Sakai and H. Tokumaru, "Autocorrelations of a certain chaos," *Acoustics, Speech and Signal Processing, IEEE Transactions on*, vol. 28, pp. 588–590, Oct. 1980.

[7] M. Eisencraft, D. M. Kato, and L. H. A. Monteiro, "Fast communication: Spectral properties of chaotic signals generated by the skew tent map," *Signal Process.*, vol. 90, no. 1, pp. 385–390, 2010.

[8] K. Feltekh, D. Fournier-Prunaret, and S. Belghith, "Analytical expressions for power spectral density issued from one-dimensional continuous piecewise linear maps with three slopes," *Signal Processing*, vol. 94, no. 0, pp. 149 – 157, 2014.

[9] R. A. Costa, M. B. Loiola, and M. Eisencraft, "Spectral properties of chaotic signals generated by the bernoulli map," *Journal of Engineering Science and Technology Review*, vol. 8, no. 2, pp. 12–16, 2015.

[10] M. Hénon, "A two-dimensional mapping with a strange attractor," *Communications in Mathematical Physics*, vol. 50, pp. 69–77, 1976. 10.1007/BF01608556.

[11] A. Lasota and M. MacKey, *Probabilistic properties of deterministic systems*. Cambridge University Press, 1985.

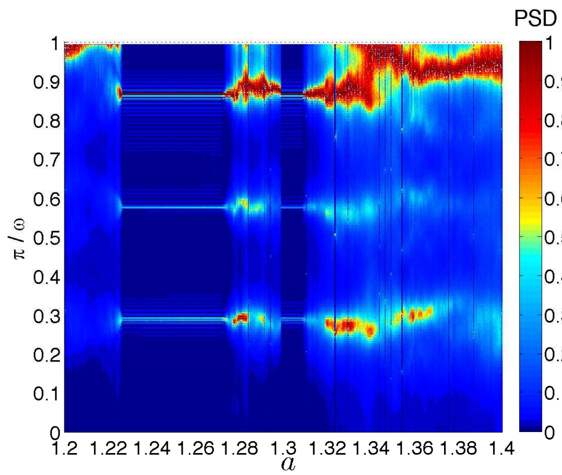

Figure 5 – PSD of orbits of the Hénon map for $b = 0.3$ **and** $1.2 < a \leq 1.4$**.**