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ESTIMATION OF DYNAMICAL PHASE MODELS FOR CHAOTIC OSCILLATORS

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Abstract: Despite the advances in synchronization theory, a complete understanding about the best coupling variable to synchronize chaotic oscillators, in a general way, is still lacking. Dealing with periodic oscillators, one of the most successful approaches is the phase description method, where a dynamical model is obtained to describe the evolution of the phase of the system. Conclusions about synchronous regimes of the system (e.g. Arnold tongues) can be estimated by analyzing only the phase model. This work presents an attempt to extend this approach to chaotic oscillators, by means of modelling its phase dynamics, based on the evolution of the states. Phase models for the Rössler and Lorenz systems were estimated from data using system identification techniques. The paper investigates the feasibility of choosing the most adequate variable for coupling oscillators by analyzing the phase model – structure and parameters. This could lead to a more systematic way of investigating *aspects of synchronizability*.

keywords: Synchronization in Nonlinear Systems, Modeling, Numerical Simulation and Optimization, Control in Complex Systems.

INTRODUCTION

A matter of importance in synchronization theory is to describe synchronization conditions of an autonomous self-sustained oscillator and its sensitivity to an external force. For limit cycles, the most successful approach is the *phase description method* [1] which, based on the concepts of *isochrones* [2, 3], makes it possible to calculate those conditions and, besides that, a quantification of the influence due to an external force upon the oscillator.

For limit cycles, *isochrones* can be defined as a family of Poincaré sections on which, after one oscillation period T, the state converges to the same section [2, 3]. Dealing with chaotic oscillators, it is not even possible to define a regular period of oscillation and it is difficult to define an *isochrone*. Nevertheless, one can try to work with the concept of *isophases* [4].

The main idea of this work is to define a *phase-like* variable[1] and identify models of that variable, based on the states of the system, in such a way that it is possible to conclude something about the *synchronizability* of the system. These conclusions will be based only on information (data) about the unperturbed system, and the type of perturbation acting upon it. This procedure aims at describing the behavior of the system upon an external force, but not the micro-dynamics, on time scales smaller that the *mean* period. Numerical simulations and analysis is presented for Rössler and Lorenz systems.

PURPOSE

Consider the nonlinear oscillator with an external force:

$$\dot{\vec{x}}(t) = \boldsymbol{f}\big(\vec{x}(t)\big) + \varepsilon \boldsymbol{p}(t), \tag{1}$$

where $\vec{x} \in \mathbb{R}^n$ is the state vector, $\boldsymbol{f}$ the vector field, ε the coupling strength and $\boldsymbol{p}(t)$ the external force.

[1] A *phase-like* variable for chaotic oscillators can be defined in quite different ways, including angle measurements on the attractor, Hilbert transform, Poincaré sections, Fourier transform, Wavelets. For detailed discussion see [5] and the references therein.

Suppose that a phase-like variable can be described by:

$$\phi(t) := \phi\big(\vec{x}(t)\big). \tag{2}$$

The phase dynamics of the perturbed system can be calculated as:

$$\begin{aligned}\dot{\phi} &= \mathcal{L}_{\boldsymbol{f}}\phi(\vec{x}) + \mathcal{L}_{\boldsymbol{p}}\phi(\vec{x})\\ &= \nabla^T\phi(\vec{x}).\boldsymbol{f}(\vec{x}) + \varepsilon\nabla^T\phi(\vec{x}).\boldsymbol{p}(t)\\ &= \omega_0(t) + \varepsilon\nabla^T\phi(\vec{x}).\boldsymbol{p}(t),\end{aligned} \tag{3}$$

where $\mathcal{L}$ denotes the Lie-derivative, $\nabla^T\phi(\vec{x})$ the transposed gradient of ϕ with respect to the state vector $\vec{x}$, $\omega_0(t)$ the natural (not regular) frequency of the unperturbed system.

Suppose that the external force has *constant* frequency ω, the phase difference is given by:

$$\psi(t) := \phi\big(\vec{x}(t)\big) - \omega t, \tag{4}$$

and its dynamics is given by:

$$\begin{aligned}\dot{\psi} &= \dot{\phi} - \omega\\ &= \omega_0(t) - \omega + \varepsilon\nabla^T\phi(\vec{x}).\boldsymbol{p}(t)\\ &= \nu(t) + \varepsilon q(t),\end{aligned} \tag{5}$$

where $\nu(t) := \omega_0(t) - \omega$ is the frequency mismatch and $q(t) := \nabla^T\phi(\vec{x}).\boldsymbol{p}(t)$ measures the influence of the perturbation on the phase [6].

The synchronous state is achieved when the phase difference dynamics (5) is asymptotically stable, and it occurs when the external influence $\varepsilon q(t)$ cancels the frequency mismatch $\nu(t)$. This work is based on the conjecture that, by estimating a model $\hat{q}(t)$, may be possible to analyze which is the best coupling scheme, e.g. what is the state variable that needs less effort (ε) to synchronize with the external force.

The following problem is stated: in a master-slave topology, is it possible to figure out which is the coupling variable that requires less effort (ε) to achieve complete synchronization?

METHODS

Consider a pairwise dataset Z containing the full state $\vec{x}(t)$ and a phase-like variable $\tilde{\phi}(t)$ for the unperturbed system:

$$Z = \begin{bmatrix}\vec{x}^T(1) & \tilde{\phi}(1)\\ \vec{x}^T(2) & \tilde{\phi}(2)\\ \vdots & \vdots\\ \vec{x}^T(N) & \tilde{\phi}(N)\end{bmatrix}, \tag{6}$$

where all data on Z was normalized to zero mean and unit variance, $Z \sim (0,1)$.

From Eq. (2), a static model is estimated over dataset Z, using standard least-squares solution:

$$\boldsymbol{\theta} = (Z_{\vec{x}}^T Z_{\vec{x}})^{-1} Z_{\vec{x}} Z_{\tilde{\phi}}, \tag{7}$$

where $Z_{\vec{x}}$ represents the first n columns of Z and $Z_{\tilde{\phi}}$ the last column. The following model is obtained:

$$\hat{\phi}(t) = \vec{x}^T\boldsymbol{\theta}, \tag{8}$$

where $\boldsymbol{\theta} \in \mathbb{R}^n$ is a parameter vector. We propose composing $q(t)$ in (5) as:

$$q(t) = \boldsymbol{\theta}\boldsymbol{p}(t). \tag{9}$$

Based on the values of the parameters $\boldsymbol{\theta}$, by a single numerical comparison, it is possible to determine which state has greater influence on the phase dynamics.

RESULTS

Numerical results are presented for the Rössler and Lorenz oscillators.

Rössler system

The Rössler system:

$$\begin{aligned}\dot{x} &= -y - z\\ \dot{y} &= x + 0.15y\\ \dot{z} &= 0.2 + xz - 10z,\end{aligned} \tag{10}$$

was used as the first example. A well known phase-like variable can be calculated by [7, 8]:

$$\tilde{\phi}(t) = \tan^{-1}\left(\frac{y}{x}\right). \tag{11}$$

A proper dataset (6) was acquired by simulating the unperturbed system (10) with random initial conditions given by:

$$\vec{x}_0 = [0 \;\; \mathcal{U}(-16.4, -4.7) \;\; 0]^T, \tag{12}$$

where $\mathcal{U}(a, b)$ denotes the uniform distribution between the values a and b. The system was simulated for 150 seconds, with integration time of 0.01, where the first 50 seconds were discarded from the analysis.

One hundred different datasets were obtained and the boxplot of the parameter values is shown in Figure 1. A simple comparison between the magnitude of the parameters shows that the state y has the greater weight in determining the phase dynamics (remember that the data are normalized). Some difference of absolute values can be observed between θ_x and θ_z, in which the former is greater than the latter[2].

The synchronization of two Rössler systems, in a master-slave topology, was simulated numerically:

$$\dot{\vec{x}}_{1,2} = \boldsymbol{f}_{1,2}(\vec{x}_{1,2}) + \boldsymbol{p}_2(t) \tag{13}$$

where the subindex 1 denotes the master system and 2 denotes the slave one, hence only the slave system is subject to an external force. In order to verify which is the state

[2] The statistical t-Student test was made, at a significance level of 99%, it is possible to say that θ_x is greater than θ_z.

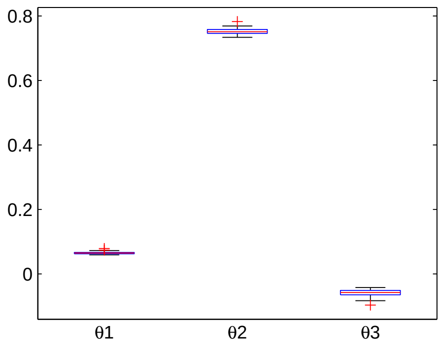

Figure 1 – Boxplot of estimated parameters over 100 different datasets of the Rössler system.

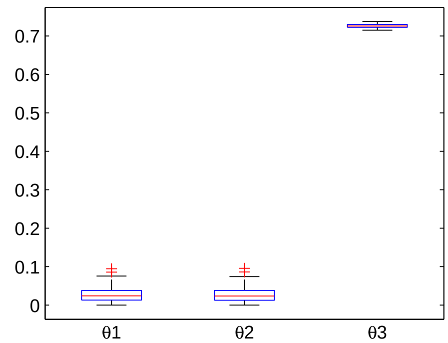

Figure 2 – Boxplot of estimated parameters over 100 different datasets of the Lorenz system.

that needs less effort to synchronize, three different coupling functions were used:

$$\begin{aligned}\boldsymbol{p}_{2x} &= [(x_1 - x_2) \;\; 0 \;\; 0]^T, \\ \boldsymbol{p}_{2y} &= [0 \;\; (y_1 - y_2) \;\; 0]^T, \\ \boldsymbol{p}_{2z} &= [0 \;\; 0 \;\; (z_1 - z_2)]^T. \end{aligned} \tag{14}$$

The minimum effort necessary for the complete synchronization $\varepsilon_x^{\min}$, $\varepsilon_y^{\min}$ and $\varepsilon_z^{\min}$ was computed, empirically, for each coupling function[3]. The results are summarized in Table 1.

Lorenz system

The second example used the Lorenz system:

$$\begin{aligned}\dot{x} &= 10(y - x) \\ \dot{y} &= 28x - y - xz \\ \dot{z} &= -\frac{8}{3}z + xy. \end{aligned} \tag{15}$$

The phase-like variable is [7, 8]:

$$\tilde{\phi}(t) = \tan^{-1}\left(\frac{z - 27}{\sqrt{x^2 - y^2} - 12}\right). \tag{16}$$

A proper dataset (6) was acquired by simulating the unperturbed system (15) with random initial conditions given by:

$$\vec{x}_0 = [\mathcal{U}_1(-5, 5) \;\; x_0 \;\; 0]^T. \tag{17}$$

The system was simulated in the same conditions as the previous example, i.e. simulation time of 150 seconds, integration time of 0.01 – with the first 50 seconds discarded from the analysis, and one hundred different datasets used for parameter estimation. Figure 2 shows the boxplot of the parameter values. In this system, the variable z seems to make major influence upon the phase dynamics. The difference of values between θ_y and θ_x was not significant[4].

[3]The minimal coupling strength required for synchronization was recently used as a threshold measure in a related study [9].

[4]The statistical t-Student test was made, at a significance level of 99%.

As the previous example, the synchronization of two Lorenz systems, in a master-slave topology, was simulated numerically (Eq. 13) and the same three coupling functions were used for comparison (Eq. 14). The minimum effort necessary for the complete synchronization are summarized on Table 1.

Table 1 – Results of synchronization limits estimated by the parameters values and empirically.

			Rössler		
Parameters	θ_y	$\gg$	θ_x	$>$	θ_z
Simulation	$\varepsilon_y^{\min} = 0.17$	$<$	$\varepsilon_x^{\min} = 0.21$	$<$	$\varepsilon_z^{\min} = \infty$
			Lorenz		
Parameters	θ_z	$\gg$	θ_y	$\approx$	θ_x
Simulation	$\varepsilon_z^{\min} = 1.45$	$<$	$\varepsilon_y^{\min} = 2.75$	$<$	$\varepsilon_x^{\min} = 7.83$

DISCUSSION

For the Rössler system, the absolute value of the estimated parameters showed the exact reverse order as the empirical minimum synchronization effort. But the difference between the synchronization effort $\varepsilon_x^{\min}$ and $\varepsilon_z^{\min}$ is poor related to the difference between parameters θ_x and θ_y. For the Lorenz system, the estimated parameters showed with success the best coupling state z, but the remaining difference between $\varepsilon_y^{\min}$ and $\varepsilon_x^{\min}$ was not related to the parameters.

In both cases it was possible to detect the coupling that permitted synchronization with less effort, but was not successful on predicting the correct order of the remaining states. Some possible causes of this can be i) the simplicity of the models structure (Eq. 8), which could estimate the major behavior but not the dynamics in details; ii) the definition of the *phase-like* variable (Eqs. 11, 16), which carries on itself some simplification of the phase definition; iii) the dynamic behavior of the frequency mismatch $\nu(t)$ (Eq. 3), which it was not considered in the analysis; iv) the assumption that the external frequency ω is nearly constant, which is not the case for chaotic oscillators.

Some open questions about the procedure are: i) could a more sophisticated model structure yield better estimates about the behavior? ii) The usage of others phase definitions (e.g. Poincaré section, Hilbert transform, sinusoidal fit [5]) as *phase-like* variable could lead to better estimates? iii) Whats the very influence of the dynamic behavior of $\nu(t)$ in this procedure? Is it valid for complete, phase, generalized synchronization cases? iv) In which cases the assumption that the external frequency ω is nearly constant can be considered? Those questions will be addressed on future works.

CONCLUSIONS

In this work we estimate an algebraic function that maps the state of an oscillator to the phase or a phase-like quantity. This will be called the phase model below. The function was estimated using a linear model structure and the parameters estimated by least squares. The main aim in obtaining this function is *not* to have yet another estimate of the phase, but rather to be able to decide which variable to use for (dissipative) coupling two oscillators in a master-slave configuration.

The analysis is based on the unperturbed system to estimate the phase model and from this model it is immediate to see which state variable has the greatest influence on the phase. The work continues by conjecturing, based on Eqs. 3–5, that the variable that most influences the phase is the best candidate to couple two oscillators in a master-slave configuration with a view to synchronization.

The above conjecture was tested by numerical simulation. Two examples are presented: the master-slave dissipative coupling of two identical Rössler oscillators and of two identical Lorenz oscillators. In both cases, the variable that has greatest influence on the phase as determined by the estimated phase model (y for Rössler and z for Lorenz) did provide the dissipative coupling that achieved *complete* synchronization with the smallest coupling strength.

Such preliminary results are promising, and suggest a number of possible avenues of research. For instance, although the best coupling variable in both simulated examples is the same as the one with greatest relative importance in the phase model, the relation of the other two variables is not so clear. Possible reasons for this are provided in the text. These will be addressed in future works.

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