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CHAOTIC PROPERTIES OF THE HÉNON MAP WITH A LINEAR FILTER

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Abstract: A practical way of limiting the bandwidth of the transmitted signal in a chaos-based communication scheme is to use linear discrete-time filters in the feedback loop. It was recently proved that such filters do not disturb chaotic synchronization. However, there is no guarantee that the generated signals remain chaotic. In this paper, we consider the Hénon map plus a linear time-invariant finite impulsive response filter. We numerically access its largest Lyapunov exponent as a function of the filter coefficients, obtaining regions where chaotic, periodic or unbounded orbits are present.

keywords: Analysis and Control of Nonlinear Dynamical Systems with Practical Applications, Nonlinear Dynamics and Complex Systems, Discrete Dynamical Systems, Time series Analysis.

1. INTRODUCTION

The motivation for using chaotic signals in a communication system is related to their sensitive dependence on initial conditions (SDIC) [1]. The SDIC can improve security issues involving information coding [2, 3]. This way, if the system structure or the transmitted message modifies the chaotic map, as in the system proposed in [4] and adapted to discrete-time by [5], it is necessary to determine whether the transmitted signal remains chaotic.

Chaotic signals are often broadband and the physical transmission channels are always bandlimited. This way, the authors of [5] proposed to place linear discrete-time lowpass filters [6] in the feedback loops in order to limit the bandwidth of the chaotic transmitted signals. The synchronization conditions for this bandlimited system have been analytically determined in [7]. However, it is not clear if the transmitted signals remain chaotic. Preliminary numerical results presented in [8] show that the insertion of finite impulse response (FIR) filters can turn originally chaotic signals into periodic or even unbounded signals depending on the filter coefficients.

In this work, we extend the results of [8], numerically accessing the effect of the filters coefficients on the nature of the transmitted signal, considering as chaotic generator the Hénon map [9].

2. HÉNON MAP WITH A LINEAR FILTER

The two-dimensional Hénon map can be expressed as [1]

$$\boldsymbol{x}(n+1) = f_H\left(\boldsymbol{x}(n)\right) = \begin{bmatrix} \alpha - x_1^2(n) + \beta x_2(n) \\ x_1(n) \end{bmatrix} \quad (1)$$

where $\{\alpha, \beta\} \in \mathbb{R}$ are parameters and $\boldsymbol{x}(n) = \left[\, x_1(n) \; x_2(n) \,\right]^T$.

In the system proposed in [5], the state $x_1(n)$ is filtered through a FIR filter with N_S coefficients before been fed back in the nonlinearity. The filtered Hénon map, is then described by

$$\boldsymbol{x}(n+1) = \begin{bmatrix} x_1(n+1) \\ x_2(n+1) \\ x_3(n+1) \end{bmatrix} = \begin{bmatrix} \alpha - x_3^2(n) + \beta x_2(n) \\ x_1(n) \\ \sum_{j=0}^{N_S-1} c_j x_1(n+1-j) \end{bmatrix} \quad (2)$$

where $\boldsymbol{x}(n) = \left[\, x_1(n) \; x_2(n) \; x_3(n) \,\right]^T$, and c_j, $0 \leq j \leq N_S - 1$, are the filter coefficients.

In this extended abstract we analyse the cases $N_S = 1$ and $N_S = 2$ postponing the analysis of filters with more coefficients for a complete paper.

3. LYAPUNOV EXPONENTS FOR $N_S = 1$ AND $N_S = 2$

For $N_S = 1$, (3) can be rewritten as

$$\boldsymbol{x}(n+1) = \begin{bmatrix} \alpha - x_3^2(n) + \beta x_2(n) \\ x_1(n) \\ c_0\left(\alpha - x_3^2(n) + \beta x_2(n)\right) \end{bmatrix}. \qquad (3)$$

In particular, for $c_0 = 1$, $x_3(n) = x_1(n)$ and the dynamics is that of the original Hénon map of (1). For the usual parameter $\beta = 0.3$ [9], Figure 1 shows the largest Lyapunov exponent and the bifurcation diagram of $x_1(n)$ for $0 < \alpha \leq 1.4$. We can see that after a period-doubling cascading, we have a range of values of α that generate chaotic signals. In particular for the usual $\alpha = 1.4$ we have chaos.

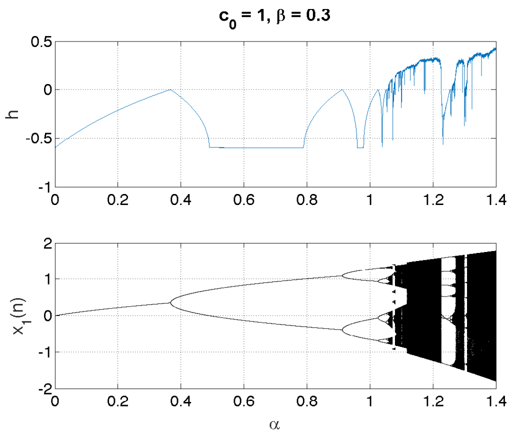

Figure 1 – The largest Lyapunov exponent, h and the orbit of $x_1(n)$ as a function of α for $\beta = 0.3$ and $c_0 = 1$.

For $0 < c_0 \leq 1$, Figure 2 shows the largest Lyapunov exponent and the bifurcation diagram of $x_1(n)$ as function of c_0 for $\alpha = 1.4$ and $\beta = 0.3$. Both graphics are distorted versions of the ones in Figure 1. We clearly see that depending on the value of c_0 the chaotic condition presented for $c_0 = 1$ can vanish. It is possible to obtain chaos for $c_0 > 0.87$.

For $N_S = 2$, (3) can be rewritten as

$$\boldsymbol{x}(n+1) = \begin{bmatrix} \alpha - x_3^2(n) + \beta x_2(n) \\ x_1(n) \\ c_0\left(\alpha - x_3^2(n) + \beta x_2(n)\right) + c_1 x_1(n) \end{bmatrix}. \qquad (4)$$

Figure 3 shows regions of positive and negative largest Lyapunov exponent h in the $c_0 \times c_1$ space. Purple (darker) regions represent $h < 0$ and yellow (lighter) regions represent $h > 0$. In the blank regions the orbits diverge. For low values of c_0 and c_1 chaos is lost and periodic orbits are found. It is possible to obtain chaotic signals for c_0 or c_1 sufficiently large. However, if both are simultaneously large, the orbits can diverge as is shown by the blank area in Figure 3.

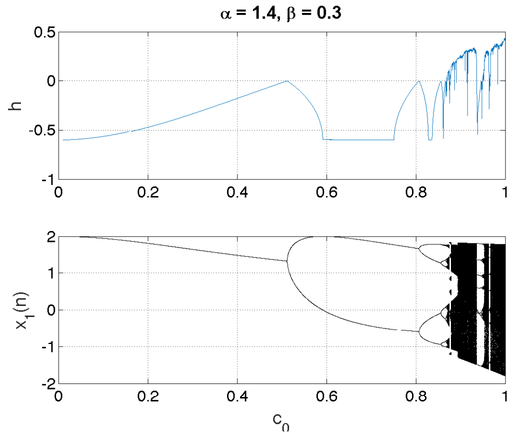

Figure 2 – The largest Lyapunov exponent, h and the orbit of $x_1(n)$ as a function of c_0 for $\alpha = 1.4$ and $\beta = 0.3$.

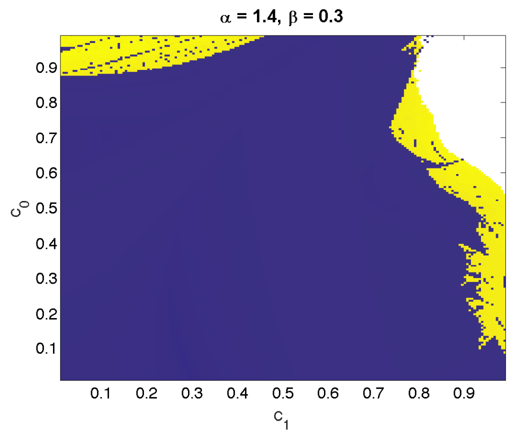

Figure 3 – Regions of positive and negative largest Lyapunov exponent h in the $c_0 \times c_1$ space for $\alpha = 1.4$ and $\beta = 0.3$ in (4). Purple (dark) regions represent $h < 0$ and yellow (light) regions represent $h > 0$. In the blank regions the orbits diverge.

A more profound analysis in the parameter space will be given in a full paper.

4. CONCLUSIONS

The numerical simulation presented show that filtering a Hénon map can modify its chaotic regions. This way, bandlimited chaos-based communication systems must be carefully projected to guarantee that the generated signals remain chaotic. In our present research we are studying the influence of filters with more coefficients.

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