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NUMERICAL IMPRECISION AND ITS IMPACT ON DISCRETE SYSTEMS SUCH AS LOGISTIC MAP

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Abstract: A well-known property of chaotic systems is the sensitivity to initial conditions that are typically observed by means of numerical methods. However, usually a computational tool is used to observe this chaotic behavior, which may interfer in the outcome due to its limited precision. In this work we investigate the impact of the round mode as defined in IEEE 754-2008 standard. We show that the standard mode, that is, round to the nearest, is not always the best approach to simulate nonlinear dynamical systems.

keywords: Discrete Dynamical Systems, Chaos, Numerical Simulation.

1. INTRODUCTION

The field of nonlinear dynamic systems has attracted the attention of researchers around the world [7]. Over the past five decades, the scientific community has been looking for ways to understand the behaviour of chaotic nonlinear systems. It is noted that the nonlinear discrete equations or recursive functions have a great importance in many processes. As stated in [2] that recursive functions, described as the following equations

$$x_{n+1} = f(x_n), \tag{1}$$

are used to describe a variety of problems.

Examples of important recursive functions have been one-dimensional maps, especially the logistic map, which according to [5] is used in a large number of applications and investigations.

The bifurcation diagram has been used in many works as an important tool to describe the general behaviour of the logistic map. It seems surprisingly, however, that we could not find a rigorous approach to undertake such task, that is,to build the bifurcation diagram [6]. The procedure suggested in [6] does not take into account numerical problems due to rounding off. And this is also stated in [1], in which a hypothesis about the possible interference of the computer in chaos study is raised.

2. PURPOSE

This paper presents a very brief investigation on the effects of rounding off in the simulation of chaotic systems. We focus our attention on round mode guided by IEEE 754-2008 standard. We show different results subject to the rounding mode. These results agree with what is expected by the theorem of convergence of recursive functions on computers presented in [4].

3. METHODS

Consider discrete logistic model [3] as defined in Equation (2). Let the interval $I = \{x \in \mathbb{R} \mid 0 < x < 1\}$, $f : I \to \mathbb{R}$ a continuous function and $\varepsilon > 0$. Consider f in the following form

$$x_{n+1} = f(x_n) = r x_n(1 - x_n), \tag{2}$$

where $n \in \mathbb{N}$ and $r \in \mathbb{R}$ and $r > 1$

Let $A \subset I$ be defined as

$$A = \{a \in I \mid a = 1/r\}. \tag{3}$$

It is easy to show that when $x_0 = a$

$$x_1 = f(a) = r\frac{1}{r}\left(1 - \frac{1}{r}\right) = \frac{r-1}{r}, \tag{4}$$

and $x_1 = x_2 = \ldots = x_n$ and so $d(f^p(a), x^*) \to 0$ as $p \to \infty$. In particular, for $a = 1/r$ $d(f^p(a), x^*) = 0$ when $p = 1$. It is known that $x^* = (r-1)/r$ is a fixed point of (2).

Let the definition of fixed point proposed in [4]

Definition 1: If $d(\widehat{f}_n(\widehat{x}^*), \widehat{f}_{n-1}(\widehat{x}^*)) > \delta_n + \delta_{n+1}$, then $\widehat{x}^*$ is a fixed point, where $\widehat{f}_n$ and $\widehat{f}_{n-1}$ be an approximation of f_n and f_{n-1} respectively, $\widehat{x}^*$ approximation of x^*, δ_n and δ_{n+1} error obtained by iterating n and $n+1$.

It means that when the distance between two consecutive points is within the tolerance given by the accumulated error, the simulation must be interrupted as we may reach a fixed point.

We used the following Matlab functions to set the different round modes.

- Used to round towards $-\infty$:

```
system_dependent('setround',-Inf)
```

- Used to round towards $+\infty$:

```
system_dependent('setround',+Inf)
```

- Used to round to the nearest;

```
system_dependent('setround',0.5)
```

The default round mode for the majority of numerical simulation software is to the nearest.

4. RESULTS

Let $r = 4$ and initial condition $x_0 = 1/r$. Therefore analytical analysis we have the following result.

$$x_1 = f(1/4) = 4 \cdot \frac{1}{4}\left(1 - \frac{1}{4}\right) = \frac{4-1}{4} = \frac{3}{4} \quad (5)$$

$$x_2 = f(3/4) = 4 \cdot \frac{3}{4}\left(1 - \frac{3}{4}\right) = \frac{4-1}{4} = \frac{3}{4} \quad (6)$$

and $x_1 = x_2 = \ldots = x_n$ and so $d(f^p(1/4, x^*) \to 0$ as $p \to \infty$. In particular, $d(f^p(1/4), x^*) = 0$ when $p = 1$. It is known that $x^* = 3/4$ is a fixed point of (5).

Algorithm 1 (see Tab. 1) is used to simulate the logistic map using the round mode to the nearest. The result is presented in Figure 1. From what has been shown in Fig. 1 and by means of the Lyapunov exponent, this series is easily identified as chaotic.

Now, we present the results of simulation using the round mode towards $-\infty$ and $+\infty$, as indicated by the Algorithm 2 (see Tab. 2). Figure 2 shows the simulation for the round mode towards to $-\infty$. Table 3 to 5 show the first computed values for the three round modes under investigation.

Algorithm 1
$r \leftarrow 4$;
$x_0 \leftarrow 1/4$;
$N \leftarrow 1500$;
for $n \leftarrow 0$ **to** N **do**
$\mid x_{n+1} \leftarrow r \cdot x_n(1 - x_n)$;
end

Table 1 – Algorithm for simulation of logistic map.

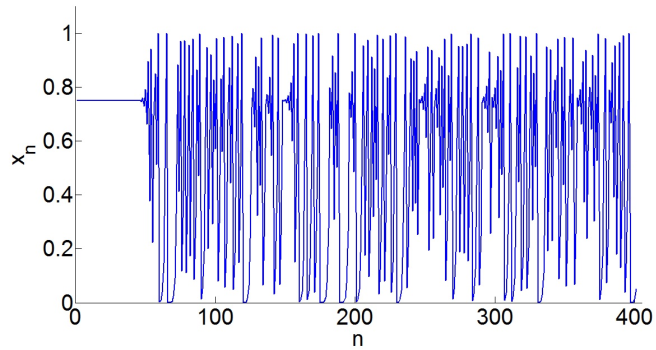

Figure 1 – Simulation of Logistic Map as the initial condition is $x_0 = 1/r$, to $r = 4$ with round mode to the nearest.

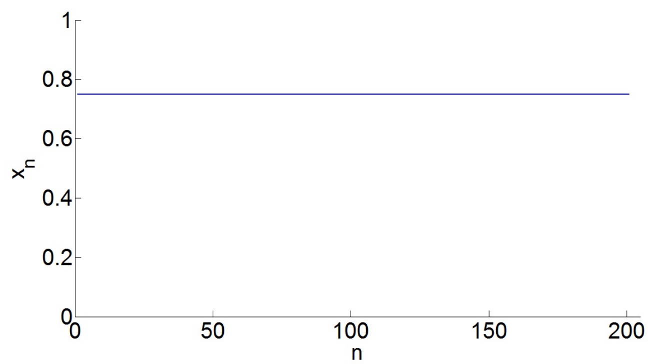

Figure 2 – Simulation of Logistic Map as the initial condition is $x_0 = 1/r$, to $r = 4$, when rounding technique is to $-\infty$, with 200 iterations.

5. DISCUSSION

It is possible to see that only the round mode towards $-\infty$ presents the correct result. The results obtained using the

Algorithm 2
$r \leftarrow 4$;
$x_0 \leftarrow 1/4$;
$N \leftarrow 1500$;
for $n \leftarrow 0$ **to** N **do**
$\mid x_{n+1} \leftarrow round^p(r \cdot x_n(1 - x_n))$;
end

Table 2 – Algorithm for simulation of logistic map using round mode towards $p = -\infty$ or $p = +\infty$.

n	x_n (decimal)	x_n (hexadecimal)
0	0.250000000000000	3fd0000000000000
1	0.750000000000000	3fe8000000000002
2	0.750000000000000	3fe7fffffffffffe
3	0.750000000000001	3fe8000000000006
4	0.749999999999999	3fe7fffffffffff6

Table 3 – Initial computed values for the round mode towards $+\infty$.

n	x_n (decimal)	x_n (hexadecimal)
0	0.250000000000000	3fd0000000000000
1	0.750000000000000	3fe8000000000002
2	0.750000000000000	3fe7fffffffffffe
3	0.750000000000001	3fe8000000000006
4	0.749999999999999	3fe7fffffffffff6

Table 4 – Initial computed values for the round mode to the nearest.

other two round modes, presents the wrong result at the first computed value.

However if the theorem proposed by [4] for simulation of recursive functions on computers is applied it would be possible calculate the error at each iteration, for example, the calculation error would x_1 equals $3.053113317719181 \cdot 10^{-16}$ and each iteration of this error increases because it is cumulative. Using this approach, it would be possible to interrupt the simulation for the two modes that do not converge to the right value.

To ensure that the simulation should be stopped is observed the hexadecimal value. From Tab. 3 and 4 you can see that the values are different. However, this difference is less than the ulp (unit in last place) which equals $22.2204460492503131 \cdot 10^{-16}$ for the standard 64-bit double. One way to confirm this hypothesis is that if the difference was greater than that ulp in module the value in decimal would be equal to 0.750000000000001 or 0.749999999999999.

As the accumulated error is greater than the distance between x_2 and x_1, remembering that the errors arising from the calculation of x_1 and x_2 were not considered, we must stop the simulation and take the value obtained in the third iteration as answer.

6. CONCLUSION

This short note presents the relevance of taking into account the method of rounding. The majority of software use the round mode to the nearest. It is clear that it is the best solution when one noticing only one arithmetic operation or when it is used to store a real number. However, this is not the case when the most important thing in the computation is a function. There is no easy to way to establish a correct round mode to a function, which in our opinion deservers further research.

n	x_n (decimal)	x_n (hexadecimal)
0	0.250000000000000	3fd0000000000000
1	0.750000000000000	3fe8000000000000
2	0.750000000000000	3fe8000000000000
3	0.750000000000000	3fe8000000000000
4	0.750000000000000	3fe8000000000000

Table 5 – Initial computed values for the round mode towards $-\infty$.

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