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CONDITIONAL LYAPUNOV EXPONENTS FOR IZHIKEVICH NEURONAL MODEL: PRELIMINARY RESULTS

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Abstract: The Izhikevich neuronal model has been adopted as an important benchmark in theoretical neuroscience, and exhibits one of the best relations between biological plausibility and computational cost. The present work aims to analyze the synchronization between unidirectional coupled Izhikevich neurons by means of the conditional Lyapunov exponent evaluation. In order to overcome the analytical difficulties imposed by its discontinuous structure, we apply a saltation matrix theory to the variational equations. The generalization of these results for bidirectional couplings in electrical and chemical contexts stand as natural perspective of this work.

keywords: Conditional Lyapunov Exponents, Discontinuous Dynamical Systems, Izhikevich model, Nonlinear Systems and Neural Dynamics, Synchronization in Nonlinear Systems.

1. INTRODUCTION

Due to its simplicity and versatility, the Izhikevich (IZ) neuronal model has been recently adopted as a main benchmark unit in many studies in theoretical neuroscience, ranging from isolated cells to complex realistic neuronal networks or even in the neural fields context. In fact, it has been shown that the IZ model exhibits one of the best relations between biological plausibility and computational cost [1]. Despite of that, just recently, a series of works [2, 3] have been devoted to rigorously analyze the model in the light of Lyapunov exponents, which is extremely important for identifying chaotic behavior or even the amount of information generated by its state equations, attractors topological structure, etc. In particular, such works have overcome the analytical difficulties imposed by its discontinuous structure by applying saltation matrix theory to the variational equations, as elegantly exposed in [4]. Having these ideas in mind, the present work aims to analyze the stability of the synchronism of unidirectional coupled IZ neurons by means of the conditional Lyapunov exponent evaluation, which implies in mapping both neurons discontinuities to a new set of state equations obtained after the classical coordinate transforms for defining the error between the emitter and receiving neurons. The results obtained here illustrate the effect of coupling increasing on the synchronism stability, which not necessarily monotonically favors the synchronous state and tends to be stronger after a critical coupling value.

This work is organized as it follows: section 2 presents the conditional Lyapunov exponent evaluation approach. Section 3 presents the obtained results and section 4 presents the discussion, conclusions and perspectives.

2. CONDITIONAL LYAPUNOV EXPONENTS FOR COUPLED IZHIKEVICH NEURONS

The Lyapunov exponents, in its classical form, can be evaluated by means of the divergent or convergent behavior obtained under the application of tangent map to a set of orthogonal vectors that spans all the state space. This can be described by [5, 6]:

$$\begin{aligned}\dot{\mathbf{\Phi}}(\mathbf{x},t) &= \mathbf{J}(\mathbf{x},t)\mathbf{\Phi}(\mathbf{x},t)\\ \mathbf{\Phi}(\mathbf{x},t_0) &= \mathbf{I}_N\end{aligned} \tag{1}$$

in which $\mathbf{\Phi}(\mathbf{x}, t)$ is the tangent space underlying the dynamical system, $\mathbf{J}(\mathbf{x}, t)$ is the jacobian of dynamical system and $\mathbf{I}_N$ is the $N \times N$ identity matrix, being N the system order. Whenever a discontinuity in the form of a Filippov type occurs [4] in t_1, the variational needs to be corrected by an appropriate saltation matrix S in the form:

$$\mathbf{\Phi}(\mathbf{x}, t) = \mathbf{\Phi}_2(\mathbf{x}, t_1) S \mathbf{\Phi}_1(\mathbf{x}, t_1) \tag{2}$$

in which $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are, respectively, the tangent space defined immediately before and after the discontinuity. According to [2], the saltation matrix is given by:

$$\begin{aligned} S &= M_x(\mathbf{x}^-) + ... \\ &... + \frac{\left[f_2\left(M(\mathbf{x}^-)\right) - M_x(\mathbf{x}^-) f_1(\mathbf{x}^-)\right] \nabla^T h|_{(\mathbf{x}^-, t_1)}}{\nabla^T h|_{(\mathbf{x}^-, t_1)} f_1(\mathbf{x}^-) + \frac{\partial h}{\partial t}|_{(\mathbf{x}^-, t_1)}} \end{aligned} \tag{3}$$

in which $\mathbf{x}^-$ is the state just before the discontinuity; $M(.)$ is the state mapping at the discontinuity, e.g. $\mathbf{x}^+ = M(\mathbf{x}^-)$; $M_x(.)$ is the jacobian of $M(.)$; $f_1(.)$ and $f_2(.)$ refer to the state mapping before and after the discontinuity; $h(\mathbf{x}, t)$ is an event manifold that indicates if a state $\mathbf{x}(t)$ belongs to the $\Sigma(t)$ - a surface that splits the smooth vector fields defined by f_1 and f_2 - whenever $h(\mathbf{x}, t) = 0$.

After obtaining, and, eventually correcting $\mathbf{\Phi}(\mathbf{x}, t)$, the system is allowed to time-evolution for an interval T, being the tangent map applied to a set of independent vector $\boldsymbol{\delta}_{1x}^{(1)} = \mathbf{\Phi}(\mathbf{x}, t)\mathbf{u}_1^{(0)}$, with $\mathbf{u}_1^{(0)} = \boldsymbol{\delta}_{1x}^{(0)} / \|\boldsymbol{\delta}_{1x}^{(0)}\|$ to evaluate the divergent (or convergent) behavior of perturbations related to the fiducial trajectory. The Gram-Schmidt Reorthonormalization (GSR) is then applied to a new set of vector $\{\mathbf{v}_1^{(1)}, \mathbf{v}_2^{(1)}, ..., \mathbf{v}_n^{(1)}\}$ and its normalized version $\{\mathbf{u}_1^{(1)}, \mathbf{u}_2^{(1)}, ..., \mathbf{u}_n^{(1)}\}$ from $\{\boldsymbol{\delta}_{1x}^{(1)}, \boldsymbol{\delta}_{2x}^{(1)}, ..., \boldsymbol{\delta}_{nx}^{(1)}\}$. The algorithm is repeated K times, and for K large enough, the i^{th} (for $i = 1, ..., N$) Lyapunov exponent can be obtained by [5]:

$$\lambda_i = \lim_{K \to \infty} \frac{1}{KT} \sum_{k=1}^{K} \ln \left\| \mathbf{v}_n^{(k)} \right\| \tag{4}$$

The described approach with saltation matrix corrections has been used in [2] for computing the whole set of Lyapunov for the original Izhikevich model. Here, we aim to expand this strategy also for computing the conditional Lyapunov exponents for coupled neurons with resetting conditions. To accomplish this task, let us consider the IZ model given by:

$$\begin{aligned} \dot{v}_1 &= 0.04 v_1^2 + 5 v_1 + 140 - u_1 + I_1 \\ \dot{u}_1 &= a\left(b v_1 - u_1\right) \end{aligned} \tag{5}$$

Under the resetting condition, i.e., when $v_1 \geq 30$ [mV], then:

$$\begin{cases} v_1 = c \\ u_1 = u_1 + d \end{cases} \tag{6}$$

with constants a, b, c and d. I_1 defines a bias excitation. Moreover, consider a second identical neuron electrically and unidirectionally coupled to the first in the form:

$$\begin{aligned} \dot{v}_2 &= 0.04 v_2^2 + 5 v_2 + 140 - u_2 + I_2 + \alpha_{1,2}\left(v_1 - v_2\right) \\ \dot{u}_2 &= a\left(b v_2 - u_2\right) \end{aligned} \tag{7}$$

under the same resetting condition, with $\alpha_{1,2}$ defining the coupling conductance from neuron 1 to 2. In this case, the divergent behavior of the neurons can be defined by an error system by means of the transformation $e_1 = v1 - v_2$ and $e_2 = u_1 - u_2$, which leads to the following state equations:

$$\begin{aligned} \dot{e}_1 &= 0.04\left[v_1^2 - \left(v_1 - e_1\right)^2\right] + 5 e_1 - e_2 + I_1 - I_2 - \alpha_{1,2} e_1 \\ \dot{e}_2 &= a\left(b e_1 - e_2\right) \end{aligned} \tag{8}$$

In this case, whenever one (or both) of the neurons resets, such introduced membrane potential discontinuity is mapped onto the error system, giving rise to a specific saltation matrix to be applied for the variational system correction. In fact, if one of the neurons resets, the error system will exhibit a specific discontinuity to be capture by the saltation matrix approach. For instance, when neuron 1 resets, such transition can be mapped onto the error system in the form:

$$\begin{aligned} e_1 &= c - v_2 = c - v_1 + e_1 \\ e_2 &= u_1 + d - u_2 = e_2 + d \end{aligned} \tag{9}$$

for which the jacobian is given by:

$$\mathbf{J}_{e_1, e_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{10}$$

with an event manifold $h(\mathbf{e}, t) = 30 - (v_1 - e_1) = 0$, which provides $\nabla h\left(\mathbf{e}, t\right) = [1,\ 0]$. In this case, the saltation matrix S can be obtained by:

$$\begin{aligned} S &= \mathbf{I}_{2\times2} + \frac{\left[\begin{bmatrix} \dot{e}_1^+ \\ \dot{e}_2^+ \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{e}_1^- \\ \dot{e}_2^- \end{bmatrix}\right]\begin{bmatrix} 1 & 0 \end{bmatrix}}{\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{e}_1^- \\ \dot{e}_2^- \end{bmatrix}} \\ &= \mathbf{I}_{2\times2} + \begin{bmatrix} \frac{\dot{e}_1^+ - \dot{e}_1^-}{\dot{e}_1^-} & 0 \\ \frac{\dot{e}_2^+ - \dot{e}_2^-}{\dot{e}_1^-} & 0 \end{bmatrix} \end{aligned} \tag{11}$$

It is also possible to proof that if neuron 2 resets, the same S matrix presented on xx is obtained. Finally if both neurons resets, the discontinuities mapped on the error system leads to the following S:

$$S = \mathbf{I}_{2\times2} + \begin{bmatrix} \frac{\dot{e}_1^+}{\dot{e}_1^-} & 0 \\ \frac{\dot{e}_2^+ - \dot{e}_2^-}{\dot{e}_1^-} & 0 \end{bmatrix} \tag{12}$$

In the following, it is shown the results concerning the obtained conditional Lyapunov exponents on different coupling scenarios.

3. RESULTS

In all following simulations $a = 0.2$, $b = 2$, $c = -56$, $d = -15$, $I_1 = I_2 = -99$. Euler integration method was used with step time equal to 0.01.

Figures 1 and 2 show, respectively, the time series and both Lyapunov exponents when the coupling $\alpha_{1,2} = 0.0$, i.e. when the neurons operate independent from each other. As it was expected, there is no synchronous behavior, which is clear from the obtained membrane potentials time course and also from the proposed error measure ($e_f = \sqrt{e_1^2 + e_2^2}$) which is considered here aiming to better describe the synchronism. In this case, complete synchronism is only attained when $e_f \to 0$ for $t \to \infty$. The nonsynchronous state is corroborated by the positive conditional Lyapunov exponent obtained in such condition as shown in Figure 2.

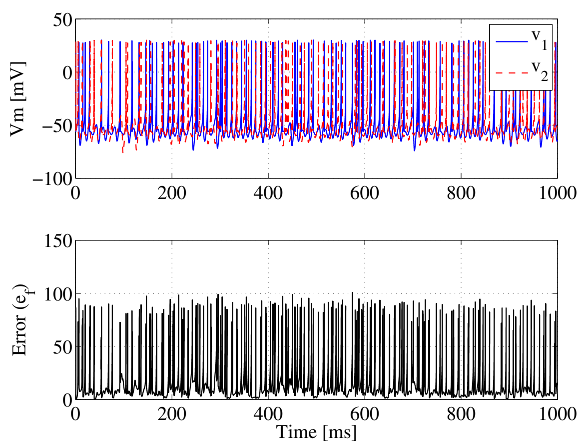

Figure 1 – (Upper) Membrane potentials v_1 and v_2 for $\alpha_{1,2} = 0.0$. (Lower) e_f for the neurons.

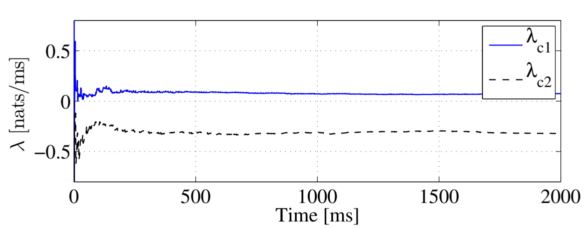

Figure 2 – Time course of both conditional Lyapunov exponents for $\alpha_{1,2} = 0$.

When the parameter $\alpha_{1,2}$ increases to 0.2, it is possible to observe that neuron 1 tends to establish the behavior of neuron 2 (Figure 3 - Upper), implying in longer time intervals in which $e_f = 0$ (Figure 3 - Lower), although complete synchronism is not attained as indicated for error peaks and also by the positive conditional Lyapunov exponent (Figure 4).

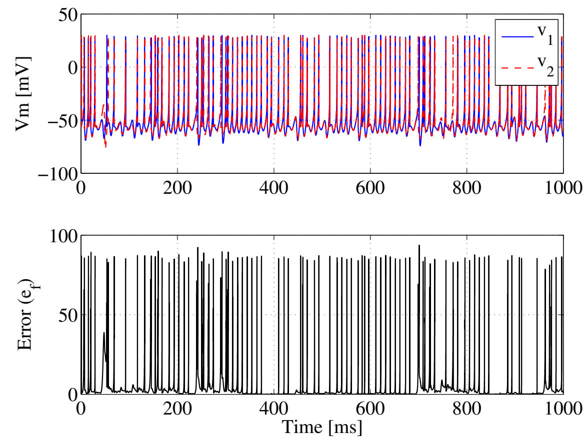

Figure 3 – (Upper) Membrane potentials v_1 and v_2 for $\alpha_{1,2} = 0.2$. (Lower) e_f for the neurons.

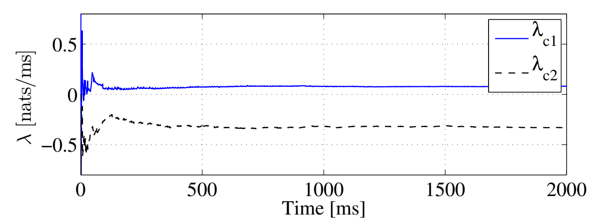

Figure 4 – Time course of both conditional Lyapunov exponents for $\alpha_{1,2} = 0.2$.

Finally, when $\alpha_{1,2} = 0.5$, the synchronous state is attained by the neurons membrane potentials (Figure 5 - Upper), converging to null e_f in steady-state (Figure 5 - Lower) and also leading to negative conditional Lyapunov exponents, as it should be.

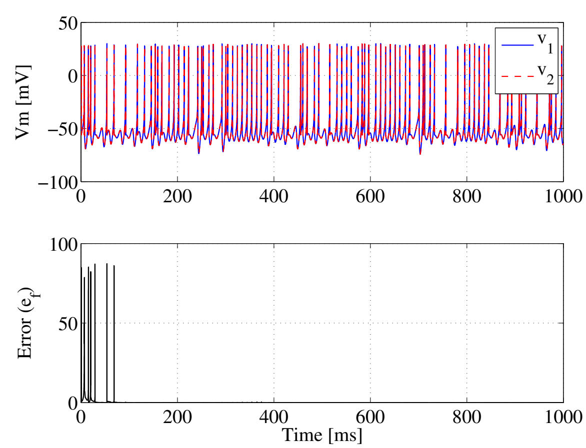

Figure 5 – (Upper) Membrane potentials v_1 and v_2 for $\alpha_{1,2} = 0.5$. (Lower) e_f for the neurons.

Figure 7 shows a detailed bifurcation diagram with the respective maximum conditional Lyapunov exponent (λ_{c1}) when $\alpha_{1,2}$ is increased in steps of 0.005 from 0 to 0.5.

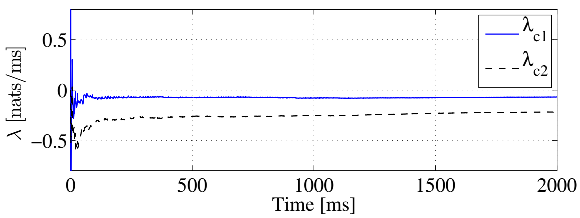

Figure 6 – Time course of both conditional Lyapunov exponents for $\alpha_{1,2} = 0.5$.

Clearly, the bifurcation diagram (Upper) - built on sampling e_1 points whenever $|e_2| = 0.01$ - is in agreement with the maximum evaluated Lyapunov exponent, providing a better insight of how coupling increasing affects the synchronism between the neurons. In this case, it is possible to realize a non-monotonic behavior with peak around $\alpha_{1,2} = 0.15$ and 0.2, which is probably related to a first iteration of the emitter information with the original chaotic nature of the receiver neuron, which, after that, seems to receive significant information for attaining synchronism. After $\alpha_{1,2} = 0.37$ the maximal conditional Lyapunov exponent turns to be negative, with quite few spikes errors between the neurons, which leads to a few sampled points on the bifurcation diagram. After $\alpha_{1,2} = 0.45$, even these spiked disappear, leading to no sampling points.

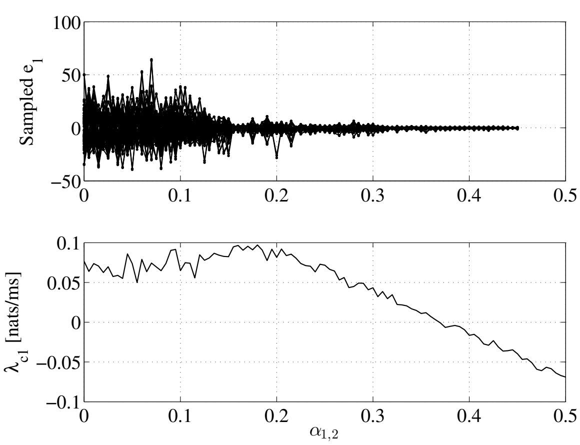

Figure 7 – (Upper) Bifurcation diagram obtained by taking $\alpha_{1,2}$ as the control parameter. (Lower) Maximal conditional Lyapunov exponent.

4. CONCLUSIONS AND DISCUSSIONS

In this work we have employed the saltation matrix approach [2–4] in order to compute the conditional Lyapunov exponents for electrically and unidirectionally coupled IZ neurons as performed for a single neuron in [2]. The obtained results seem coherent with the behavior exhibited by the error between the neurons and widens the perspective for study how neurons exchange information and the possible role of a critical coupling value for complete neuronal synchronization and its biological implications. Currently, studies have been conducted for a better characterization of bidirectional coupling in electrical and chemical conditions with the aid of GPU computing [7], also aiming a better understand of how the synchronism in these conditions can affected by bias mismatch between the neurons.

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