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ORDER-CHAOS-ORDER TRANSITION IN A SPRING PENDULUM

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Abstract: The dynamics of a spring pendulum varies according to its total energy and a parameter, called f, that accounts for its physical characteristics: pendulum mass, spring stiffness constant and spring unstretched length. We build the Poincaré sections of the system and analyze the order-chaos-order transition that it undergoes as we change the control parameter f.

keywords: Analysis and Control of Non-linear Dynamical Systems with Practical Applications, Bifurcation Analysis and Applications, Chaos and Global Nonlinear Dynamics.

1. INTRODUCTION

The spring pendulum (also known as extensible pendulum) is a paradigm for low-dimensional Hamiltonian systems with nonlinear characteristics, and thus interesting phenomena like the transition to chaos as we present in this work. It was introduced by Vitt and Gorelik in 1933 [1] and has since been extensively studied in the literature from the 70's onward [2–6]. Several important features of this system's dynamics have been studied, such as the Lyapunov exponents of its chaotic trajectories [6, 7], power spectra and the applicability of the KAM theorem [7] and approximate analytical solutions [4].

Besides this vast number of interesting dynamical features, the spring pendulum is also relevant due to its qualitative representation of several systems of great physical interest. Between these, some examples are the orbits of celestial bodies, such as satellites (both natural and artificial) and asteroids, and the classical analogue of vibrational modes of triatomic molecule.

Although the behavior of individual trajectories has been extensively studied, and their dynamics is fairly well understood today, little is known about the general behavior of the spring pendulum. In this work we present one small part of what can be found by integrating the non-linear equation of motion.

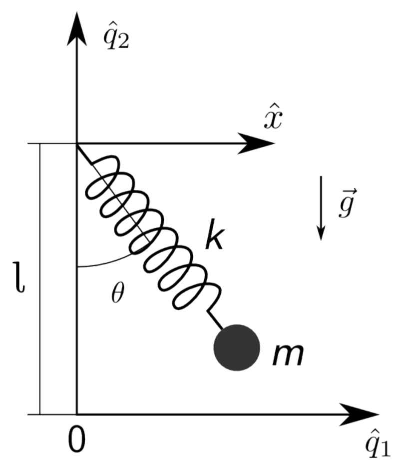

Figure 1 – Schematic diagram of an extensible pendulum.

2. THE EQUATIONS OF MOTION

A mass m attached to a spring with stiffness constant k and rest length l_0 in a vertical plane, with one edge of the spring fixed at the center of the Cartesian coordinate system $(x, y) = (0, 0)$ and free to move in the (x, y) plane only, we have a system denoted by extensible pendulum, as depicted in figure 1.

The stable equilibrium position of this system is $(x, y) = (0, -l)$, where $l = l_0 + mg/k$ corresponds to the equilibrium position, and g to the acceleration of gravity. In the Cartesian coordinates, as written in Ref. [7], the Hamiltonian H of the spring pendulum is given by

$$E_T = H(x, y, p_x, p_y) = \frac{p_x^2 + p_y^2}{2m} + mgy + \frac{k}{2}\left(\sqrt{x^2 + y^2} - l_0\right)^2, \quad (1)$$

where E_T is the total energy of the system.

Using the dimensionless quantities $H/kl^2 \rightarrow H$,

$x/l \rightarrow\ q_1$, $(y+l)/l \rightarrow q_2$, $\sqrt{k/m}t \rightarrow t$, $f = mg/kl$, we move the origin of the coordinate system to the stable equilibrium position and also write the dimensionless polar coordinate system

$$\rho = f - 1 + \sqrt{q_1^2 + (q_2-1)^2}, \tag{2}$$

$$\theta = \arctan\frac{q_1}{1-q_2}, \tag{3}$$

where ρ is the radial displacement. Therefore, we rewrite the Hamiltonian (1) into polar coordinates as

$$E_T = H(\rho,\theta,p_\rho,p_\theta) = \frac{1}{2}\left[p_\rho^2 + \frac{p_\theta^2}{(\rho+1-f)^2}\right] + \\ + \frac{\rho^2}{2} - f(\rho+1-f)\cos\theta \tag{4}$$

Despite of the richness of the results obtained under such constrictive conditions, most of the interesting features are displayed when the equations of motion are integrated numerically using Cartesian and polar coordinates systems for the complete set of values of energy and control parameter f, as showed by van der Weele[8].

By having two degrees of freedom, the spring direction (ρ) and the pendulum direction (θ), and only one constant of motion (the total energy E_T), we deal with a non-integrable system that exhibit a chaotic behavior. In this work, we use polar coordinate system to show the order-chaos-order transition.

3. ORDER-CHAOS-ORDER WITH POINCARÉ MAPS

Depicted in figure 2, the Poincaré sections show the (θ, p_θ) plane at the moments when the spring pendulum passes at the equilibrium position $(\theta,\rho) = (0, f)$, with positive angular momentum $p_\rho \geq 0$. We choose the same value of energy as reported in the work Ref.[9], $E = 0.275$, to illustrate some aspects of the system's energy distribution presented in that work.

Starting from (a) with $f = 0.01$, we see the well known eye shaped phase-space portrait characteristic of simple pendulum, but a bit tilted if compared to the usual phase portrait of the simple pendulum. We got closed orbits, which denote oscillatory motion, and also open curves for pivotal rotation, but no chaos. By raising f of a factor of 10, $f = 0.1$, from the figure 2(b), we see the presence of the chaos among with two main structures: four star shaped islands and other two regular islands and open curves. Increasing the parameter to $f = 0.425$ we see on the figure 2(c) a rearrangement of the islands around the center $(\theta, p_\theta) = (0,0)$, even as the formation of a new two island chain. Noticeable, the system is confined to the pendulum and spring motion, but no rotation around the pivot since θ lays inside $[-2, 2]$ interval. Going further on increasing the control parameter to $f = 0.9$, the islands distribute uniformly around the center $(\theta, p_\theta) = (0,0)$ following the closed orbits and the center of two island chain go near to the edge.

4. CONCLUSION

The dynamics of a spring pendulum changes according to its total energy E_T and the control parameter f, that accounts for the pendulum mass, spring stiffness constant and spring unstretched length. We built the Poincaré sections of the system for a fixed value of energy and analyzed the order-chaos-order transition that it undergoes as we have changed the control parameter f.

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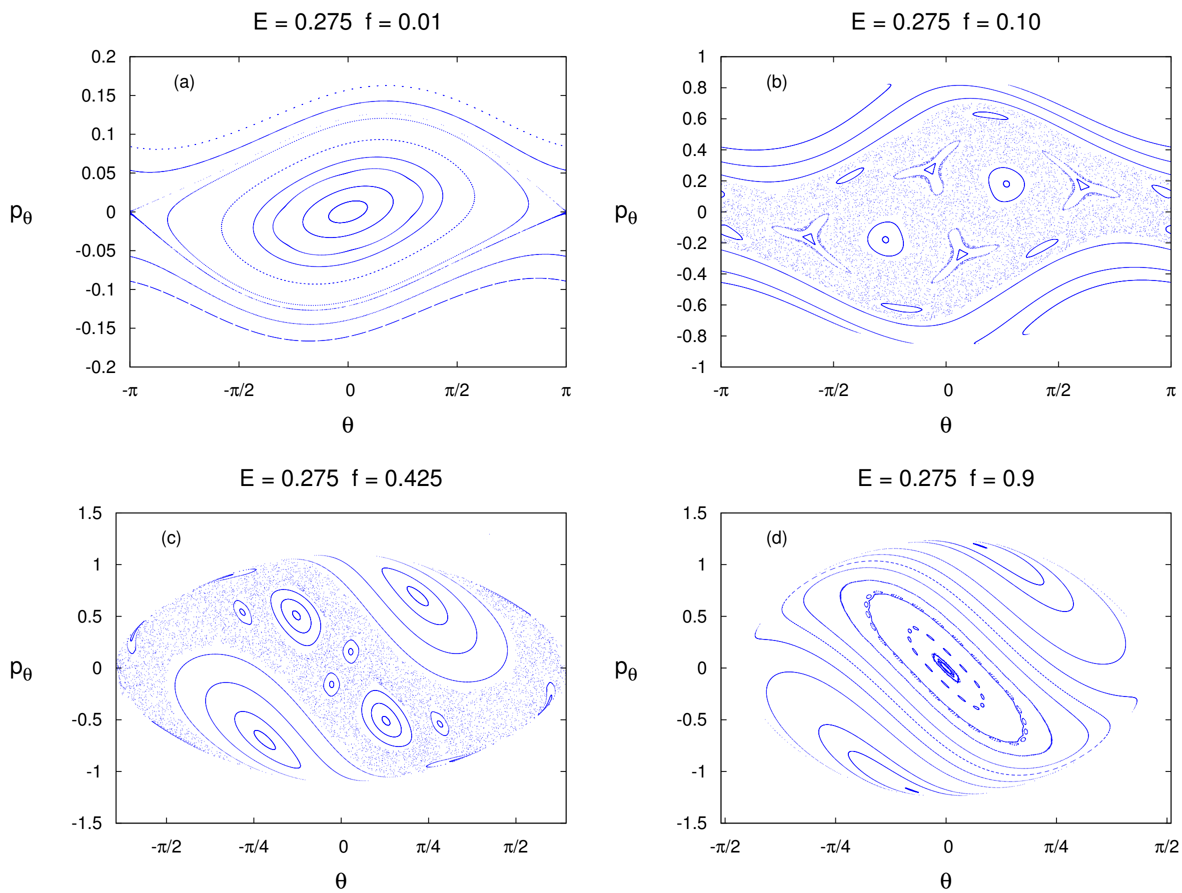

Figure 2 – Order-chaos-order displayed by four Poincaré maps with $E = 0.275$. With four different values for the control parameter, (a) $f = 0.01$, (b) $f = 0.1$,(c) $f = 0.425$,(d) $f = 0.9$, we see the transition from order to chaos and then back to order, but displaying a different configuration for the higher value of f.