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ENERGY DISTRIBUTION IN A SPRING PENDULUM

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Abstract: We investigate how the energy distribution of a spring pendulum varies as a function of its parameters. We split the total energy of the system into three terms: spring, pendulum and coupling. We show numerically that the energy distribution follows a well defined pattern. The maximum values of the spring and coupling energy terms vary regularly with the parameters, as well as the minimum values of the pendulum energy term.

keywords: Chaos and Global Nonlinear Dynamics, Time Series Analysis, Analysis and Control of Nonlinear Dynamical Systems with Practical Applications, Spring Pendulum, Energy Distribution.

1. INTRODUCTION

The spring pendulum is a rather simple mechanical system that presents a complex behavior, including resonances and chaos. It is a paradigm for the study of nonlinear Hamiltonian systems with applications in different areas of Physics, such as celestial mechanics [1–3], plasma physics [4, 5] and nonlinear optics [4, 6].

The spring pendulum has been extensively studied in the literature and several of its characteristic are well known [1, 7–12]. However, these studies focus more on the dynamics of individual trajectories, especially for the cases of small oscillations, than in its general behavior. Some insight into the general dynamics of the spring pendulum can be found in Ref. [13], where we discuss how the system undergoes a transition to chaos and then back to order as we increase the values of its parameters.

In this paper, we investigate the dynamics of the spring pendulum as a whole and we determine some statistical properties of the system. We consider that the total energy of the system is divided into three different terms that we call *spring*, *pendulum* and *coupling*. We analyze how this energy distribution varies according to the parameters of the system and we observe that it presents a well defined pattern. The maximum in the spring and coupling energy terms coincides with the minimum value of the pendulum energy term, and the position of these maxima and minima is a regular function of the parameters of the system.

The paper is organized as follows: In Section 2, we describe the spring pendulum and we write its Hamiltonian. In Section 3, we discuss how the total energy distributes itself among the energy terms we define. In Section 4, we present our conclusions.

2. DESCRIPTION OF THE SPRING PENDULUM

Suppose a mass m attached to a spring with stiffness constant k and unstretched length l_0. The other extremity of the spring is fixed at the center of the Cartesian coordinate system $(x, y) = (0, 0)$, and the spring pendulum is allowed to move only in the xOy plane. The stable equilibrium position of this system corresponds to $(x, y) = (0, -l)$, where $l = l_0 + mg/k$ and g is the free-fall acceleration.

In the Cartesian coordinates, the Hamiltonian of the spring pendulum is given by

$$\begin{aligned} E_T = H(x, y, p_x, p_y) =& \frac{p_x^2 + p_y^2}{2m} + mgy \\ &+ \frac{k}{2}(\sqrt{x^2 + y^2} - l_0)^2, \end{aligned} \tag{1}$$

where E_T is the total energy of the system.

Using the dimensionless quantities $H/kl^2 \to H$, $x/l \to x$, $(y + l)/l \to y$, $\sqrt{k/m}\, t \to t$, $f = mg/kl$, and the dimensionless coordinate system

$$\begin{aligned} \rho &= f - 1 + \sqrt{x^2 + (y-1)^2}\,, \\ \theta &= \arctan\frac{x}{1-y}\,, \end{aligned} \tag{2}$$

we rewrite Hamiltonian (1) as

$$\begin{aligned} E_T = H(\rho, \theta, p_\rho, p_\theta) =& \frac{1}{2}\left[p_\rho^2 + \frac{p_\theta^2}{(\rho + 1 - f)^2}\right] + \frac{\rho^2}{2} \\ &- f(\rho + 1 - f)\cos\theta\,, \end{aligned} \tag{3}$$

where ρ represents the spring extension or compression from its unstretched length and θ is the angle formed between the mass m and the vertical axis pointing down.

3. ENERGY DISTRIBUTION

In this paper, we consider that the total energy E_T of the spring pendulum is distributed among three terms. The energy E_S corresponds to the energy associated with the spring. E_P is the energy associated with the pendulum, and E_C is the energy representing the coupling between the spring and the pendulum, such that

$$E_T = E_S + E_P + E_C. \tag{4}$$

The spring pendulum described by Hamiltonian (3) presents just two parameters, E_T and f. For each value of these parameters, we calculate the average energy terms in the time interval $t = [0, 250]$ for 3264 initial conditions evenly distributed throughout the phase space. We normalize the average energies for each component (spring, pendulum and coupling) to the interval [0, 1] according to the following expressions:

$$\begin{aligned} |\overline{E}_S|_{normalized} &= \frac{|\overline{E}_S|}{|\overline{E}_S| + |\overline{E}_P| + |\overline{E}_C|}, \\ |\overline{E}_P|_{normalized} &= \frac{|\overline{E}_P|}{|\overline{E}_S| + |\overline{E}_P| + |\overline{E}_C|}, \\ |\overline{E}_C|_{normalized} &= \frac{|\overline{E}_C|}{|\overline{E}_S| + |\overline{E}_P| + |\overline{E}_C|}. \end{aligned} \tag{5}$$

Figure 1 shows how expressions (5) vary as a function of the parameter f for $E_T = 0.275$. Whenever the total energy E_T lies in the interval [0.17, 0.8], the curves described by the energy terms are similar to those represented in this figure. From Figure 1.a, we observe that the normalized average energy of the spring $|\overline{E}_S|_{normalized}$ presents a maximum that depends on f. Correspondingly, Figure 1.c shows that the maximum point of the energy $|\overline{E}_C|_{normalized}$ occurs in the same location with respect to the parameter f, while we have $|\overline{E}_P|_{normalized} = 0$ for this values of f (Figure 1.b).

From Figure 1.a, we observe that the normalized average energy of the spring is always above 0.4, while the normalized average energy of the pendulum component lies below these values (Figure 1.b). The energy of the coupling shown in Figure 1.c varies between 0.0 and 0.3 for the analyzed range of E_T.

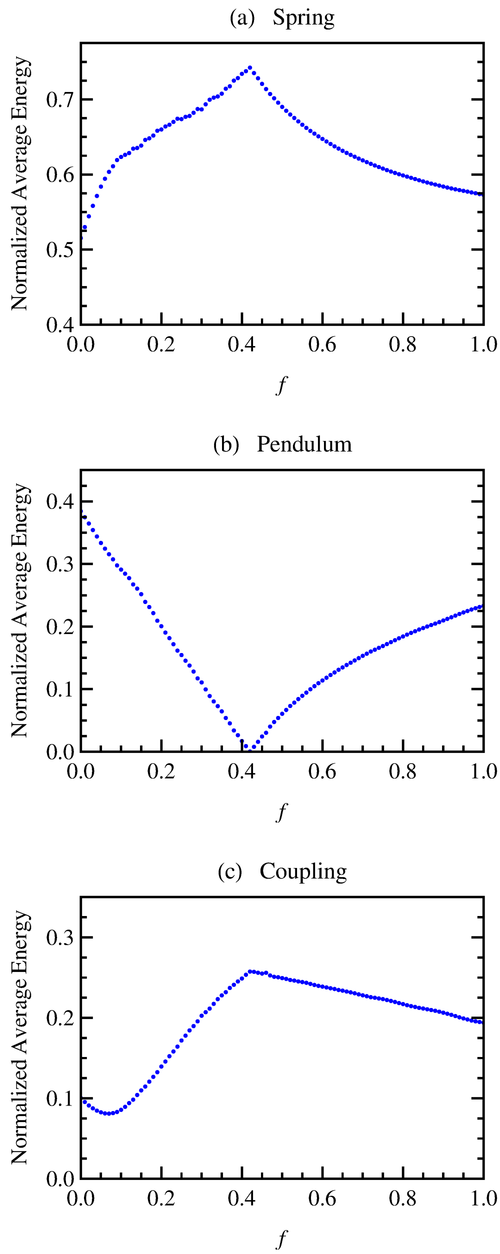

Figure 1 – Normalized average energy for the (a) spring, (b) pendulum, and (c) coupling components as a function of the parameter f for $E_T = 0.275$.

4. CONCLUSIONS

We analyzed the dynamics of a spring pendulum as a function of its dimensionless total energy and a parameter, called f, that accounts for the physical characteristics of the system, i.e., the pendulum mass, the spring stiffness constant and the spring unstretched length. We divided the total energy of the system into three different terms: spring, pendulum and coupling. When the total energy lies in the interval [0.17, 0.8], we observed that the energy terms associated with the spring and the coupling present a maximum value that varies regularly according to the total energy and the parameter f. On the other hand, the energy term that corresponds to the pendulum is minimum for the same values of the total energy and the parameter f.

These results show us that the energy distribution of a spring pendulum varies according to the configuration of the system. By adjusting its parameters properly, one finds that great part of the energy is stored in the spring, while the energy of the pendulum is very low.

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References

[1] R. Broucke and P.A. Baxa, "Periodic Solutions of a Spring-Pendulum System," Celest. Mech., Vol. 8, pp. 261, 1973.

[2] G. Contopoulos, "Resonance Cases and Small Divisors in a Third Integral of Motion, I," Astronom. J., Vol. 68, pp. 763, 1963.

[3] G. Hori, "Theory of General Perturbation with Unspecified Canonical Variables," Publ. Astron. Soc. Japan, Vol. 18, pp. 287, 1966.

[4] H.N. Núñez-Yépez, A.L. Salas-Brito, C.A. Vargas, and L.Vicente, "Onset of chaos in an extensible pendulum," Phys. Lett. A, Vol. 145, pp. 101, 1990.

[5] R.Z. Sagdeev and A. A. Galeev, edited by T.M. O'Neil and D.L. Book, "Nonlinear Plasma Theory," (Benjamin, New York, 1969, Chap. I).

[6] J.A. Armstrong, N. Bloembergen, J. Ducuing, and P.S. Pershan, "Interactions Between Light Waves in a Nonlinear Dielectric," Phys. Rev., Vol. 127, pp. 1918, 1962.

[7] A. Vitt and G. Gorelik, "Oscillations of an Elastic Pendulum as an Example of the Oscillations of Two Parametrically Coupled Linear Systems," J. Tech. Phys., Vol. 3, pp. 294, 1933.

[8] D.L. Hitzl, "The Swinging Spring - Approximate Analyses for Low and Very High Energy II," Celest. Mech., Vol. 12, pp. 359, 1974.

[9] H.N. Núñez-Yépez, A.L. Salas-Brito, C.A. Vargas, and L. Vicente, "Chaos in a Dripping Faucet," Eur. J. Phys., Vol. 10, pp. 99, 1989.

[10] E. Breitenberger and R.D. Mueller, "The Elastic Pendulum: A Nonlinear Paradigm," J. Math. Phys., Vol. 22, pp. 1196, 1981.

[11] R. Carretero-González, H.N. Núñez-Yépez ,and A. L. Salas-Brito, "Regular and Chaotic Behaviour in an Extensibe Pendulum," Eur. J. Phys., Vol. 15, pp. 139, 1994.

[12] R. Cuerno, A.F. Rañada, and J.J. Ruiz-Lorenzo, "Deterministic Chaos in the Elastic Pendulum: A Simple Laboratory for Nonlinear Dynamics," Am. J. Phys., Vol. 60, pp. 73, 1992.

[13] F.A. Marcus, M.C. de Sousa, and I.L. Caldas, "Order-Chaos-Order Transition in a Spring Pendulum," to appear in the Proceedings of the 6^{th} International Conference on Nonlinear Science and Complexity, São José dos Campos, May 2016.