INPE – National Institute for Space Research
São José dos Campos – SP – Brazil – May 16-20, 2016

STOCHASTIC DYNAMICS WITH MULTIPLICATIVE NOISE: AN ANALYSIS ON TIME REVERSIBILITY

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Abstract: In this work, we study equilibrium properties of stochastic differential equations with multiplicative noise. We use a general prescription α for considering the stochastic integration and we also represent the stochastic process in a functional Grassmann formalism. We carefully define a time reversal transformation taking into account that the asymptotic stationary probability distribution depends on the prescription. We show that, using a careful definition of equilibrium distribution and the appropriate time reversal transformation, usual equilibrium properties are satisfied for any prescription.

keywords: Stochastic Models; Nonlinear Dynamics and Complex Systems; Modeling, Numerical Simulation and Optimization

1. INTRODUCTION

Stochastic differential equations and its applications are a subject of great interest for scientific research. In this area, Langevin and Fokker-Planck formalisms are extensively used. Systems modeled by differential stochastic equations with additive noise have been largely studied and are the most popular models. However, the understanding of the stochastic dynamics and the evolution to equilibrium for systems dealing with multiplicative noise is difficult and there is a lack of general tools for its characterization. In particular, for the multiplicative noise case, the Fokker-Planck equation does depend on the chosen prescription for the stochastic integration of the associated Langevin equation.

In this work [1–3], we study equilibrium properties of stochastic differential equations with multiplicative noise. We use the Langevin equation (1) as a prototype of these processes. Due to the Gaussian white-noise distribution, the continuum limit of the discretized time evolution is not unique. In fact, there are different prescriptions to perform this limit, and the asymptotic stationary probability distribution depends on the prescription. Moreover, stochastic trajectories commonly evolve with different prescriptions in one direction and in the reverse one. Bearing that in mind, we carefully define the time reversal transformation for this kind of processes. In this context, we show that usual equilibrium properties are satisfied for any prescription.

2. MULTIPLICATIVE WHITE–NOISE STOCHASTIC EVOLUTION

We consider a single random variable $x(t)$ satisfying a first order differential equation (Langevin equation) given by

$$\frac{dx(t)}{dt} = f(x(t)) + g(x(t))\eta(t), \tag{1}$$

where $\eta(t)$ is a Gaussian white-noise, $\langle \eta(t) \rangle = 0$, $\langle \eta(t)\eta(t') \rangle = \delta(t - t')$. The drift force $f(x)$ and the diffusion function $g(x)$ are, in principle, arbitrary smooth functions of $x(t)$. This kind of equations is commonly used to describe many diffusion processes occurring in nature under some random influence.

To correctly define Eq. (1), it must be specified a prescription for performing the stochastic integration. We consider the Generalized Stratonovich convention [4] or "α-

convention" [5], where α is defined as a continuous parameter, $0 \leq \alpha \leq 1$. Each of its values corresponds with a different discretization rule for the stochastic differential equation. $\alpha = 0$ corresponds with Itô prescription at the time that $\alpha = 1/2$ corresponds with Stratonovich one.

Equilibrium is closely related with the concept of time reversal symmetry. In multiplicative white-noise processes, the forward and backward stochastic trajectories evolve with different prescriptions (except when considering Stratonovich convention). For instance, if Itô convention ($\alpha = 0$) is defined for the forward evolution, the backward trajectory evolves with the so called Hänggi-Klimontovich prescription ($\alpha = 1$). In general, we will show that the prescription α is the time reversal conjugate of $(1 - \alpha)$. In addition, since the asymptotic stationary probability distribution also depends on the chosen prescription, the correct definition of time reversal transformation, compatible with a unique equilibrium distribution, is quite involved.

Instead of the Langevin or Fokker-Planck approach, we represent the stochastic process in a functional Grassmann formalism [1, 6, 7]. Using this formalism, we analyze equilibrium properties, such as detailed balance relations, microscopic reversibility and entropy production. We show that, by using a careful definition of equilibrium distribution and taking into account the appropriate time reversal transformation, usual equilibrium properties are satisfied for *any value of* α.

3. EQUILIBRIUM DISTRIBUTION

Using equation (1) and the α-prescription, a Fokker-Planck equation is obtained [8] for the probability distribution $P(x,t)$:

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left\{-f - \alpha g g'\right\} P(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(g^2 P(x,t)\right), \tag{2}$$

where $g' = dg/dx$.

By specifying initial and boundary conditions, and defining the equilibrium state as the solution of the stationary Fokker-Planck equation with zero current probability, the probability density tends, at long times, to the equilibrium distribution

$$P_{\rm eq}(x) = N e^{-U_{\rm eq}(x)}, \tag{3}$$

with

$$U_{\rm eq}(x) = -2\int^x \frac{f(x')}{g^2(x')}dx' + (1-\alpha)\ln g^2(x). \tag{4}$$

So, the equilibrium distribution depends, not only on the given functions $(f(x), g(x))$, but also on the value of the α-prescription which defines the Wiener integral. If equation (1) is used to model a conservative physical system, *i. e.*, if the otherwise deterministic system is characterized by an energy potential $V(x)$, and which is expected to asymptotically converge to thermodynamic equilibrium, the only possible choice for the stochastic prescription is the Hänggi-Klimontovich one, $\alpha = 1$. With any other choice, the equilibrium potential $U_{\rm eq}(x) \neq V(x)$, including the usual Itô ($\alpha = 0$) and Stratonovich ($\alpha = 1/2$) prescriptions.

At the same time, except for the Stratonovich prescription $\alpha = 1/2$, each value of α is associated with different calculus rules. Despite the Hänggi-Klimontovich convention being the only one which guarantees the convergence to the thermodynamic equilibrium, calculations can be particularly cumbersome in this interpretation. In this work, we consider a stochastic process completely defined by fixing $(f(x), g(x), \alpha)$ in equation (1). Thus, the probability distribution $P(x,t)$ flows, at long times, to an α-dependent equilibrium distribution given by equation (4). In this way, we are able to study, in a unique formalism, model systems with general equilibrium distributions that go from Boltzmann thermal equilibrium to power-law distributions.

4. TIME REVERSAL

The α-prescription introduces some difficulties in the proper definition of time reversal evolution. Given the Langevin equation (1), the backward evolution of the stochastic variable $x(t)$ cannot be naively computed by just changing the sign on the velocity $\dot{x} \to -\dot{x}$. The stochastic integrals need to be carefully analyzed. After some manipulations of the stochastic integrals, it is obtained that the time reversed stochastic evolution is characterized by the transformations

$$x(t) \to x(-t) \quad \text{and} \quad \alpha \leftrightarrow (1-\alpha). \tag{5}$$

In this sense, we say that the prescription $(1 - \alpha)$ is the time reversal conjugate (TRC) of α. Then, the post-point Hänggi-Klimontovich interpretation is the TRC of the pre-point Itô one, and vice versa. The only *time reversal invariant prescription* is the Stratonovich one, $\alpha = 1/2$. This means that, except for the Stratonovich case, the backward and forward stochastic paths do not have the same end points. This is illustrated in figure (1), where we compute a "time-loop" evolution.

We see that, in general, $\Delta_\alpha x(t) \neq 0$ since the backward and forward evolutions progress with different dual prescriptions. This fact is reflected in the α-dependence of the equilibrium stationary state $U_{eq}(x)$ (equation (4)). The stationary state in the forward evolution is different from the stationary state reached by the backward one (except for the Stratonovich prescription). Thus, this is not the backward evolution we are interested in. The key point is to determine the backward stochastic process which, at long times, converges to the same equilibrium distribution as the forward one.

Working with the time reversed Fokker-Planck equation and considering appropriate transformations we finally obtained the time reversal transformation $\mathcal{T}$ which makes physical sense, meaning that it produces a backward evolution which converges to a unique equilibrium distribution. This transformation is given by

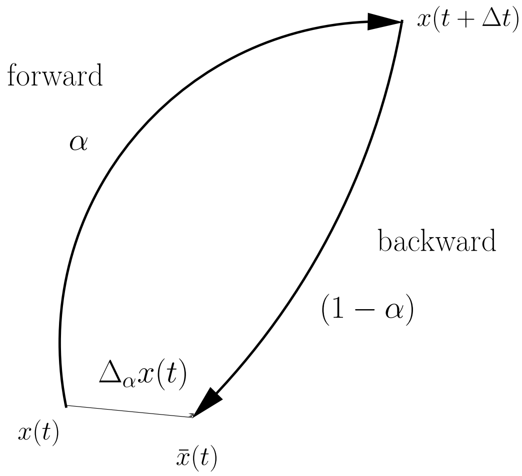

Figure 1 – Sketch of a "time-loop" evolution of Langevin equation (1). The forward evolution $t \to t + \Delta t$ is performed with a prescription α, while the backward trajectory $t + \Delta t \to t$ evolves with the time reversal conjugate prescription $(1-\alpha)$. Except for the Stratonovich convention $\alpha = 1/2$, $\Delta x(t)_\alpha \neq 0$.

$$\mathcal{T} = \begin{cases} x(t) & \to \quad x(-t) \\ \alpha & \to \quad (1-\alpha) \\ f & \to \quad f - (1-2\alpha)\; gg' \end{cases} \tag{6}$$

We checked that the concept of α-dependent equilibrium and the introduced time reversal transformation are consistent with the usual concepts of detailed balance, microscopic reversibility and entropy production. To do this, we use the path integral formalism [1, 6, 7], which turns out to be very convenient for handling stochastic trajectories.

5. CONCLUSION

We found an asymptotic equilibrium distribution by solving the stationary Fokker-Planck equation. This conduces to an equilibrium potential $U_{\rm eq}(x)$ which explicitly depends on α. Regarding time reversibility, we shown that, if the forward stochastic trajectory evolves with a definite value of α, the time reversed trajectory evolves with the conjugated prescription $(1-\alpha)$. Therefore, in order to have a unique equilibrium distribution, the definition of time reversed stochastic process is given by a specific transformation, where we need to change not only the velocity sign, but also the prescription and the drift force.

ACKNOWLEDGMENTS

The Brazilian agencies FAPERJ and CNPq are acknowledged for partial financial support.

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