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DYNAMICAL POTENTIALS FOR NON-EQUILIBRIUM STATIONARY STATES DRIVEN BY MULTIPLICATIVE STOCHASTIC PROCESSES

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Abstract: We present a study on out-of-equilibrium phase transitions induced by multiplicative noise. Based on a recently developed path integral formalism, we built up a “dynamical potential’ written in terms of an order parameter capable to describe non-equilibrium phase transitions. As an example, we applied our formalism to a particularly simple model which captures the physics of non-equilibrium phase transition. The model is defined by a set of stochastic variables arranged in a hyper-cubic lattice satisfying a system of interacting Langevin equations with multiplicative noise. We computed a potential for the non-equilibrium stationary state in the saddle-point plus Gaussian fluctuations approximation. We discovered a phase transition with reentrant behavior for sufficiently strong lattice coupling. We computed the phase diagram for different dimensions and for different values of the stochastic prescription used to define the multiplicative stochastic process. We found that, at the level of this approximation, the phase transition is continuous and we computed critical exponents. Even thought, the concept of universality is not completely developed in out-of-equilibrium transitions, the computed exponents are in the universality class of the dynamical Ising model.

keywords: Stochastic processes, multiplicative noise, non-equilibrium stationary states, phase transitions.

1. INTRODUCTION

The theory of stochastic processes [1] is one of the natural approaches to describe out-of-equilibrium statistical mechanics since there is no closed theoretical framework to deal with this type of systems. Stochastic processes with multiplicative noise often leads to stationary out-of-equilibrium states. They are characterized by the presence of probability currents and, in general, time-reversal is a broken symmetry and usual equilibrium properties, such as detailed balance, are not satisfied. In this type of stationary states, symmetry-breaking phase transitions could take place, induced by noise[2]. That is, for weak noise the stationary state is usually disordered. However, an ordered state sets in when the noise intensity is increased. There are two necessary ingredients to produce this class of phase transitions: multiplicative noise and out-of-equilibrium stationary states[3]. In this work, we present a study on out-of-equilibrium phase transitions induced by multiplicative noise. Recently, we have presented a functional formalism[4–7] to compute correlations functions in these systems. Based on that, we built up a “dynamical potential” written in terms of an order parameter capable to describe non-equilibrium phase transitions. As an example, we applied our formalism to a particularly simple lattice model, which captures the physics of non-equilibrium phase transition. We computed a “dynamical potential” for the stationary state in the saddle-point plus Gaussian fluctuations approximation. We have built up a phase-diagram in terms of the lattice interaction and the noise. We discovered a phase transition with reentrant behavior for sufficiently strong lattice coupling. We computed the phase diagram for different values of the stochastic prescription that defines the multiplicative stochastic process. At the level of this approximation we found that the phase transition is continuous and we computed critical exponents.

2. NON-EQUILIBRIUM POTENTIALS

We consider a system of Langevin equations given by

$$\frac{dx_i(t)}{dt} = f_i(\mathbf{x}(t)) + g_{ij}(\mathbf{x}(t))\eta_j(t) \ , \tag{1}$$

where $i = 1, \ldots, n$, $j = 1, \ldots, m$, $\mathbf{x} \in \Re^n$ and $\eta_j(t)$ are m independent Gaussian white noises: $\langle \eta_i(t) \rangle = 0$, $\langle \eta_i(t), \eta_j(t') \rangle = \sigma^2 \delta_{ij} \delta(t - t')$, where σ^2 measures the noise intensity. The drift force $f_i(\mathbf{x})$ and the diffusion matrix $g_{ij}(\mathbf{x})$ are, in principle, arbitrary smooth functions of $\mathbf{x}(t)$. We use the Generalized Stratonovich prescription, parametrized by a real number $0 \leq \alpha \leq 1$ [6]. Time correlation functions can be computed by performing functional derivatives of a generating functional written in terms of functional integrals, that we have recently implemented [7]. The generating functional can be written in terms of a functional integral over the vector variable $\mathbf{x}(t)$,

$$Z[\mathbf{J}] = \int \mathcal{D}\mathbf{x} \, \det{}^{-1}(g) \, e^{-\frac{1}{\sigma^2}\left\{S[\mathbf{x}] - \int_{-\infty}^{\infty} dt' \mathbf{J}\cdot\mathbf{x}(t')\right\}} \ , \tag{2}$$

where the "action" is given by

$$\begin{aligned} S = \int_{t_i}^{t_f} dt \ &\left\{ \frac{1}{2} \left[\dot{x}_\ell - \Gamma_\ell\right] [g^2]^{-1}_{\ell m} \left[\dot{x}_m - \Gamma_m\right] + \alpha\sigma^2 \partial_k f_k \right. \\ + \quad &\left. \frac{1}{2}\alpha^2\sigma^4 \left[\partial_m g_{kj}(x)\partial_k g_{mj}(x) - \partial_m g_{mj}(x)\partial_i g_{ij}(x)\right] \right\} , \end{aligned} \tag{3}$$

with $\Gamma_\ell(\mathbf{x}) = f_\ell(\mathbf{x}) - \alpha\sigma^2 g_{\ell j}(\mathbf{x})\partial_i g_{ij}(\mathbf{x})$. The second and third terms of Eq. (3), proportional to α and α^2 respectively, comes from the non-trivial Jacobian associated with the change of variables $\eta_i(t) \to x_i(t)$. Formally, $Z[\mathbf{J}(t)]$ plays the same role as the partition function in equilibrium statistical mechanics. Then, it is immediate to define the functional $F[\mathbf{J}(t)] = -\sigma^2 \ln Z[\mathbf{J}]$, from which we can compute a "local order parameter"

$$M_i(t) \equiv \langle x_i(t) \rangle = -\frac{\delta F[\mathbf{J}]}{\delta J_i(t)}. \tag{4}$$

In order to define a generating functional in terms of the order parameter, we perform a Legendre transformation in the following way,

$$G[\mathbf{M}(t)] = \int dt \ \mathbf{J}(t)\cdot\mathbf{M}(t) + F[\mathbf{J}(t)] \ , \tag{5}$$

in which $\mathbf{J}(t) \equiv \mathbf{J}[\mathbf{M}(t)]$ is obtained by inverting Eq. (4). It is immediate to verify $\delta G[\mathbf{M}]/\delta M_i(t) = J_i(t)$, that can be interpreted as the nonequilibrium state equation.

Assuming that, at long times, the system reaches an homogeneous stationary state, we can define the nonequilibrium potential as

$$G_{\rm st}(M) = \lim_{t\to\infty} \lim_{N\to\infty} \frac{1}{N} G[\mathbf{M}(t)] \ , \tag{6}$$

where N is the number of degrees of freedom and the order parameter $M = (1/N) \lim_{t\to\infty} \sum_{i=1}^N \langle x_i(t) \rangle$.

The expressions in Eqs. (5) and (6) are the main proposals of this work. The potential $G_{\rm st}(M)$ is the analog of the Gibbs free energy for describing nonequilibrium stationary states. The critical behavior of the nonequilibrium state should be codified in the analytical properties of $G_{\rm st}(M)$.

3. STOCHASTIC LATTICE MODEL IN THE WEAK NOISE EXPANSION

The functional integral can be computed in the saddle-point plus Gaussian fluctuations approximation, for small σ. Retaining the leading order terms in σ, we find the expression

$$G[\mathbf{M}(t)] = S[\mathbf{M}(t)] + \frac{\sigma^2}{2} {\rm Tr} \ln \left\{ \mathbf{S}^{(2)}[\mathbf{M}(t)] \right\} \ . \tag{7}$$

where the components of the fluctuation matrix $\mathbf{S}^{(2)}$ are given by

$$S^{(2)}_{ij}(t,t') = \left. \frac{\delta^2 S[\mathbf{x}]}{\delta x_i(t) \delta x_j(t')} \right|_{\mathbf{x(t)} = \mathbf{x}^0(t)} , \tag{8}$$

in which $x^0[J]$ minimize the exponent in (2). Eq. (7) is the explicit expression of the generating functional of the local order parameter in the Gaussian fluctuations approximation.

We computed $G[\mathbf{M}(t)]$ for a model consisting in a set of N stochastic variables whose dynamics are driven by the same drift $f(x)$ and the same diffusion function $g(x)$. Each degree of freedom is arranged in a d-dimensional hypercubic lattice and we consider short-range lattice couplings. Then, in Eq. (1), we take $f_i(\mathbf{x}) = f(x_i) + F_i(\mathbf{x})$ and $g_{ij}(\mathbf{x}) = g(x_i)\delta_{ij}$, where $F_i(\mathbf{x})$ represent the lattice couplings. In the absence of couplings, $F_i(\mathbf{x}) = 0$, the system converges, at long times, to an equilibrium state. Imposing the Einstein relation $f(x) = -(1/2)g(x)^2 dV(x)/dx$, where $V(x)$ is a classical potential, the equilibrium distribution is $P_{\rm eq} \sim \exp\{-(1/\sigma^2)U_{\rm eq}(x)\}$, with the potential $U_{\rm eq}(x) = V(x) + (1-\alpha)\ln g^2(x)$ [6]. We consider the simplest lattice interaction,

$$F_i(\mathbf{x}) = \left(\frac{D}{2d}\right) \sum_{x_j \in n(x_i)} (x_j - x_i) \ , \tag{9}$$

where $n(x_i)$ denotes the set of first neighbors of x_i, D is the coupling constant and d is the number of dimensions of the hypercubic lattice. Due to interactions, the fluctuation matrix $S^{(2)}_{ij}(t,t')$ is not diagonal, neither in time, nor in lattice indexes. For this reason, ${\rm Tr} \ln \mathbf{S}^{(2)}$ in Eq. (7) is a cumbersome evaluation. Fortunately, in the stationary and homogeneous limit the coefficients are constants and it is possible

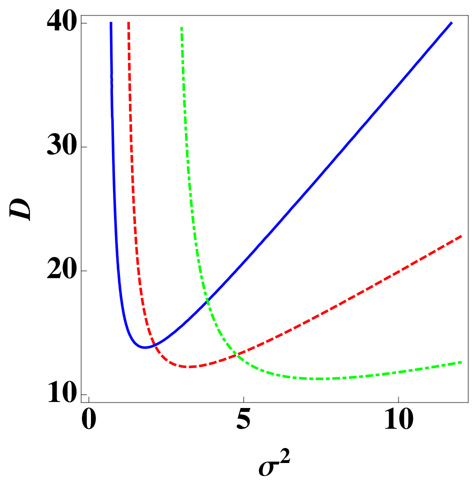

Figure 1 – Phase diagram. Critical lines are defined by $\chi_0^{-1}(\sigma^2, D) = 0$. In the exterior part of each line the system is disordered, $M = 0$, while in the interior, $M \neq 0$. Continuous, dashed and dot-dashed lines correspond to $\alpha = 0, 1/2, 0.8$, respectively. We fixed the parameters $d = 2, \Omega = 1$.

to diagonalize $\mathbf{S}^{(2)}$ by Fourier transforming in time and lattice indexes. In the long distance limit, the fluctuation kernel takes form $S^{(2)}(\omega, \vec{k}) = \omega^2 + A|\vec{k}|^2 + \Sigma$, where A and Σ are constants. Then, we can finally write the nonequilibrium potential as,

$$G_{\rm st}(M) = S_{\rm st}(M) + \frac{\sigma^2}{2}\int \frac{d\omega}{2\pi}\frac{d^dk}{(2\pi)^d}\ln\left(\omega^2 + A|\vec{k}|^2 + \Sigma\right). \tag{10}$$

where the coefficients $A(M)$ and $\Sigma(M)$ are functions of the order parameter and can be computed from the local properties of $V(x)$ and the diffusion function $g(x)$, near $x = 0$. To explore possible phase transitions, we compute the inverse susceptibility in the disordered phase, $\chi_0^{-1} = d^2G_{\rm st}(M)/dM^2\big|_{M=0}$. To be specific, we choose an harmonic oscillator potential, $V(x) = \Omega^2x^2$, and a diffusion function $g(x) = 1 + x^2$. The critical line is defined by $\chi_0^{-1}(\sigma^2, D) = 0$. In Figure (1) we plot the critical line in $d = 2$, for different values of the stochastic prescription. Above a minimum lattice coupling $D_{\rm min}$, we found two continuum phase transitions. For very weak noise the system is disordered, $M = 0$. At a threshold σ^2_{c1}, the system orders, $M \neq 0$, breaking in this way Z_2 symmetry. For higher values of the noise, $\sigma^2_{c2} > \sigma^2_{c1}$ it gets disordered again. We observe that the ordered area in the $D - \sigma^2$ plane growths with α. On the other hand, σ^2_{c1} as well as σ^2_{c2} are increasing functions of α, making the phase transition harder to reach. In fact, in the anti-Itô prescription, $\alpha = 1$, χ_0^{-1} is positive definite and there is no phase transition.

4. CONCLUSIONS

We have presented a path integral formalism to compute potentials for non-equilibrium steady states, reached at long times by multiplicative Langevin dynamics. The formalism is completely general and can be applied to study a variety of models presenting interesting features, such as noise-induced phase transitions, stochastic resonance and pattern formation. We have also developed a controlled weak noise expansion which correctly captures fluctuation-induced phenomena. In particular, we have analyzed the physics of noise induced phase transitions by computing the stationary state potential for a general class of lattice models. We believe that the results presented in this work open many interesting possibilities to advance our understanding of out-of-equilibrium critical phenomena.

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