INPE – National Institute for Space Research
São José dos Campos – SP – Brazil – May 16-20, 2016

ANOMALOUS SEA SURFACE STRUCTURES (ROGUE WAVES) AS AN OBJECT OF STATISTICAL TOPOGRAPHY

***V. I. Klyatskin*[1]**, *K. V. Koshel*[2,3]

[1]A.M. Obukhov Atmospheric Physics Institute of RAS, Moscow, Russia, klyatskin@yandex.ru
[2]V.I.Il`ichev Pacific Oceanological Institute of RAS, Vladivostok, Russia, kvkoshel@poi.dvo.ru
[3]Far Eastern Federal University, Vladivostok, Russia, koshelkv@gmail.com

Abstract: We study the stochastic problem of anomalous structures emergence on the sea surface. The kinematic boundary condition on the surface is assumed to be a closed stochastic quasi-linear equation. We derive an equation for a spatially single-point, simultaneous joint probability density of the surface elevation field and its gradient. We show the emergence of anomalous high structures on the sea surface almost in every realization of a stochastic velocity field.
keywords: Fluid dynamics, Stochastic Models, Geophysical Nonlinear Dynamics, Statistical topography, Clustering.

1. INTRODUCTION

By now, a great body of literature is concerned with anomalously high structures which are observable in a vast range of media including random ones. For instance, rogue waves in the ocean [1-4], capillary waves [5], the branching of electron flows in semiconductor devices [6, 7], extreme acoustic [8] and optical [9,10] waves. In this work, we are mainly interested in surface ocean structures. Despite significant efforts (see reviews [1,3] and with an emphasis on statistical properties [11-13]), there is still no exhaustive explanation on the way by which those structures are born and what are their main statistical properties. Partial explanations, based on the observation that such events correspond to heavy tails of distribution other than Gaussian, can be found in [14].

It is worth noting, that one of the approaches to investigating rogue waves is to consider them as structures that emerge at random places in random instants of time. The causes and mechanisms of rogue wave appearance attract the attention of many researchers (see, for example, [14-16], and references therein). Some of them analyze dynamical systems on the basis of the Schrödinger equation which can be treated analytically or numerically. Usually, when dealing with random process realizations in time, the emergence of rare but intense impulses is identified as rogue waves. This identification seems to be debatable (see monographs [17,18]).

In this work, we adhere to the idea that rogue waves can be generated as a result of clustering that appears in uniform random fields. We suggest addressing the anomalous structures on the sea surface problem based on the ideas of statistical topography [17,19-21], which describe the formation of stochastic structures in random fields [4,18]. The main goal of the paper is to study the generation of anomalous sea surface structures by making use of developed methods that have been shown as fruitful in solving stochastic problems with parametric excitation [18,20].

2. PROBLEM STATEMENT

Let us consider the water boundary elevation $z=\xi(\mathbf{R},t)$ on the sea surface as a kinematic boundary condition

$$\frac{d}{dt}\xi(\mathbf{R},t)=w(\mathbf{R},z;t)\Big|_{z=\xi(\mathbf{R},t)},$$

where $\mathbf{R}$ is the horizontal coordinates, z is the vertical coordinates and $\mathbf{U}(\mathbf{r};t)=(\mathbf{u}(\mathbf{R},z;t),w(\mathbf{R},z;t))$ is the three-dimensional velocity field with horizontal $u_\alpha(\mathbf{R},z;t)$ and vertical $w(\mathbf{R},z;t)$ components (see fig. 1).

The boundary condition can be considered as a closed stochastic quasi-linear equation within the kinematic approximation, i.e. with prescribed statistical features of the velocity fields $\mathbf{u}(\mathbf{R},z;t)$, and $w(\mathbf{R},z;t)$:

$$\frac{\partial\xi}{\partial t}+u_\alpha(\mathbf{R},\xi;t)\frac{\partial\xi}{\partial R_\alpha}=w(\mathbf{R},\xi;t) \qquad (1)$$

with initial condition $\xi(\mathbf{R},0)=\xi_0(\mathbf{R})$. We accept the Einstein notation over summation. We can obtain equation for the elevation gradient $p_\beta(\mathbf{R},t) = \dfrac{\partial \xi(\mathbf{R},t)}{\partial R_\beta}$ by differentiating (1) over $\mathbf{R}$:

$$\frac{\partial p_\beta(\mathbf{R},t)}{\partial t} + u_\alpha(\mathbf{R},\xi(\mathbf{R},t),t)\frac{\partial p_\alpha(\mathbf{R},t)}{\partial R_\beta} + \left[\left.\frac{\partial u_\alpha(\mathbf{R},z;t)}{\partial R_\beta}\right|_{z=\xi} + \frac{\partial u_\alpha(\mathbf{R},\xi(\mathbf{R},t);t)}{\partial z} p_\beta(\mathbf{R},t)\right] p_\alpha(\mathbf{R},t) = \left.\frac{\partial w(\mathbf{R},z;t)}{\partial R_\beta}\right|_{z=\xi} + \frac{\partial w(\mathbf{R},\xi(\mathbf{R},t);t)}{\partial z} p_\beta(\mathbf{R},t),$$

with initial conditions $\mathbf{p}(\mathbf{R},0) = \mathbf{p}_0(\mathbf{R}) = \dfrac{\partial \xi_0(\mathbf{R})}{\partial \mathbf{R}}$.

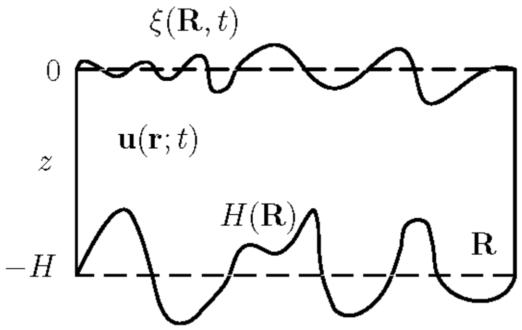

Figure 1 – Sea surface fluctuations

3. DERIVATION OF PROBABILITY DENSITY EQUATION

3.1. Liouville equation

Let us introduce a joint indicator function of surface elevation and its gradient,

$$\varphi(\mathbf{R},t;\xi,\mathbf{p}) = \delta\left(\xi(\mathbf{R},t) - \xi\right)\delta\left(\mathbf{p}(\mathbf{R},t) - \mathbf{p}\right)$$

Taking into account dynamical conditions, one can derive a linear Liouville equation [4,18],

$$\frac{\partial \varphi(\mathbf{R},t;\xi,\mathbf{p})}{\partial t} = -\frac{\partial}{\partial \xi} w(\mathbf{R},\xi;t)\varphi(\mathbf{R},t;\xi,\mathbf{p}) - \left[u_\alpha(\mathbf{R},\xi;t)\frac{\partial}{\partial R_\alpha} - \frac{\partial u_\alpha(\mathbf{R},\xi;t)}{\partial \xi} p_\alpha\right]\varphi(\mathbf{R},t;\xi,\mathbf{p}) - \frac{\partial}{\partial p_\beta}\left[\frac{\partial u_\alpha(\mathbf{R},\xi;t)}{\partial R_\beta} + \frac{\partial u_\alpha(\mathbf{R},\xi;t)}{\partial \xi} p_\beta\right] p_\alpha \varphi(\mathbf{R},t;\xi,\mathbf{p}) - \frac{\partial}{\partial p_\beta}\left[\frac{\partial w(\mathbf{R},\xi;t)}{\partial R_\beta} + \frac{\partial w(\mathbf{R},\xi;t)}{\partial \xi} p_\beta\right]\varphi(\mathbf{R},t;\xi,\mathbf{p}),$$

$$\varphi(\mathbf{R},0;\xi,\mathbf{p}) = \delta\left(\xi - \xi_0(\mathbf{R})\right)\delta(\mathbf{p} - \mathbf{p}_0(\mathbf{R})).$$

Let us consider the random hydrodynamic velocity field in the form of a random Gaussian field which is statistically uniform and isotropic in space and statistically stationary in time. Its correlation and spectral functions read

$$B_{ij}(\mathbf{r}-\mathbf{r}', t-t') = \int d\mathbf{k} E^{s}(k,t-t')\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')},$$

in the case of turbulent velocity, and

$$B_{ij}(\mathbf{r},t) = \int d\mathbf{k}\frac{k_i k_j}{k^2} E^{p}(k,t) e^{-\lambda k^2 t}\cos\left(\mathbf{kr} - \omega(\mathbf{k})t\right).$$

in the case of wave turbulence. Here $\omega = \omega(\mathbf{k}) > 0$ determines a dispersion curve of the wave motion, and λ indicates the wave attenuation. In both cases the joint probability density of the surface elevation field and its gradient is the joint indicator function averaged over an ensemble of the random velocity field realizations.

3.2. Statistical averaging

Now, one can obtain an equation for the joint probability density of surface elevation and its gradient. To do that, one must average Liouville equation over an ensemble of the random velocity field realization. To divide correlations of velocity $\mathbf{u}(\mathbf{r},t)$ with arbitrary fuctional of it, one can make use of the Furutsu-Novikov theorem [17,22,23], taking into account the sea bottom irregularities $H(\mathbf{R})$ and diffusion approximation [18]:

$$\left\langle U_i(\mathbf{R},\xi,t) R\left[\mathbf{U}(\tilde{\mathbf{R}},\tilde{\xi},\tau)\right]\right\rangle_{\mathbf{U}} = \int d\mathbf{R}' \int_{-\infty}^{\xi+0} \theta\left(\xi' - H(\mathbf{R})\right) d\xi' \times B_{ij}(\mathbf{R}-\mathbf{R}', \xi-\xi')\left\langle \frac{\delta R\left[\mathbf{U}(\tilde{\mathbf{R}},\tilde{z},\tau)\right]}{\delta U_j(\mathbf{R}',\xi',t-0)}\right\rangle_{\mathbf{U}}, \tag{2}$$

where $B_{ij}(\mathbf{R},\xi) = \int_0^\infty d\tau B_{ij}(\mathbf{R},\xi,\tau)$.

Let bottom topography has mean value $\langle H(\mathbf{R})\rangle = -H$, where H is the mean depth of the sea as on fig. 1. Taking into account that topographic irregularities are statistically independent on the hydrodynamical velocity field and by averaging (2) over an ensemble of realizations, one gets

$$\left\langle U_i(\mathbf{R},\xi,t) R\left[\mathbf{U}(\tilde{\mathbf{R}},\tilde{\xi},\tau)\right]\right\rangle_{\mathbf{U}} = \int d\mathbf{R}' \int_{-\infty}^{\xi+0} P\left(H,\xi'\right) d\xi' \times B_{ij}(\mathbf{R}-\mathbf{R}', \xi-\xi')\left\langle \frac{\delta R\left[\mathbf{U}(\tilde{\mathbf{R}},\tilde{z},\tau)\right]}{\delta U_j(\mathbf{R}',\xi',t-0)}\right\rangle_{\mathbf{U}}.$$

Here $P\left(H,\xi\right) = P\{\xi > H(\mathbf{R})\} = \left\langle \theta\left(\xi - H(\mathbf{R})\right)\right\rangle_H$ is an integral probability function of the distribution of topographic irregularities and we consider the case of a statistically uniform random field $H(\mathbf{R})$, so this function does not depend on spatial coordinates $\mathbf{R}$. In the case of a model of an infinitely deep sea the function $P\left(\infty,\xi\right) = 1$ and we will consider such a case.

3.3. Surface elevation probability density

After some calculation (see detail in works [24,25]) we obtain for surface elevation probability density very unexpected result

$$\frac{\partial}{\partial t}P(t;\xi)=B_{ww}(\mathbf{0},0)\frac{\partial^2}{\partial \xi^2}P(t;\xi).$$

So, the probability density of the surface elevation random field $\xi(\mathbf{R},t)$ is the Gaussian distribution

$$P(t;\xi)=\frac{1}{\sqrt{4\pi B_{ww}(\mathbf{0},0)t}}\exp\left\{-\frac{\xi^2}{4B_{ww}(\mathbf{0},0)t}\right\} \qquad (3)$$

not depending on non-linearity of initial equation with the variance $\left\langle \xi^2(\mathbf{R},t)\right\rangle = 2B_{ww}(\mathbf{0},0)t$. Diffusion coefficient $B_{ww}(\mathbf{0},0)$ is naturally linked to the variance of the random hydrodynamic velocity field,

$$B_{ww}(\mathbf{0},0)=\frac{2}{3}\int d\mathbf{k}\,E^{s}(k,0)=\frac{1}{3}\sigma_{\mathbf{U}}^2.$$

It is worth to mention that surface elevation not correlated with its gradient [24,25].

Summarizing, one can draw a conclusion that the complex structure of a hydrodynamic velocity field in the deep sea cannot be a direct cause of a stochastic structure on the sea surface.

From solution (3) the expressions for conditional averages follow

$$\left\langle \xi(\mathbf{R},t)\,|\,\xi>0\right\rangle_{\xi}=\int_0^{\infty} d\xi\,\xi P(t;\xi)=\sqrt{\frac{1}{\pi}B_{ww}(\mathbf{0},0)t},$$

$$\left\langle \xi(\mathbf{R},t)\,|\,\xi<0\right\rangle_{\xi}=\int_{-\infty}^{0} d\xi\,\xi P(t;\xi)=-\sqrt{\frac{1}{\pi}B_{ww}(\mathbf{0},0)t}.$$

3.4. Probability density of the surface elevation gradient

To simplify, we go to the random field $I(\mathbf{R},t)=\mathbf{p}^2(\mathbf{R},t)$ and obtain the equation, with $P(t;I)$ being the probability density,

$$\frac{\partial}{\partial \tau}P(\tau;I)=\frac{\partial}{\partial I}I\left(1+4\frac{\partial}{\partial I}I+I\right)P(\tau;I)+$$
$$2\frac{\partial}{\partial I}I\frac{\partial}{\partial I}I^2\,P(\tau;I)+2\frac{\partial}{\partial I}I\frac{\partial}{\partial I}P(\tau;I)$$

with dimensionless time $\tau=\frac{8}{15}D^{s}t$ in the case of hydrodynamic turbulence and $\tau=\frac{2}{15}D^{p}t$ in the case of wave turbulence. Here we denote

$$D^{s}=\frac{1}{2}\int_0^{\infty} d\tau\langle\boldsymbol{\omega}(\mathbf{r},t+\tau)\boldsymbol{\omega}(\mathbf{r},t)\rangle,$$

$$D^{p}=\int_0^{\infty} d\tau\left\langle\frac{\partial\mathbf{U}(\mathbf{r},t+\tau)}{\partial\mathbf{r}}\frac{\partial\mathbf{U}(\mathbf{r},t)}{\partial\mathbf{r}}\right\rangle$$

where $\boldsymbol{\omega}(\mathbf{r},t)=\operatorname{curl}\mathbf{U}(\mathbf{r},t)$ is the velocity field curl and $\frac{\partial\mathbf{u}(\mathbf{r},t)}{\partial\mathbf{r}}$ is the divergence of the velocity field.

So we again have the very unexpected result, the equation for probability density of the surface elevation gradient does not depend from statistical nature of velocity field.

4. STATISTICAL TOPOGRAPHY OF RANDOM FIELD FOR THE SURFACE ELEVATION GRADIENT

The probability density equation for the random field $I(\tau;\mathbf{R})$, provided a spatial uniformity, govern the statistical properties of the surface elevation gradient field. At any fixed point in the space $\tilde{\mathbf{R}}$, function $I(\tau;\tilde{\mathbf{R}})$ is a random process in time with the simultaneous probability density independent on $\tilde{\mathbf{R}}$ governed by the obtained equations. In the physical space $\{\mathbf{R}\}$, there may appear the structure as a physical object. This structure appears as closed regions with high gradient concentration conventionally called as cluster. The equations, describing this process, are fairly complex due to two concurrent phenomena. On one hand, the field $I(\tau;\mathbf{R})$ is induced by a random Gaussian field from the zero initial condition, on the other hand, there is a parametric excitation of the field. Certain statistical parameters, characterizing this excitation within separate realizations of the random field, can be obtained analytically.

4.1. Moment equation and the Lyapunov characteristic parameter

Let us consider the n-th moment $\left\langle I^n(\tau;\mathbf{R})\right\rangle$ of the function $I(\tau;\mathbf{R})$. By multiplying the probability density equation by I^n, and integrating over I, one obtains

$$\frac{\partial}{\partial \tau}\left\langle I^n(\tau;\mathbf{R})\right\rangle=n(4n-1)\left\langle I^n(\tau;\mathbf{R})\right\rangle+$$
$$n(2n-1)\left\langle I^{n+1}(\tau;\mathbf{R})\right\rangle+2n^2\left\langle I^{n-1}(\tau;\mathbf{R})\right\rangle$$

In particular, as $n=1$, the equation for $\left\langle I(\tau;\mathbf{R})\right\rangle=\sigma_{\mathbf{p}}^2(\tau;\mathbf{R})$ the surface elevation gradient variance is

$$\frac{\partial}{\partial \tau}\left\langle I(\tau;\mathbf{R})\right\rangle=3\left\langle I(\tau;\mathbf{R})\right\rangle+\left\langle I^2(\tau;\mathbf{R})\right\rangle+2$$

It means that there is no steady-state distribution as $\tau\to\infty$.

By same procedure, one obtains the equation for the Lyapunov characteristic parameter, $\frac{\partial}{\partial\tau}\left\langle \ln I(\tau)\right\rangle$,

$$\frac{\partial}{\partial \tau}\left\langle \ln I(\tau;\mathbf{R})\right\rangle=-1-\left\langle I(\tau;\mathbf{R})\right\rangle$$

i.e. the Lyapunov exponent decreases rapidly in time as $\tau\to\infty$,

$$e^{\langle \ln I(\tau;\mathbf{R})\rangle}\sim\exp\left\{-\tau-\int_0^{\tau} d\tau'\left\langle I(\tau';\mathbf{R})\right\rangle\right\}$$

It follows that the field $I(\tau;\mathbf{R})$ decreases rapidly almost at every point in space as time τ increases, therefore the field may cluster in small space areas.

4.2. Integral probability distribution equation

The integral probability distribution function for the probability density $P(\tau;I)$ is defined as

$$\Phi(\tau,I)=\int_0^{I} dI'\,P(\tau;I')=\left\langle \theta\left(I-I(\tau;\mathbf{R})\right)\right\rangle_{\mathbf{U}}.$$

Due to parametric excitation, that function rapidly approaches unity as time increases. So, the value

$$s_{\text{hom}}\left(\tau,\bar{I}\right)=\left\langle\theta\left(I\left(\tau;\mathbf{R}\right)-\bar{I}\right)\right\rangle_{\mathbf{U}} \qquad (4)$$

is the probability of an event $\mathrm{P}\left(I\left(\tau;\mathbf{R}\right)>\bar{I}\right)$. Then, in accordance with statistical topography theory [17,18], this function in the case of a statistically uniform field delineates geometrically a specific area, where the field $I\left(\tau;\mathbf{R}\right)$ exceeds any given value $\bar{I}$. As the field decreases almost at every point in space, this probability approaches zero, indicating that the basic statistic characteristics (moments) concentrate inside this small area. As the specific gradient squared, localized inside the area, satisfies the equation,

$$\left\langle i\left(\tau;I>\bar{I}\right)\right\rangle=\left\langle I(\tau)\right\rangle-\int_0^{\bar{I}}dI\; IP(\tau;I)$$

Let us consider the process of formation of structure of the sea elevation. We must study the probability density of field $\xi(\mathbf{R};t)$ in areas with big value of gradient, i.e. areas from (4). As the fields $\xi(\mathbf{R};t)$ and $I(\mathbf{R};t)$ are statistically independent, the conditional average

$$\int d\mathbf{R}\left\langle\xi(\mathbf{R},t)\theta(\xi(\mathbf{R},t)-0)\theta(I\left(\tau;\mathbf{R}\right)-\bar{I})\right\rangle_{\xi,I}=$$
$$\int d\mathbf{R}\left\langle\xi(\mathbf{R},t)\theta(\xi(\mathbf{R},t)-0)\right\rangle_{\xi}\left\langle\theta(I\left(\tau;\mathbf{R}\right)-\bar{I})\right\rangle_{I}$$

for statistically uniform and isotropic in space problem has the form

$$\left\langle\xi(\mathbf{R},t)\theta(\xi(\mathbf{R},t)-0)\theta(I\left(\tau;\mathbf{R}\right)-\bar{I})\right\rangle_{\xi,I}=$$
$$\int_0^{\infty}\xi P(t;\xi)d\xi\; s_{\text{hom}}\left(\tau,\bar{I}\right)=\sqrt{\frac{1}{\pi}B_{ww}(\mathbf{0},0)t}\; s_{\text{hom}}\left(\tau,\bar{I}\right).$$

We can see, that the average elevation of sea surface in the area of gradient clustering will grows at short time and then will decreases due to decreasing of clustering area $s_{\text{hom}}\left(\tau,\bar{I}\right)$.

By the same way we can consider the conditional average $\int d\mathbf{R}\left\langle\xi(\mathbf{R},t)\theta(0-\xi(\mathbf{R},t))\theta(I\left(\tau;\mathbf{R}\right)-\bar{I})\right\rangle_{\xi,I}$.

5. CONCLUSION

The main result of the paper is that we derived equations for the joint simultaneous probability density for a perturbed surface field $\xi(\mathbf{R};t)$ and its gradient $p(\mathbf{R};t)$ under the influence of the vertical component in the hydrodynamic velocity field. And secondly, we have analyzed statistical features of these equations. In the case of a spatially uniform medium in the deep sea, the fields $\xi(\mathbf{R};t)$ and $p(\mathbf{R};t)$ do not correlate.

Then making use of the equations, we have obtained an equation for the Lyapunov characteristic exponent. These equations show, that the Lyapunov exponent rapidly decreases with time. It means the clustering of gradient field in small areas. We show that the surface elevation in that area increases at small time and then decreases as the clusters area decreases. It results in the emergence of rare events such as anomalous high structures and deep gaps on the sea surface almost in every realization of a random velocity field. Such structures exist only a finite time and restricted by amplitude.

ACKNOWLEDGMENTS

The paper has been supported by the RSF: Nos 14-27-00134 and RFBR: Nos 16-05-00238.

References

[1] C. Kharif, E. Pelinovsky, A. Slunyaev, Rogue Waves in the Ocean, Springer, Berlin, 2009.

[2] K. Dysthe, H. E. Krogstad, and P. Muller, Annu. Rev. Fluid Mech. Vol. 40, p. 287, 2008.

[3] E. J. Heller, L. Kaplan, and A. Dahlen, J. Geophys. Res. Vol. 113, P. C09023, 2008.

[4] V. I. Klyatskin, Theor. Math. Phys. Vol. 180, P. 850, 2014.

[5] M. Shats, H. Punzmann, and H. Xia, Phys. Rev. Lett. Vol. 104, P. 1, 2010.

[6] M. A. Topinka, B. J. LeRoy, R. M. Westervelt, S. E. J. Shaw, R. Fleischmann, E. J. Heller, K. D. Maranowski and A. C. Gossard, Nature (London), Vol. 410, P. 183, 2001.

[7] B. Liu, and E. J. Heller, Phys. Rev. Lett., Vol. 111, P. 236804, 2013.

[8] M. A. Wolfson, and S. Tomsovic, J. Acoust. Soc. Am., Vol. 109, P. 2693, 2001.

[9] C. Bonatto, M. Feyereisen, S. Barland, M. Giudici, C. Masoller, J. R. Rios Leite, and J. R. Tredicce, Phys. Rev. Lett., Vol. 107, P. 053901, 2011.

[10] D.R. Solli, C. Ropers, P. Koonath, and B. Jalali, Nature, Vol. 450, P. 1054, 2007.

[11] V. I. Klyatskin, Stochastic Equations through the Eye of the Physicist: Basic Concepts, Exact Results and Asymptotic Approximations, Elsevier, 2005.

[12] Y. A. Kravtsov, Rep. Prog. Phys., Vol. 55, P. 39, 1992.

[13] S. M. Rytov, Y. A. Kravtsov, and V. I. Tatarskii, Principles of Statistical Radiophysics: Wave Propagation through Random Media, Springer, Berlin, 1989.

[14] J. J. Metzger, R. Fleischmann, and T. Geisel, Phys. Rev. Lett., Vol. 112, P. 203903, 2014.

[15] V. P. Ruban, JETP Letters, Vol. 97, P. 686, 2013.

[16] V. E. Zakharov, and A. A. Gelash, Phys. Rev. Lett., Vol. 111, P. 054101, 2013.

[17] V. I. Klyatskin, Lectures on Dynamics of Stochastic Systems, Elsevier, Boston, 2010.

[18] V. I. Klyatskin, Stochastic Equations: Theory and Applications in Acoustics, Hydrodynamics, Magnetohydrodynamics, and Radiophysics, Vol. 1,2, Helderberg, Springer, 2014.

[19] M. B. Isichenko, Rev. Modern Phys., Vol. 64, P. 961, 1992.

[20] V. I. Klyatskin, Phys. Usp., Vol. 55, P. 1152, 2012.

[21] V. I. Klyatskin, Rus. Jour. Math. Phys., Vol. 29, P. 295, 2013.

[22] K. Furutsu, J. Res. N.B.S., Vol. D-67, P. 303, 1963.

[23] E. A. Novikov, Sov. Phys. J.E.T.P., Vol. 20, P. 1290, 1963.

[24] V. I. Klyatskin, K.V. Koshel. Phys. Rev. E., Vol. 91, P. 063003, 2015.

[25] V. I. Klyatskin, K. V. Koshel. Theor. Math. Phys., Vol. 186, P. 410, 2016.