DINCON 2017
CONFERÊNCIA BRASILEIRA DE DINÂMICA, CONTROLE E APLICAÇÕES
30 de outubro a 01 de novembro de 2017 – São José do Rio Preto/SP

# Application of Horner's method to simulation of the polynomial NARMAX

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**Abstract**. This paper reports the use of Horner's method in order to decrease the error propagation on simulation of recursive functions on digital computers, with particular attention to polynomial NARMAX. Three case studies were investigated: Logistic Map, Sine Map and Chua's Circuit. For each case, the Lower Bound Error of two pseudo orbits was calculated and it was possible to verify that the error decreases when Horner's method is applied.

**Key-words**. Horner's method, Lower Bound Error, Numerical Computation.

## 1 Introduction

Numerical computing has became a vital part of modern scientific infrastructure and it is essential in several subjects. Computers are used to solve equations that model different kind of systems: the universe expansion, an atom micro-structure, chaotic systems simulation, to cite a few examples [9].

Virtually, all numeric computation uses floating-point arithmetic with most current computers making use of the IEEE standard for such [3]. However, all computational arithmetic is susceptible to errors. This happens because floating point is a systematic approximation of real numbers arithmetic and is represented by a finite real numbers subset [9]. Consequently, some arithmetic properties of these numbers are not guaranteed

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for floating point arithmetic [3]. As indicated by [4], there are many published works in which the reliability of numerical results are not carefully evaluated.

In Nepomuceno (2014) [6], the author presented a study in one of the most used software in the area of research of Systems Identification, *Matlab*, using double precision, in which a simple sequence of iterations converged to the wrong answer. In continuity to the work of this line, in some works, such [10], interval analysis were used to reproduce the same results obtained by [6], showing once again that the calculation of the fixed points may require careful attention with regard to computation numerical value. In addition, [7] determine the propagation of the error in calculating the recursive function through polynomial NARMAX. By means of the concept of multiple pseudo-orbits and interval analysis, these authors defined a simple strategy, starting with interval extensions, that allow the estimation of the numerical calculation error.

As shown on all works cited previously, the error may grow with to the number of arithmetical operations realized on each iteration. Therefore, in order to retard the error propagation, Horner's method can be utilized [2]. To validate this preposition, Horner's method was applied to three well known chaotic systems available on the literature: Sine Map, Logistic Map and Chua's Circuit.

From this, in this paper we shall discuss about comparing numerical simulation errors by means of the implementation of the method proposed by [2] in the systems cited above. The results showed that Horner's method was able to reduce the error propagation in the three cases studied, once the number of iterations has decreased.

# 2 Preliminary Concepts

## 2.1 Lower Bound Error

If it is true that the propagation of errors in the simulation affects the performance of some systems, it is no less true that there are researches that seek to improve the reliability of the simulations, by decreasing errors accumulation. One of the techniques used is the introduction of a lower error limit in the simulation of the NARMAX (Nonlinear AutoRegressive Moving Average model with eXogenous input) polynomial simulation [7]. The procedure is based on the comparison of two pseudo-orbits produced from two mathematical and equivalent models, but different from floating point representation. This technique can be used to interrupt a simulation if a required precision is greater than the lower error limit, increasing numerical reliability in the NARMAX polynomial simulation. The lower bound error (LBE) shall be used to observe the error inherent in the simulation evolution.

Let $\hat{x}_{a,n}$ and $\hat{x}_{b,n}$ be two pseudo-orbits derived from two interval extensions.

$$\delta_{\alpha,n} = \frac{|\hat{x}_{a,n} - \hat{x}_{b,n}|}{2} \tag{1}$$

is the map error lower bound $f(x)$ when $\delta_{a,n} \geq \delta_{\alpha,n}$ or $\delta_{b,n} \geq \delta_{\alpha,n}$, in which $\delta_{a,n}$ e $\delta_{b,n}$ are the errors pertaining to each pseudo-orbit.

## 2.2 Horner's Method

Given the polynomial

$$p(x) = \sum_{1 \equiv 0}^{n} a_i x^i = a_o + a_1 x + a_2 x^2 + \cdots + a_n x^n, \tag{2}$$

where $a_0, \ldots, a_n$ are real numbers, we wish to evaluate the polynomial at a specific value of $x$ said to be $x_0$, thus:

$$p(x_0) = \sum_{1 \equiv 0}^{n} a_i {x_0}^i = a_o + a_1 x_0 + a_2 {x_0}^2 + \cdots + a_n {x_0}^n. \tag{3}$$

It results in $n$ additions and $\frac{n(n+1)}{2}$ multiplications. It can be a bit complicated when a lot of calculations is needed. To reduce the number of operations, we define a new sequence of constants as follows:

$$b_n = a_n \tag{4}$$

$$b_{n-1} = a_{n-1} + b_n x_0 \quad \cdots \quad b_0 = a_0 + b_1 x_0. \tag{5}$$

To see why this works, note that the polynomial can be written in the form

$$p(x) = a_0 + x(a_1 + x(a_2 + \cdots x(a_{n-1} + a_n x))).$$

Thus, by iteratively substituting the $b_i$ into the expression,

$$p(x_0) = a_0 + x_0(a_1 + x_0(a_2 + \cdots x_0(a_{n-1} + b_n x_0))) \;\cdots\; = a_0 + x_0 b_1 = b_0$$

This method results in $n$ additions and $n$ multiplications.

## 2.3 Logistic Map

The logistic map was originally investigated by Robert May [5]. The magnitude of the population in generation $t+1$, $X_{t+1}$, is related to the magnitude of the population in the preceding generation $t$, $X_t$. Such a relationship may be expressed in the general form, $X_{t+1} = f(X_t)$, where the function $F(X)$ is a first order, nonlinear difference equation. The logistic difference equation in a canonical form is given by

$$X_{t+1} = rX_t(1 - X_t), \tag{6}$$

where $0 < Xt < 1$ represents the ratio of existing population to the maximum possible population at generation $t$. [5] investigates the results of (6) by varying the parameter $r$. In general, the nth iteration of (6) can be written as: $X_{t+n} = F^{(n)}(X_t)$. Then, this Equation is called Recursive Function (RF).

## 2.4 Sine Map

A unidimensional sine map is defined as, $x_{t+1} = \alpha \sin(x_n)$, where $\alpha = 1.2\pi$. A polynomial NARMAX identified for this system is given by [8]

$$x_{t+1} = 2.6868 x_n - 0.2462 x_{n^3} \tag{7}$$

### 2.5 Chua's Circuit

The Chua's Circuit can be modelled by a set of nonlinear differential equations, commonly represented as:

$$\dot{x} = \alpha(y - x - g(x)) \tag{8}$$

$$\dot{y} = x - y + z \tag{9}$$

$$\dot{z} = -\beta y \tag{10}$$

Considering only Equation (8), a polynomial NARMAX identified for this circuit is given by [1], p.18-19. For further reference, this equation is identified as Z.

## 3 Method

From what has been presented on the previous section, we may apply it by rewriting one variable of the equation of the system of interest, as introduced by [7]. We shall do it for the Logistic Map, Sine Map and Chua's Circuit, as shown in the following topics.

### 3.1 Logistic Map

Given Equation (6), we show two distinct ways to write that equation: one using equation showed in [7] and the other considering [2], as represented by Equations (11) and (13), respectively. Furthermore, using Horner's method, we rewrite Equations (11) and (13), in order to compute less arithmetic operations on each iteration. The result is given by Equations (12) and (14) :

$$E_{t+1} = rX_t - rX_t^2 \tag{11}$$

$$G_{t+1} = rX_t(1 - X_t) \tag{12}$$

$$F_{t+1} = rX_t - r \cdot X_t \cdot X_t \tag{13}$$

$$H_{t+1} = r(X_t(1 - X_t)). \tag{14}$$

### 3.2 Sine Map

Likewise, we rewrote Sine Map equivalent polynomial NARMAX equation - same step adopted previously. As soon, we seek decreasing amount of operations. Therefore, Equation (7) (represented again as Equation (15)) became (16). Equation (17) is defined as showed in [7], and can be changed to (18).

$$A_{n+1} = 2.6868x_n - 0.2462x_n^3 \tag{15}$$

$$B_{n+1} = x_n(2.6868 - 0.2462x_n^2) \tag{16}$$

$$C_{n+1} = 2.6868x_n - (0.2462x_n)(x_n)^2 \tag{17}$$

$$D_{n+1} = x_n(2.6868 - (0.2462x_n)x_n). \tag{18}$$

### 3.3 Chua's Circuit

As done before for Logistic Map and Sine Map, Chua's Circuit equivalent polynomial NARMAX was rewritten. Hence, three other forms of $Z$ given by [1], p.18-19, can be defined as follows. To get the first interval extension, the terms $X_k^n$ of Z were rewritten as $X_k \cdots X_1$. This extension will be called Y. In order to apply Horner's method and thus reduce the amount of arithmetic operations, we define the second extension, where terms $X_k(10^{-1})$ of Z were substituted for $0.1X$. This interval extension will be identified as W. The same was to done to define the third extension, where terms $X_k^n 10^{-1}$ of $Z$ were rewritten as $0.1X_k \cdots X_1$. This equation will be defined as $X$.

$$\begin{aligned} z(k) \;=\; & 0.17799 \cdot 10 z_{(k-1)} + 0.46703 z_{(k-3)} z_{(k-4)^2} - 0.87039 z_{(k-3)} \\ & +0.91053 \cdot 10^{-1} z_{(k-1)^2} z_{(k-4)} + 0.63361 z_{(k-5)} \\ & +0.13195 \cdot 10^{-2} z_{(k-4)^2} z_{(k-5)} + 0.57917 z_{(k-1)} z_{(k-4)} z_{(k-5)} \\ & -0.34940 z_{(k-4)} - 0.91067 \cdot 10^{-1} z_{(k-1)^3} + 0.81258 \cdot 10^{-1} z_{(k-1)^2} z_{(k-2)} \\ & -0.10603 z_{(k-5)^3} - 0.16898 z_{(k-1)^2} z_{(k-5)} + 0.73569 \cdot 10^{-1} z_{(k-2)} z_{(k-5)^2} \\ & -0.12979 \cdot 10^{-1} z_{(k-1} z_{(k-2)^2} - 0.39490 z_{(k-1)} z_{(k-5)^2} \\ & -0.31911 \cdot 10^{-1} z_{(k-2)} z_{(k-4)^2} + 0.22167 \cdot 10^{-1} z_{(k-1)^2} z_{(k-3)} \\ & +0.56608 z_{(k-3)} z_{(k-5)^2} - 0.78874 z_{(k-3)} z_{(k-4)} z_{(k-5)} \\ & -0.40725 z_{(k-1)} z_{(k-4)^2} + 0.19216 z_{(k-1)} z_{(k-3)} z_{(k-5)} \\ & -0.15206 z_{(k-3)^2} z_{(k-5)} \end{aligned} \tag{19}$$

### 3.4 Error Comparison

We shall draw a comparison for the error's evolution by measuring lower bound error (LBE) among the equations above, for example, in the Sine Map: Let $LBE_{s1}$ be the lower bound error among Equations (15) and (16) - $LBE_{s1} = \left|\frac{B_{n+1} - D_{n+1}}{2}\right|$ and $LBE_{s2} = \left|\frac{C_{n+1} - A_{n+1}}{2}\right|$. In this regard, we simulate both LBE's on time domain and compare step by step the results. Then, the same logic was used for the others two cases of study in this paper.

## 4 Results

### 4.1 LBE Comparison

In this section, we present the lower bound error comparison applied for two case studies, which exhibit nonlinear dynamics, such as chaos. The idea is to produce two pseudo-orbits from two different models obtained from literature. We select two equations that are equivalent, according to the explanation above. The two chosen unidimensional models are Sine Map and Logistic Map.

Figure (1) shows the error evolution comparison (logarithm scale) of a simulation performed for Logistic Map. Figure (2) shows the same comparison for Sine Map. Finally, Figure (3) shows the comparison for Chua's Circuit.

The conditions used in Logistic Map performance is parameter $r = 3.9$ and initial value $x_0 = 0.1$ same that we used in Sine Map.

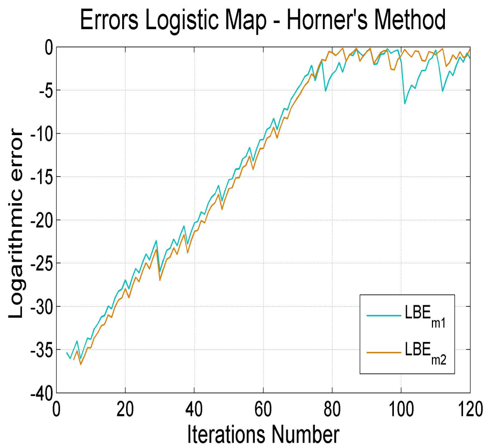


Figure 1: Logistic Map - LBE comparison.

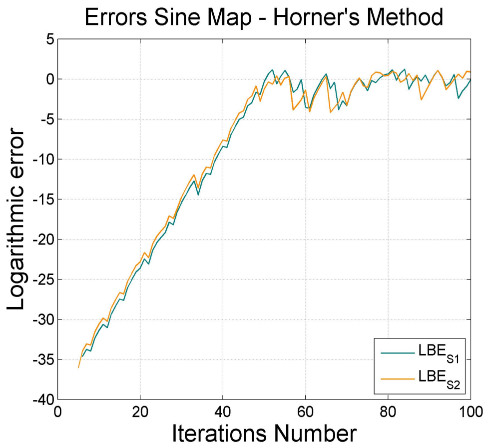


Figure 2: Sine Map - LBE comparison.

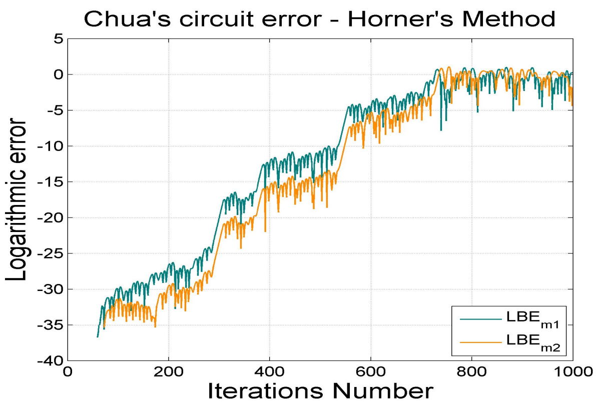


Figure 3: Chua's Circuit - LBE comparison.

## 4.2 Conclusion

As we explained above, the simulation of mathematical equivalent equations yields to different results. It can be seem clearly in Figures (1), (2) and (3) that a simple modification in the order of operations change all results. In this way, a normal problem is very present at researchers of nonlinear dynamic systems daily when it is necessary to reproduce a result found at literature. Since chaotic systems are very sensitive, any change is decisive to final results.

Thereby, we have to write the equation seeking to decrease arithmetical operations, and consequently the error associate is decreased. Taking that into account, Horner's method was applied. As simulation results show, we can see a little shift between the two pseudo-orbits, so the error has been reduced when the method presented in this paper was applied. For example, in the Logistic Map case, we could observe that inaccuracy had a logarithmic value of -2.7742 on iteration 93 before the application of the method. After Horner's method was applied, the error presented was -1.9704 on the same iteration. Thereby, the error diminished about 71% in this point. An important question for future studies is to extend the present work for multidimensional systems, that is, for cases where polynomial NARMAX presents more than one regressor.

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