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Reliability analysis of the Logistic Map's pseudo-orbits through the Delta-Epsilon limit definition

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Abstract. This paper reports the use of Delta-Epsilon limit definition as a way of evaluating a maximum number of iterations for the simulation of Logistic Map. This was possible by showing that logistic map is continuous and the error propagation is related to the nth composed function. It has been shown an error relationship with the number of iterations. Thus, the proposed method seeks to relate an arbitrary precision value to the number of iterations. With an analysis of the results, it is clear that for a result with a certain degree of precision one must be consider the number of iterations.

Key-Words. Logistic Map, IEEE 754, Nonlinear Dynamic Systems, Chaos, Delta-Epsilon, Limit Definition.

1 Introduction

The logistic map was originally investigated by Robert May [4].The magnitude of a population on the generation $t+1$, X_{t+1}, is related to the magnitude of the population on the previous generation t, X_t. This relation can be expressed as:

$$X_{t+1} = rX_t(1 - X_t) \tag{1}$$

$0 < X_t < 1$ represents the proportion of the existing population in relation to the maximum population possible in the generation t. The author in [4] investigates the results of (1)

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varying the r parameter. In particular, May investigates what happens with the fixed points, in other words, the values of X the $X_{t+1} = X_t = X^*$, solving the equation: $X^* = F(X^*)$. If $X^* = \lim_{t\to\infty} X_t$ exist, then X^* satisfies the equation above and it is called "fixed point". The second iteration of the equation (1) can be written as $X_{t+2} = F(F(X_t))$ or equivalently as $X_{t+2} = F^{(2)}(X_t)$

In general, the nth iteration of (1) can be written as: $X_{t+n} = F^{(n)}(X_t)$. This Equation is called Recursive Function (RF).

The Equation (1) can be seen as a discrete dynamic system that present complex dynamic [1], like chaos, according to specific values of r. On the literature, it is possible that, for $0 < r < 1$, the population dies, that is, $X^* = 0$. However, it is curious that for $x_0 = 1/r$ the systems converges to the fixed point $X^* = (r-1)/r$, even when $0 < r < 1$. Another aspect that deserves attention is the fact that the iterated map leads to a polynomial composition, which is a polynomial. Therefore, for all $n \in \mathbb{N}$, the iterated function will be continuous. But any continuous function which the domain is an interval $[0,1]$ is delimited above. In other words, for all initial conditions of the interval $[0,1]$, other result will not diverge to infinite when $r > 4$, although it has pointed a divergence of the logistic map to infinite when $r > 4$ for almost all initial conditions.

Despite the projects developed explain and investigate a computational reliability, for example, Nepomuceno (2014) [6] presents a theorem that identifies a reliability of calculations performed at fixed points, in Nepomuceno (2016) [7] a technique was developed to reject a simulation if an accuracy required for greater than the lower limit error, increasing numerical reliability in the simulation, furthermore, in Nepomuceno (2017) [8] was developed a method to determine a lower bound error and the critical time for the pseudo-orbits. Usually, the methods to evaluate the error are based on computational approaches, such as in [7]. Less attention has been given to develop procedures to achieve this task by means of analytical techniques. Thereby, we present a way to show analically that Logistic Map is a convergent sequence by using $\delta - \epsilon$ limit definition, proposed by Bolzano and Cauchy. Thus, a definition for an estimate of the error of a pseudo-orbit of the logistic map was analytically obtained, what provides another alternative to increase the computational reliability of recursive functions.

2 Preliminary Concepts

In this section a brief review on important mathematical concepts are given along.

2.1 Theorems and Definitions

Definition 1: The extended real number system consists of the entire $\mathbb{R}$ field and also consists of the addition of two symbols, $+\infty$ e $-\infty$. Thus, the original order in $\mathbb{R}$ can be preserved, and $+\infty < x < -\infty$ can be definied for each $x \epsilon \mathbb{R}$.

Definition 2: Let $f : X \to \mathbb{R}$ be a function with real values, defined on a subset $X \subset \mathbb{R}$. In this case, one can say that f is a real function of a real variable. Let $a \in \mathbb{R}$ be an accumulation point of X, that is, $a \in X'$. One can say that the real number L is the limit of $f(x)$ when x goes to a. That can be written as $lim_{x \to a} f(x) = L$, which has the following meaning: $\forall\, \epsilon > 0\; \exists\, \delta > 0;\; x \in X,\; 0 < |x - a| < \delta \therefore |f(x) - L| < \epsilon$.

Definition 3: Let $I \subset \mathbb{R}$ be an interval, $x \in I$ and $f : I \to \mathbb{R}$ is a function. Thus, the following affirmation are valid: f is continuous in x; for every sequence $(z)_{n=1}^{\infty}$ with $z_n \in I$ for all $n \in \mathbb{N}$, thus $\lim_{n\to\infty} z_n = x$ and so, $\lim_{n\to\infty} f(z_n) = f(x)$; for all $\epsilon > 0$ existis a $\delta > 0$ such that for all $z \in I$ exists $|z - x| < \delta$ such that $z \in I$, $|f(z) - f(x)| < \epsilon$.

Definition 4: A function $f : X \to \mathbb{R}$ is called continuous in the point $a \in X$ when it is possible to make $f(x)$ arbitrarily close to $f(a)$, whereas x is taken sufficiently close to a. Therefore, it can be said that $f : X \to \mathbb{R}$ is continuous in the point $a \in x$ when, for all $\epsilon > 0$ arbitrarily chosen, $\delta > 0$ can be found such that $x \in X$ and $|x - a| < \delta$ imply $|f(x) - f(a)| < \epsilon$.

Definition 5: Let f be a real function continuous on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there is a point $x \in (a, b)$ such that $f(x) = c$.

Definition 6: Let $f : A \to B$ and $g : A \to C$ functions such that the domain of g is equal to the co-domain of f. Thus, the composed function $(g\ o\ f) : A \to C$ can be defined, which consists on applying firstly f and then g. More precisely, $(g\ o\ f)(x) = g(f(x))$ for all $x \in A$. Yet, on the same meaning, gives the functions $f : A \to B$ and $g : A \to C$ and $h : C \to D$, the associative law is valid $(h\ o\ g)(x)\ o\ f(x) = h(g(f(x)))$.

Definition 7: Let $(a)_{n=1}^{\infty}$ be a sequence of real numbers. Then $(a)_{n=1}^{\infty}$ is called Cauchy's sequence if, for all $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that for every $m, n > N$, $|a_n - a_m| < \epsilon$.

Definition 8: The compost of a continuous function is also continuos until its nth composition.

3 Methodology

3.1 Continuity of the Map

As mentioned in Definition 8, the composition of a continuous function is always continuous. As the logistic map is a continuous and recursive function, and being its composition formed by its own map, its *nth* composition can be affirmed as continuous.

Mathematically, let $I \subseteq \mathbb{R}$ be an open interval and $x \in I$. A function $F : I \to \mathbb{R}$ is continuous if it satisfies: $\forall\, \epsilon > 0\; \exists\, \delta > 0;\; x \in X,\; 0 < |x - a| < \delta \therefore |f(x) - L| < \epsilon$.

Thereby, to show that a function is continuous, it is enough to prove that it satisfies this condition. So, for $z \in I$, $|x - z| < \delta$, $|f(x) - f(z)| < \epsilon$. In the logistic map

$$\begin{aligned}
|f(x) - f(z)| &= |rx(1-x) - rz(1-z)| \\
&= \left|r(x-z) + r(z^2 - x^2)\right| \\
&= |r(x-z) + r(z-x)(z+x)| \\
&= |r(x-z) + (-1)(-1)r(z-x)(z+x)| \\
&= |x-z|\,|r - r(z+x)| \\
&\leq |x-z|\,(|r| + |r(z+x)|) \text{ Triangular inequality} \\
&\leq |x-z|\,(|r| + |r(2z - z + x)|) \\
&\leq |x-z|\,(|r|\,|(x-z)| + |r|\,|2z| + |r|\,.
\end{aligned}$$

Supposing $|x - z| < 1$ e $r > 0$

$$\begin{aligned}
|f(x) - f(z)| &\leq |z-x|\,(r\,|z-x| + r\,|2z| + r) \\
&\leq |z-x|\,(2r\,|z| + 2r) < \epsilon.
\end{aligned}$$

Thus,

$$|x - z| < \frac{\epsilon}{2r\,|z| + 2r} \tag{2}$$

Therefore, arbitrarily choosing

$$\delta = min\{1, \frac{\epsilon}{2r\,|z| + 2r}\} \tag{3}$$

3.2 Continuity of *nth* composition of the Map

Let $f^p : I \to \mathbb{R}$ for $p \geq m, k, n \in \mathbb{N}$ be a continuous function, Definition 3 $f[I]$ is an interval. Thus, $f[I] : j \subseteq \mathbb{R}$ e $y, w \in j$. It can be showed that $\forall\ \delta_j\ \exists\ \epsilon_j \mid \forall\ w \in j$, $|w - y| < \delta_j$ and $|f(w) - f(y)| < \epsilon_j$. Therefore, its composition, $f\ o\ f : I \to \mathbb{R}$ is continuous on x. That can be rewritten as: $f^{(2)} : I \to \mathbb{R}$, which can be extended to all $p \in \mathbb{N}$.

Thereby, $\epsilon_2 > 0$, as $f^{(2)}$ is continuous, $\exists\ \delta_2 > 0 \mid \forall\ y \in j$ with $|y - f(x)| < \delta_2$, $|f(y) - f(f(x))| < \epsilon_2$. Since f is continuous on $x \in I$, for δ_2 there is one $\delta_1 > 0$ such that for all $z \in I$ there is $|z - x| < \delta_1$ and also

$$|f(z) - f(x)| < \delta_2 = \epsilon_1 \tag{4}$$

Thus, it is possible to apply the result obtained through (3), then:

$$\delta_1 = min\{1, \frac{\epsilon_1}{2r\,|z| + 2r}\}$$

Through equation (4), $\delta_2 = \epsilon_1$, and particularizing for the studied case of the logistic map, assuming $\epsilon_n < 1$:

$$\delta_2 = \delta_1(2r\,|z| + 2r) \qquad\qquad \epsilon_2 = \delta_2(2r\,|y| + 2r) \tag{5}$$

$$\epsilon_2 = \delta_1(2r\,|f(z)| + 2r)(2r\,|y| + 2r) \tag{6}$$

Consequently $f^{(0)} = z$, and by induction:

$$\epsilon_p = \delta_1 \prod_{q=0}^{p} (2r\,|f(z)| + 2r) \tag{7}$$

Particularizing, once more, for the specific case of our work, $0 < z < 1$ and so $0 < f(z) < 1$. Therefore:

$$\epsilon_p = \delta_1 \prod_{q=0}^{p} (2r\,|f(z)| + 2r) \leq \delta_1 (4r)^p \tag{8}$$

And, for $f(z) = 0$ we have:

$$\delta_1 \leq \frac{\epsilon_p}{(4 \cdot r)^p} \tag{9}$$

$$\delta_1 \leq \frac{\epsilon_p}{(2 \cdot r)^p} \tag{10}$$

$\forall\ p \in \mathbb{N}$, $f^{(p)} : I \to \mathbb{R}$ is continuous. Equation (9) works with the worst accuracy case, where $|f(z)| = 1$, whereas for Equation (10), $|f(z)| = 0$ which increases the upper iterations limit. Here, we chose the second case, Equation (10), for increasing upper iterations limit. In this regard, it can be modified, by isolating (p). Since the logistic map is a second order function $(ax^2 + bx)$ it is possible to analyze possible changes in the behavior of the evolution of the error through its derivative. The derivative, by definition, is the tangent line to the curve. In this sense, it is possible to notice that its derivative sometimes is increasing and sometimes it is descending. For this reason, to ensure a better number of iterations, the values are divided by 0.5, then we have:

$$p = \frac{\log(\frac{\epsilon_p}{\delta_1})}{0.5 \cdot \log(2 \cdot r)} \tag{11}$$

4 Results

4.1 Results - *Matlab*

As mentioned above, [6] presented a study where a simple sequence of iterations converged to the wrong answer - Figures (1) and (2).

In this way, it is notorious the error evolution in function of the number of iterations. On many works available in the literature, as identified by [3], the confiability of numerical results are not carefully verified. Therefore, on section 3.1, an analytical solution that aims to minimize the problem proposed by [6] was presented. Equation (10) relates final and initial precision to the number of iterations of the logistic map. And, because of that, there are limitations of the number of iterations proportionally to the final required precision, ϵ_p.

Thus, for the initial conditions specified on Section 3.1, the value of the final precision $\epsilon_p = 0.001$ and of the initial precision $\delta_1 = 6.93889390390723e{-}18$ were chosen. Replacing

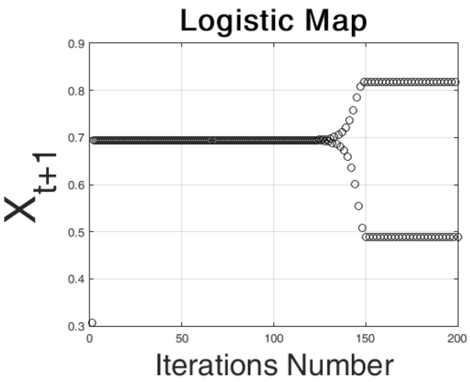

Figure 1: Iteration sequence - [6].

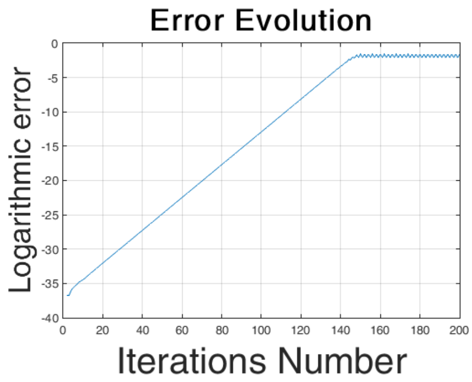

Figure 2: Error evolution - [6].

these values on Equation (11), $p = 34$. With considerations above, the number of iterations that before were arbitrarily chosen as 200, now has a maximum value of 34.

4.2 Lower Bound Error

For the logistic map it is still possible to use the LBE. In this case, the r parameter is 3.9, the initial condition ($X_0 = 0.1$), and to the ϵ_p was adopted as 0.00001. Therefore, the initial precision δ_p was calculated, resulting $6.9389e - 18$. In this way, by means of Equation (11) the number of iterations (p) is 30.

The LBE is calculated as defined in Section 2.2. The stop criteria used is given by [7]:

$$\epsilon_{\alpha,n} = \frac{|\{\hat{x}_{a,n}\} - \{\hat{x}_{b,n}\}|}{|\{\hat{x}_{a,n}\} + \{\hat{x}_{b,n}\}|}, \tag{12}$$

In comparison to the method utilized by [7], the LBE, the result was acceptable. However, by means of Equation (11), the method proposed limited the number of iterations on 30. On the other hand, LBE has its own method to limit the number of iterations, the number of iterations was 54 for a final precision of 0.0001.

5 Conclusion

In this work a method was developed to analyze the Logistic Map under specific conditions. It is known that the numerical domain of a computer, represented by the symbol $[\mathbb{D}]$, is a systematic approximation of the real set $\mathbb{R}$, that is, a finite representation of an infinite set [2]. Hence, there are limitations on the accuracy in the operations performed by a computer [9]. In this sense, we aim, in the first analysis, to show that, in opposition to what the literature presents, the number of iterations p a recursive function that is usually arbitrarily chosen is a function $p(r, \delta_1, \epsilon_p)$, dependent on three variables.

Hence, the recursive function was shown as a composition of functions - we can consider the logistic map under this category. Definition 8 says that the composition of a continuous function is always continuous. On that sense, a problem was found during the simulations. Equation (1) was inconsistent in $[\mathbb{D}]$ domain, since, after some iterations, it became noncontinuous. However, in Section 3, an analytic proof of its continuity is demonstrated. Therefore, it is possible to conclude that the definitions of conventional arithmetic shall not always perfectly translate to $[\mathbb{D}]$ domain, due the limitation of numerical representation.

Moreover, (p) Equation (10) was introduced and acts as a superior limiting factor for the number of iterations. This prevents false conclusions from being taken as a function of a large number of iterations, as shown by [6]. The Figure (2) shows the evolution of the error as a function of the number of iterations, in this way, it is clear that for a result with a certain degree of precision, one must pay attention to the number of iterations.

Consequently, by means of the considerations made above, the routine used by [6] was modified to include the unknowns δ_1 and ϵ_p. In this segment, the values of δ_1 and ϵ_p were determined, then the corresponding value of p was defined.

Therefore, the result obtained by means of Equation (10) is shown to be satisfactory since, by limiting the number of iterations, the false convergence to fixed point of period 2 was avoided, thus establishing results more consistent with the true response, increasing the reliability of the routine.

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