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Convergence to the stationary point in a discrete one-dimensional mapping

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Abstract. Convergence to asymptotic steady state in several one dimensional maps as logistic, cubic and Hassell maps are characterized by considering a phenomenological description supported by numerical simulations and confirmed by a theoretical description. We aim study the Hassell map and as the control parameter is varied, bifurcations in the stationary points appear. At transcritical bifurcation the convergence to the fixed point is characterized by a decay exponent β and near the bifurcation, the decay to the fixed point is exponential with a relaxation time given by a power law, which the slope is independent of the nonlinearity. The formalism is general and can be extended to other dissipative mappings.

Key-words. Hassell mapping, transcritical bifurcation, Lyapunov exponents, fixed points and relaxations.

1 Introduction

Studies in dynamical systems, discrete maps and nonlinear systems has increased to the long of decades. Moreover, the properties of these systems are connected the cascades of bifurcations and stationary points that can describe mathematically the nature [1–3]. The type of dynamics certainly depends on the control parameter. Varying the parameter, bifurcations appear, consequently, changing the dynamics of the steady state of the

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system [4] . From that studies of May [5], with his work applied in biology [6], several another papers were increasing with different applications as in physics [7–9] mathematics, chemistry, engineer and several other areas. [10–21].

In this work we consider of Hassell map [22–26] and we aim understand and to describe the behavior of the convergence to the stationary point at the bifurcation and near it. We will use the bifurcation diagram to analyse qualitative the system, seeing cascades of bifurcations and focusing at the transcritical bifurcation, where we intend to study the dynamical behavior of the system. Then, we use a set of numerical simulations and a theoretical investigation to show that, at the bifurcation, a convergence to the fixed point is given by a power law [27] and we also intend to investigate the relaxation to the fixed point around this bifurcation. From that numerical and theoretical studies we intend to solve analytically and, consequently, to compare the results analytical with the numerical.

This paper is organized as follows for a better comprehension. First the mapping, the orbit diagram, stationary point and the convergence to the fixed point are described. Then, we discuss the analytical approach transforming the difference equation into a differential equation. Lastly, we present our discussions and conclusions about studies described in the letter.

2 The map and Results

The Hassell map is written as

$$N_{n+1} = \frac{rN_n}{(1+aN_n)^{\gamma}}, \tag{1}$$

where r, a and γ are the control parameters and N is a dynamical variable. A typical orbit diagram is shown in figure 1(a)

From the figure 1(b) the chaotic behavior is measured through Lyapunov exponent. Following the studies we can obtain the fixed points follows as

$$F(N^*) = N^*,$$

giving us

$$N_1^* = 0 \quad \text{e} \quad N_2^* = {}^{(r^{1/\gamma}-1)}/_{a}. \tag{2}$$

where, according to stability analysis, N_1^* is asymptotically stable for $r \in [0,1)$ while N_2^* is asymptotically stable for $r \in (1, 11.3)$ for any initial conditions lying $N_0 \in (0,1)$. At $r_c = 1$, where c implies that, this point is a critical point, the system experiences a transcritical bifurcation and, consequently, the fixed point N_1^* changes stability with N_2^*. The system after of the limit of N_2^* exhibits a first period-doubling bifurcation following in a sequence of period-doubling until reaches chaotic behavior. Periodic windows are also observed for large values of r.

The region we are interested in to discuss along this paper correspond to the region of the transcritical bifurcation shown in figure 1(a). We aim to understand and to describe how is the convergence to the fixed point N_1^* at the bifurcation and around of it.

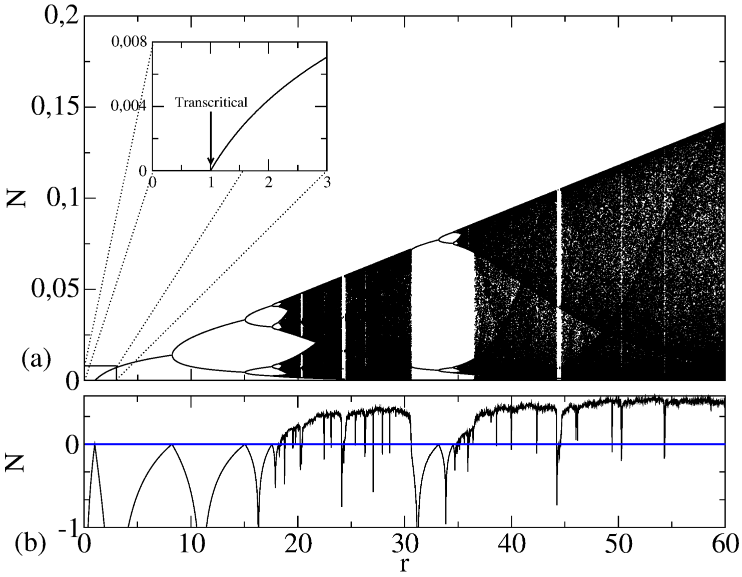

Figura 1: (a) Orbit diagram obtained for map (1) and a zoom at transcritical bifurcation, considering $a = 1$, $\gamma = 6$ and the initial condition $N_0 = 0.1$ and (b) the Lyapunov exponent characterizing the chaotic behavior.

We are then looking for describing the behavior of N approaching to N^* when we are at $r_c = 1$ and when we consider a distance μ that characterizes the nearboring of the bifurcation. Exactly like any other variable, N is a function of two entities, i.e., n which is the number of iterations and $\mu = r_c - r$. Following previous results in the literature [27,28], we start with two hypotheses:

i) For $\mu = 0$ it implies there is an algebraic decay in N so that

$$N(n, \mu = 0) \propto n^{\beta}, \tag{3}$$

where β is a critical exponent and depends on the type of bifurcation.

ii) For $\mu \neq 0$, when we are in nearboring of bifurcation, we consider the orbit relaxes to the equilibrium exponentially like in [27], according to

$$N(n, \mu) \propto e^{-n/\tau}, \tag{4}$$

where the relaxation time τ has the following form

$$\tau \propto \mu^{\delta} \tag{5}$$

and δ is a relaxation exponent.

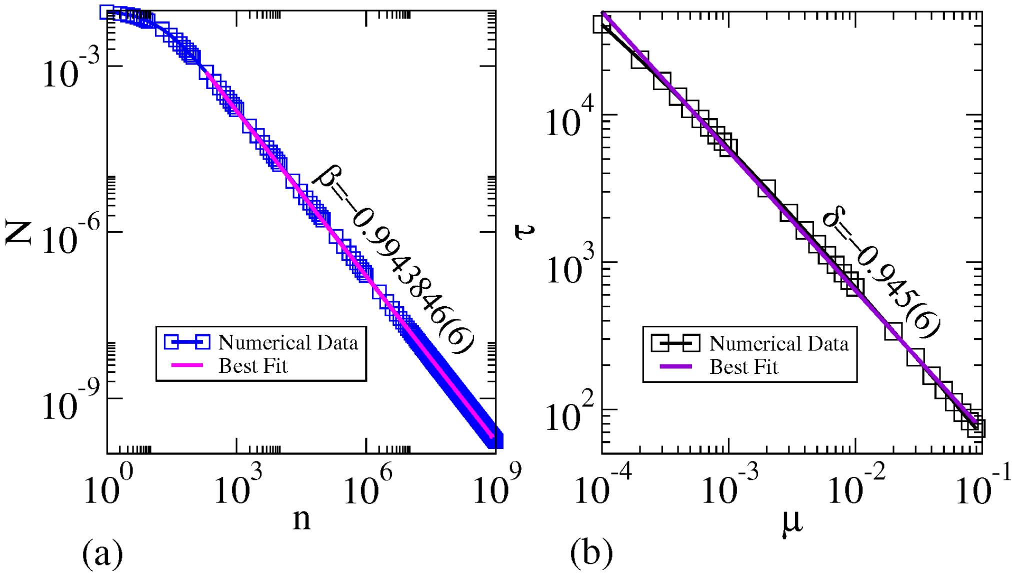

Figura 2: (a) Convergence to the stationary point $N_1^* = 0$, when $\mu = 0$, where a power law fit furnishes $\beta = -0.9943846(6)$ and (b) when $\mu \neq 0$ with a slope of $\delta = -0.945(6)$.

In figure 2 we can check what a numerical simulation furnishes to the implications above.

After doing power law fittings for the two plots on figure 2 we obtain that $\beta = -0.9943846(6)$ and $\delta = -0.945(6)$. Given the numerical results, we can consider theoretical argumentation on the characterization of the convergence to the fixed point N_1^* to the two cases *i*) and *ii*). We start first with case *i*). Expanding the mapping (1) in Taylor's series of N_n when $aN_n \ll 1$ we obtain

$$N_{n+1} = N[1 - \gamma a N_n - \frac{1}{2}\gamma a^2(-\gamma - 1)N_n^2 - O(3)]. \tag{6}$$

Look at near the fixed point, N is close to 0, then we can keep only first order term in the expansion and we obtain

$$N_{n+1} = N_n(1 - \gamma\alpha N_n), \tag{7}$$

from the condition, we suppose the dynamical variable N can be turned as a continuous variable. Thus, the equation (7) is rewritten as

$$\begin{aligned} N_{n+1} - N_n &= \frac{N_{n+1} - N_n}{(n+1) - n}\ , \\ &\approx \frac{dN}{dn} = -a\gamma N^2. \end{aligned} \tag{8}$$

Handling the terms properly we obtain the following differential equation

$$\frac{dN}{N^2} = -a\gamma dn. \tag{9}$$

The initial condition N_0 is defined for $n = 0$ while for an arbitrary n we have $N(n)$. Using these variables as limits of the integrals we have

$$\int_{N_0}^{N(n)} \frac{dN'}{N'^2} dN' = -\int_0^n a\gamma dn'. \tag{10}$$

Solving it, we obtain the analytical approach to the fixed point N_1^* given by

$$N_{(n)} = \frac{1}{(\gamma an + {}^1/{N_0})}. \tag{11}$$

Let us then discuss the implication of equation (11) for a specific range of n, when $\gamma an \gg {}^1/{N_0}$, which is equivalent to the previous section of the case $i)$ for $\mu = 0$. Then, we obtain

$$N(n) \propto n^{-1}. \tag{12}$$

After a comparison with equation (3), we see that at $r_c = 1$ the critical exponent $\beta = -1$ is well agreement to the numerical results presented in figure 2(a).

Let us discuss the implication when we consider the case $ii)$. Thus, we obtain

$$\begin{aligned} N_{n+1} - N_n &= rN_n(1 - \gamma\alpha N_n) - N_n\ , \\ &= \frac{N_{n+1} - N_n}{(n+1) - n} \approx \frac{dN}{dn}\ , \\ &= -(1-r)N - raN^2. \end{aligned} \tag{13}$$

However, near the fixed point $N_1^* = 0$ we consider the term raN^2 going to zero faster than $-(1-r)N$. Thus, the last term of equation (13) is neglected and we have

$$\frac{dN}{dn} = -\mu N, \tag{14}$$

where $\mu = 1 - r$. Considering again that for $n = 0$ the initial condition is N_0, we have to integrate the following equation

$$\int_{N_0}^{N(n)} \frac{dN'}{N'} dN = -\mu \int_0^n dn'. \tag{15}$$

After integration and grouping the terms we obtain

$$N(n) = N_0 e^{-\mu n}. \tag{16}$$

A comparison with equations (4) and (5) leads us to conclude that the relaxation exponent $\delta = -1$, it is again in well agreement with the simulation, as confirmed in figure 2(b).

3 Discussions and conclusions

We have considered the convergence to the fixed point at the transcritical bifurcation and around it. The map is characterized by control parameters, where we keep a, γ and we vary r, consequently, inducing bifurcations in the system. For the Hassell mapping, the convergence to the stationary point N_1^* at the transcritical bifurcation is given by a power law with exponent $\beta = -1$. Around of the bifurcation, the system shows an exponential decay to the equilibrium point whose relaxation time is given by a power law with the exponent, namely $\delta = -1$, that is the relaxation exponent.

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