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# A note on the Liénard-Chipart criterion and roots of some families of polynomials

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**Abstract.** We present some inequalities that provide different sufficient conditions for an univariate monic polynomial to be Hurwitz unstable. These are motivated by difficult control problems where direct application of the Liénard-Chipart criterion is not feasible. Hurwitz stability of some polynomials of degree five is also discussed. These results may be interpreted as stability results for some interval polynomials.

**Keywords**. Control Theory, Robust stability, Roots of polynomial equations

## 1 Introduction

In this note we'll say that a polynomial

$$p(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n$$

is Hurwitz stable if all roots have negative real part. Otherwise, the polynomial will be said Hurwitz unstable.

When $\alpha_i$ is positive for $i = 1, \ldots, n$, that is, all coefficients of $p(s)$ are positive, it is straight forward to prove that no real root can be strictly positive: Actually, it suffices to see that if $s > 0$ then $p(s) > 0$, but this observation also follows from Descartes' rule of signs. Furthermore if $p(s)$ has a root in $s = 0$ then $\alpha_n = 0$.

Hence, if we're trying to prove stability of a polynomial and if all coefficients $\alpha_i$ of $p(s)$ are strictly positive, we are left only to worry about the possibility of one of the complex roots to have positive real part. This still is a very complicated problem. For instance, bounds for roots like the ones derived from well-known ideas in complex analysis are not immediately useful since, a priori, all restrictions are given in absolute value (a non-trivial result that can be obtained using complex analysis is given by Routh's algorithm, see [4]).

The Liénard-Chipart stability criterion is a standard tool to understand the Hurwitz stability problem, which in turn has important consequences on the dynamics of some systems of differential equations. The theorem can be enunciated in the following way:

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**Theorem 1.1** (see [4], [6]). *Let* $p(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n$. *A necessary and sufficient condition for all roots of* $p(s)$ *to have negative real parts is that*

$$\alpha_i > 0, \forall i = 1, 2, \ldots, n \;\; and \;\; \Delta_2 > 0, \Delta_4 > 0, \ldots, \Delta_m$$

*where*

$$\Delta_i = \begin{vmatrix} \alpha_1 & \alpha_3 & \alpha_5 & \ldots & \alpha_{2i-1} \\ 1 & \alpha_2 & \alpha_4 & \ldots & \alpha_{2i-2} \\ 0 & \alpha_1 & \alpha_3 & \ldots & \alpha_{2i-3} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \ldots & \ldots & \alpha_i \end{vmatrix}$$

*with* $\alpha_k = 0$ *if* $k > n$ *and* $m$ *is the biggest even number smaller than* $n$.

This result can be thought as a simplification of the Routh-Hurwitz theorem in which, if all coefficients of $p(s)$ are positive, stability can be guaranteed by checking that $\Delta_i$ is positive for $i = 1, \ldots, n$. The Liénard-Chipart criterion can also be stated using $\Delta_i$ for $i$ odd instead of even.

In section 2 we discuss how linearization together with the Liénard-Chipart criterion may reduce the dynamic stability problem to the calculations of the $\Delta_i$. However control problems in engineering can generate, in this manner, very complicated algebraic expressions that resist to simplifications, even by means of computational algebra (see for instance [7]). Numerical simulations are often the only solution available to study the dynamics, as we resort to sampling values for the coefficients. On the other hand generic conditions on the coefficients, like the aforementioned $\alpha_i > 0$, may be physically natural, easier to evaluate or even be chosen by design.

What we aim to do in this note is simplify a condition in the Liénard-Chipart criterion as much as possible, to provide more approachable and non-trivial necessary or sufficient conditions for the Hurwitz (in)stability of some polynomials. With this in mind, and assuming that all $\alpha_i$ are positive, it is worthwhile to study the sign of $\Delta_2$ instead of $\Delta_3$, so the next term to study is $\Delta_4$, with the sign of $\Delta_6$ being apparently much harder to understand.

To study the sign of $\Delta_4$ we use a, to the best of our knowledge, new formula for $\Delta_4$ which is presented in Theorem 3.1. This is done in section 3 with its corollaries for instability being discussed in section 4 and for stability of degree five polynomials in section 5.

In [1] the authors raise the question of whether the Liénard-Chipart criterion is the simplest possible condition for real polynomials to be Hurwitz stable, regarding both the number of conditions and to the degree of the sum of the $\Delta_i$'s. Corollary 1 shows that a milder version of this question is not true: There are lower degree equations that form sufficient conditions for a polynomial to be Hurwitz unstable, although here the number of conditions is the same. The number of conditions can be only tree for some polynomials of arbitrary degree (see [3]). Conclusions are presented in section 6.

## 2 Application on control of non-linear systems

In this section we briefly recall well known results in control theory to expand on their relation with Liénard-Chipart's criterion. A starting point is the idea of stability by linearization but for non-autonomous differential equations.

**Theorem 2.1** (see [5]). *Let $x = 0$ be an equilibrium point for $\dot{x} = f(t,x)$. Assume $f : \mathbb{R}^+ \times D \to \mathbb{R}$ is continuously differentiable where $D$ is an open ball centered at the origin and that the linearization $J = [\frac{\partial f}{\partial x}]$ of $f$ is, uniformly on time, bounded and Lipschitz on $D$. Then $x = 0$ is exponentially stable for $\dot{x} = f(t,x)$ if and only if $x = 0$ is exponentially stable for $\dot{x} = A(t)x$ where $A(t) = [\frac{\partial f}{\partial x}(t,x)]_{|x=0}$*

For autonomous linear systems, exponential stability is guaranteed by negative eigenvalues. For non-autonomous linear systems this is no longer true even if the linearization is differentiable over time. Nevertheless, we have the following result.

**Theorem 2.2** (see [8]). *Assume that the entries $a_{ij}$ of $A$ are differentiable and limited for every $t \in \mathbb{R}^+$. If for every eigenvalue $\lambda(A(t))$ of $A$ we have that $Re(\lambda(A(t)) \leq \varepsilon < 0$ then there is $\delta > 0$ such that if $|\dot{a_{ij}}| < \delta$ then $x = 0$ is exponentially stable for $\dot{x} = A(t)x$*

We note now that, in order to study the stability of the equilibrium at the origin, it is not necessary to determine the spectrum of $A$, but only if the real part of its eigenvalues is negative. This takes us back to the concept of Hurwitz stability previously defined and in turn to Liénard-Chipart's criterion.

However, since the entries $a_{ij}$ of $A$ are functions of time, determination of the signs of the $\Delta_i$'s cannot in general be proven by simple inspection (see [7] for an example where the problem comes from the control of a model of a wheeled robot). It should be noted that, for most matrices, determinants have a high computational cost (consider for example the number of operations), so choosing a time grid to empirically study the sign of the $\Delta_i$'s can evolve into long computation times.

## 3 A formula for $\Delta_4$

**Theorem 3.1.** *We have that*

$$\Delta_4 = -\alpha_2(\alpha_5 - \alpha_1\alpha_4)\Delta_2 - \alpha_4\Delta_2^2 - (\alpha_5 - \alpha_1\alpha_4)^2 - (\alpha_7 - \alpha_1\alpha_6)\Delta_2$$

*where $\Delta_2 = \alpha_1\alpha_2 - \alpha_3$.*

*Proof.* By definition

$$\Delta_4 = \begin{vmatrix} \alpha_1 & \alpha_3 & \alpha_5 & \alpha_7 \\ 1 & \alpha_2 & \alpha_4 & \alpha_6 \\ 0 & \alpha_1 & \alpha_3 & \alpha_5 \\ 0 & 1 & \alpha_2 & \alpha_4 \end{vmatrix}$$

Expansion by the first column leads to the expression

$$\Delta_4 = \alpha_1\alpha_2\alpha_3\alpha_4 + 2\alpha_1\alpha_4\alpha_5 - \alpha_1\alpha_2^2\alpha_5 - \alpha_1^2\alpha_4^2 - \alpha_3^2\alpha_4 - \alpha_5^2 + \alpha_2\alpha_3\alpha_5 + \\ + \alpha_1^2\alpha_2\alpha_6 - \alpha_1\alpha_3\alpha_6 - \alpha_1\alpha_2\alpha_7 + \alpha_3\alpha_7$$

that can be obtained by developing the formula given in the statement. □

The alternative formula for $\Delta_4$ in Theorem 3.1 was originally obtained in most part by the method described in [2] for degree five (where $\alpha_6$ and $\alpha_7$ are both zero).

This formula facilitates the determination of the sign of $\Delta_4$, as we'll see bellow, and can also be easily implemented to minimize computational cost and cumulative error of non-specialized software, being particularly useful to study polynomials of degree five.

## 4 Some consequences for Hurwitz instability

**Corollary 4.1.** *Let $p(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n$, with $n \in \mathbb{N}$ greater than or equal to four. Then a sufficient condition for $p(s)$ to be unstable is that*

$$\alpha_5 - \alpha_1\alpha_4 \geq 0 \qquad \textit{and} \qquad \alpha_7 - \alpha_1\alpha_6 \geq 0$$

*Proof.* We can assume that $\alpha_i > 0$, for $i = 1, \ldots, n$ and $\Delta_2 > 0$, since otherwise $p(s)$ is automatically unstable by the Liénard-Chipart criterion (Theorem 1.1). Therefore, by Theorem 3.1, $\Delta_4 \leq 0$ so $p(s)$ is Hurwitz unstable. □

Note that the second inequality in the corollary is automatically satisfied if the degree of $p(s)$ is five, but the first condition is never satisfied when the degree of $p(s)$ is four. Still the corollary is stated for degree four polynomials since this is what's used on the Liénard-Chipart criterion.

**Corollary 4.2.** *If $\alpha_7 - \alpha_1\alpha_6 \geq 0$ and $n$ is greater than or equal to four the polynomial*

$$p(s) = s^n + \alpha_1 s^{n-1} + 2s^{n-2} + \alpha_3 s^{n-3} + s^{n-4} + \alpha_5 s^{n-5} + \ldots + \alpha_n$$

*is unstable*

*Proof.* Assume that $\Delta_2 > 0$. Since $\alpha_2 = 2$ and $\alpha_4 = 1$ we have that

$$\Delta_4 = -2(\alpha_5 - \alpha_1\alpha_4)\Delta_2 - \Delta_2^2 - (\alpha_5 - \alpha_1\alpha_4)^2 - (\alpha_7 - \alpha_1\alpha_6)\Delta_2 = \\ = -[\Delta_2 + (\alpha_5 - \alpha_1\alpha_4)]^2 - (\alpha_7 - \alpha_1\alpha_6)\Delta_2 \leq 0$$

□

Recall that if every $\alpha_i$ is strictly positive then all real roots must be strictly negative. Therefore in such a case and if $p(s)$ is as in Corollary 4.2 then $p(s)$ needs to have at least a pair of complex conjugated roots with positive real part. Also, if $n$ is odd we can assure that there will be at least one negative real root.

**Corollary 4.3.** *Let $p(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n$, with $n \in \mathbb{N}$ greater than or equal to four. Then a sufficient condition for $p(s)$ to be unstable is that*

$$\alpha_2^2 - 4\alpha_4 \leq 0 \qquad \textit{and} \qquad \alpha_7 - \alpha_1\alpha_6 \geq 0$$

*Proof.* Note that if $\alpha_5 - \alpha_1\alpha_4 = 0$ then $\Delta_4 < 0$ so there's nothing to do.Otherwise define

$$\Gamma := \frac{\Delta_4 + (\alpha_7 - \alpha_1\alpha_6)\Delta_2}{(\alpha_5 - \alpha_1\alpha_4)^2} = -\alpha_2 \frac{\Delta_2}{(\alpha_5 - \alpha_1\alpha_4)} - \alpha_4 \frac{\Delta_2^2}{(\alpha_5 - \alpha_1\alpha_4)^2} - 1$$

and note that $\Gamma \leq 0$ implies that $\Delta_4 \leq 0$. Let

$$\gamma := \frac{\Delta_2}{(\alpha_5 - \alpha_1\alpha_4)}$$

so $\Gamma = -\alpha_4\gamma^2 - \alpha_2\gamma - 1$.

As a function of $\gamma$, $\Gamma$ is concave, since we can assume that $\alpha_4 > 0$. Then for $\Gamma(\gamma)$ to be positive for some $\gamma$ we need to have distinct real roots, that is $\alpha_2^2 - 4\alpha_4 > 0$. Note that we would also need that

$$\frac{\alpha_2 + \sqrt{\alpha_2^2 - 4\alpha_4}}{-2\alpha_4} < \gamma < \frac{\alpha_2 - \sqrt{\alpha_2^2 - 4\alpha_4}}{-2\alpha_4}$$

□

An easy consequence of the previous result is that, if $\alpha_7 - \alpha_1\alpha_6 \geq 0$,

$$p(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \alpha_3 s^{n-3} + s^{n-4} + \alpha_5 s^{n-5} + \ldots + \alpha_n$$

is unstable for every $0 < \alpha_2 \leq 2$ (compare to Corollary 4.2).

# 5 Some consequences for Hurwitz stability

A reformulation of the previous corollary gives a criteria for stability.

**Corollary 5.1.** *Let $p(s) = s^5 + \alpha_1 s^4 + \alpha_2 s^3 + \alpha_3 s^2 + \alpha_4 s + \alpha_5$. Assume that $\alpha_i > 0$ for $i = 1, \ldots, 5$ and that $\Delta_2 > 0$. Then a necessary condition for $p(s)$ to be Hurwitz stable is that*

$$\alpha_2^2 - 4\alpha_4 > 0$$

Note using Corollary 4.1 that if the hypotheses of Corollary 5.1 are valid then we must have $(\alpha_5 - \alpha_1\alpha_4) < 0$. Compare this to how the Liénard-Chipart shows that some conditions of the Routh-Hurwitz theorem are not independent. Our final result is still impractical as a tool to study problems like in [7], but is provided as an alternative to direct application of Theorem 1.1 or to the final inequality in Corollary 4.3.

**Corollary 5.2.** *Let $p(s) = s^5+\alpha_1 s^4+\alpha_2 s^3+\alpha_3 s^2+\alpha_4 s+\alpha_5$ where $\alpha_i > 0$ for $i = 1, 2, \ldots, 5$. Assume that $\Delta_2 > 0$ and assume that*

$$\alpha_2^2 - 4\alpha_4 > 0$$

*Then $\alpha_5 - \alpha_1\alpha_4 < 0$ and a sufficient condition for $p(s)$ to be stable is that*

$$\frac{\alpha_1\alpha_2 - \alpha_3}{\alpha_5 - \alpha_1\alpha_4} = \frac{-\alpha_2}{2\alpha_4}$$

*Proof.* Recall as above that $\alpha_5 - \alpha_1\alpha_4 < 0$. Just note then that if $\overline{\gamma}$ denotes the vertex of $\Gamma(\gamma)$ then $\overline{\gamma} := \frac{-\alpha_2}{2\alpha_4}$. Since $\Gamma(\gamma)$ has two distinct roots then $\Gamma(\gamma) = \Gamma(\overline{\gamma}) > 0$, therefore $\Delta_4 > 0$.

It's also possible to check directly that $\Delta_4 > 0$ by studying the sign of $\Gamma$. □

Finally, we note that algebraic manipulations show that the sufficient condition above can be written as

$$\alpha_1 = \frac{2\alpha_3}{\alpha_2} - \frac{\alpha_5}{\alpha_4} \qquad \text{or as} \qquad \Delta_2 = \alpha_3 - \frac{\alpha_2\alpha_5}{\alpha_4}$$

## 6 Conclusion

In this note we propose, based on the Liénard-Chipart's criterion, simpler criteria to determine whenever the equilibrium at the origin for some non-autonomous systems is exponentially stable or unstable. We saw that, in particular, it is possible to determine the sign of $\Delta_4$ reusing the value of $\Delta_2$ or rather just its sign.

In applications where linearization leads to a characteristic polynomial of degree greater than four the inequalities provided for instability can be significantly easier to verify than routine application of Liénard-Chipart's criterion, since the coefficients are assumed time-dependent. This makes us ask the following question:

**Conjecture 6.1.** *Is there a similar formula for $\Delta_{2l}$ in terms of $\Delta_{2(l-m)}$ for $l > 2$ and $l > m$? In other words, can we decide that $\Delta_{2l} \leq 0$ using at least $2(l-1)$ variables $\alpha_k$ if we already know that $\Delta_{2(l-m)} > 0$ for $l > 2$ and $l > m$?*

Of course, an analogous question can be made for the $\Delta_i$'s with $i$ odd.

## 7 Thanks

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