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Analysis of the Critical Inclination in Orbit of Planetary Moons

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Abstract. One of the first authors to analyze the problem of critical inclination was Brouwer [3] in 1959. But even today, several papers are analyzing this problem.Here, we present an approach to analyze the critical inclination of orbits around the planetary moons considering the J_2 and C_{22} terms of the their non-uniform distribution of mass. The main focus is in the development of the equation due to the C_{22} term (equatorial ellipticity), where the explicit dependence of time is not eliminated, rather, we use a mathematical expression to maintain this time dependence in the equations. Applications are made for several moons of our solar system. We show that the value of the critical inclination depends on the longitude of the ascending node and of the time. The value of the J_2/C_{22} ratio, condition of the hydrostatic equilibrium, affects the result of the critical inclination.

Keywords. Astrodynamics, Critical Inclination, Non-sphericity, Artificial Satellite.

1 Introduction

In this work the behavior of the critical inclination of orbits around several moons of the solar system is analyzed. To this end, it is taken into account in the disturbing potential the terms due to non-sphericity (J_2 and C_{22}) of the natural satellite. We present an approach considering the effect of the J_2 and C_{22} terms, where a new equation for critical inclination is obtained, see also [16]. In [6] the behavior of critical inclinations and sun-synchronous orbits of artificial satellites orbiting planetary satellites are analyzed considering, simultaneously, the in influence of the harmonics J_2 and C_{22}, both due to the non-uniform mass distribution of the natural satellite. Here, in this work, in the equation due to the effect of C_{22} is added a new term in relation to the equation presented in [6]. In [1], the authors find a family of frozen orbits for lunar artificial satellites. For very

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low nearly circular lunar orbits, different families of the critical inclination are obtained. In [13], the effects of the first sectorial and tesseral harmonic coefficients in addition to the first zonal harmonic coefficients on the critical inclination of a lunar artificial satellite are investigated. In [5], three families of orbits, characterized by constant orbit elements on average, have been obtained. For the latter two families, the concept of critical inclination, which, in the case of only planetary oblateness, assumes the well-known values 63.43 and 116.57 deg, has been generalized adding the effect of several attracting bodies.

2 Disturbing Potential

Considering the equatorial plane of the planet (or planetary satellite) as the reference plane, the disturbing potential can be written in the form:

$$U_M = -\frac{\mu}{r}\left[\sum_{n=2}^{\infty}\left(\frac{R_p}{r}\right)^n J_n P_n(\sin\phi) - \left(\frac{R_p}{r}\right)^2 C_{22} P_{22}(\sin\phi)\cos(2\lambda)\right], \tag{1}$$

where μ is the gravitational constant of the planet, R_p is the equatorial radius of the planet, P_n are the Legendre polynomials, P_{nm} the associated Legendre polynomials, the angle ϕ is the latitude of the orbit with respect to the equator of the planet, the angle λ is the longitude measured from the direction of the longest axis of the planet. Here λ contains the time explicitly [7]. Using spherical trigonometry we have $\sin\phi = \sin i \sin(f+g)$. The Legendre polynomials for J_2 and the Legendre associated functions for the sectorial C_{22} term can be written in the form [7]

$$\begin{aligned} &P_2(\sin\phi) = \tfrac{1}{2}(3s^2\sin^2(f+g)-1), \\ &P_{22}(\sin\phi)\cos 2\lambda = 6(\xi^2\cos^2 f + \chi^2\sin^2 f + \xi\chi\sin 2f) - 3(1 - s^2\sin^2(f+g)), \end{aligned} \tag{2}$$

where we used the shortcut $\xi = \cos g\cos\Omega - c\sin g\sin\Omega$, $\chi = -\sin g\cos\Omega - c\cos g\sin\Omega$, $s = \sin i$, and $c = \cos i$. Here Ω is the selenographic node longitude of the orbit and f is true anomaly of the artificial satellite. Now, we write the potential given by equation (1) as a function of the orbital elements. Invoking equation (2) and the relation $\mu = n^2a^3$, we get

$$U_{20} = -\frac{1}{2}\frac{a^3}{r^3}\epsilon n^2(3s^2(\sin(f+g))^2 - 1) \tag{3}$$

where $\epsilon = J_2R_p^2$. Analogously, for the sectorial perturbation [7], we get

$$U_{22} = \frac{a^3}{r^3}\delta n^2(6\xi^2(\cos(f))^2 + 6\chi^2(\sin(f))^2 + 12\xi\chi\sin(2f) - 3 + 3s^2(\sin(f+g))^2) \tag{4}$$

where $\delta = C_{22}R_p^2$. To write the long-period disturbing potential, we apply the single-averaged model. This is done to eliminate the short period terms. The development of the equations is carried out in closed form to avoid expansions in eccentricity and inclination. For this, it was necessary to perform algebraic manipulations where we used known equations of celestial mechanics, namely equation (5). The average method was applied as a function of true anomaly using equation (6).

$$a/r = (1 + e\cos(f))/(1 - e^2) \tag{5}$$

$$dl = \frac{1}{\sqrt{1-e^2}}\frac{r^2}{a^2}df \tag{6}$$

where l is the mean anomaly of the spacecraft. Using equations. (3), (5) and (6), and applying the single-averaged model, after some algebraic manipulations, we get

$$< R_{J2} >= -\frac{1}{4}\frac{\epsilon}{(1-e^2)^{3/2}}n^2(3s^2 - 2) \tag{7}$$

Now, using equation (4) and the ξ and χ variables to develop the potential due to the equatorial ellipticity of the planet, we get

$$< R_{C22} >= -\frac{3}{2}\frac{\delta}{(1-e^2)^{3/2}}n^2(c^2 - 1)\cos(2\Omega) \tag{8}$$

The approach to analyze the effect of the C_{22} term is based in [11]. In equation (8), the selenographic node longitude of the orbit Ω is replaced by the expression $\Omega = h - wt$, given by [14], see also [11], where h is the right ascension of the ascending node, w is the rotation rate of the moon and t is the time. So, replacing the expression given by [14], equation (8) is written in the form

$$< R_{C_{22}} >= -\frac{3}{2}\frac{\delta}{(1-e^2)^{3/2}}n^2(c^2 - 1)\cos(2wt - 2h) \tag{9}$$

Replacing equations (7) and (9) in the Lagrange planetary equations, we get

$$\frac{dg}{dt} = -\frac{3}{4}\frac{n\left(10\,(\cos(i))^2\cos(-2\,w\,t+2\,h)\delta - 5\,(\cos(i))^2\epsilon - 6\,\delta\,\cos(-2\,w\,t+2\,h)+\epsilon\right)}{a^2(e^2-1)^2} \tag{10}$$

We have, therefore, a non-autonomous non-linear differential equation. The critical inclination (i_c) is found solving for $dg/dt = 0$. Therefore, we get

$$i_c = \arccos\left(\frac{1}{5}\frac{\sqrt{5}\sqrt{(2\,\delta\,\cos(2\,wt-2\,h)-\epsilon)(6\,\delta\,\cos(2\,wt-2\,h)-\epsilon)}}{2\,\delta\,\cos(2\,wt-2\,h)-\epsilon}\right) \tag{11}$$

$$i_c = \pi - \arccos\left(\frac{1}{5}\frac{\sqrt{5}\sqrt{(2\,\delta\,\cos(2\,wt-2\,h)-\epsilon)(6\,\delta\,\cos(2\,wt-2\,h)-\epsilon)}}{2\,\delta\,\cos(2\,wt-2\,h)-\epsilon}\right) \tag{12}$$

where equations (11) and (12) represent the critical inclination for direct and retrograde orbits, respectively. Note that when δ is neglected, the critical inclination for direct orbits is $i_c = 63.43°$ and for retrograde ones, $i_c = 116.56°$.

3 Results

Equations (11) and (12) show that the critical inclination depends on the parameters h and t. Using these equations we show the results from Figures 1-6. Note that Figures 1 and 2 were plotted in $3D$, where t is ranging from 0 (zero) to an orbital period of the moon considered in the dynamics. To generate the figures in $2D$ the time was fixed by an orbital period for each moon around your planet. The coefficients of the gravity field is given in Table 1. The assumption of hydrostatic equilibrium is that a degree-2 gravity field and moments of inertia dominated by hydrostatic rotational and tidal deformation is given by $\frac{J_2}{C_{22}} \sim \frac{10}{3}$. Note that Figures 3, 4, 5 and 6 present similar results, where the critical inclination is obtained for some values of h. But some values are not allowed, where the results are complex numbers. The fact of this similarity is that all these moons have the ratio $\frac{J_2}{C_{22}}$ close to the value of the hydrostatic equilibrium, as shown in Table 1. Now, note that Figures 3 (Phobos case) and 5 (Moon case) show that the critical inclination is obtained for all values of h, and according to Table 1, the Moon and Phobos not meet the criteria of hydrostatic equilibrium.

Table 1: Values of the spherical harmonic coefficients.

Reference	Central Body	$J_2 \times 10^{-6}$	$C_{22} \times 10^{-6}$	J_2/C_{22}
[10]	Moon	203.31	22.426	9.0655
[9]	Phobos	0105700	0014800	7.1419
[12]	Ganymede	127.53	38.26	3.3332
[4]	Europa	435.50	130.65	3.3244
[2]	Callisto	32.7	10.2	3.2059
[15]	Rhea	946	241.1	3.9075
[8]	Titan	33.462	10.022	3.3389

4 Conclusions

We show an equation to calculate the critical inclination of orbits around planetary moons considering the J_2 and C_{22} terms, where in the development of C_{22} the explicit dependence of the time was not eliminated. We use here the expression $\Omega = h - wt$, given by [14], see also [11], to maintain the explicit dependence of time in the equations. The equations obtained to calculate the critical inclination depend of the time and longitude of the ascending node (h). Applications are made considering the following moons: Earth's moon, Europa, Callisto and Ganymede (moons of Jupiter), Phobos (moon of Mars) Rhea and Titan (moons of Saturn). We found that for the moons where the ratio of J_2/C_{22} satisfies the hydrostatic equilibrium conditions the critical inclination is obtained for some values of h, but there are values that are not possible to obtain the critical inclination. Now, the moons that do not satisfy such relation, the result obtained is that the critical inclination is found for all values of h.

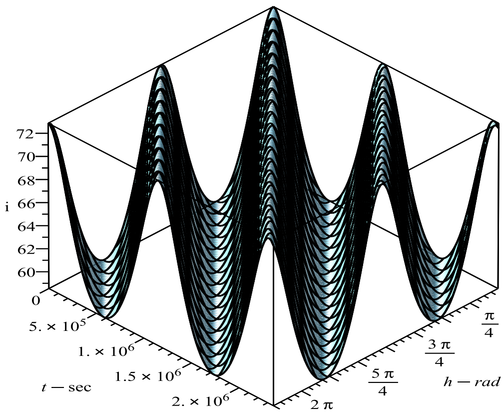

Figure 1: Diagram $i_c \times h$. (Moon). Perturbations: $J_2 + C_{22}$.

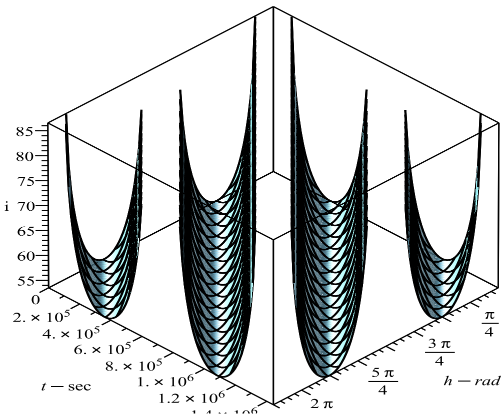

Figure 2: Diagram $i_c \times h$. (Callisto). Perturbations: $J_2 + C_{22}$.

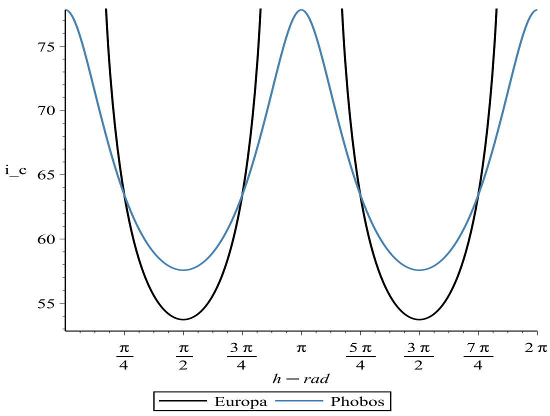

Figure 3: Diagram $i_c \times h$. Perturbations: $J_2 + C_{22}$.

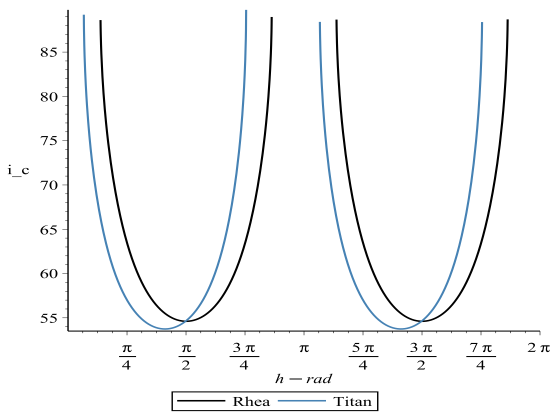

Figure 4: Diagram $i_c \times h$. Perturbations: $J_2 + C_{22}$.

The critical inclination for retrograde orbits will not be investigated here in this paper, but in an extended version, we will take into account these orbits. We will also investigate in more detail the relation J_2/C_{22}, condition of the hydrostatic equilibrium, in the study of the critical inclination of planetary moons. We must also expand the phase space, adding one more equation to the system, to eliminate the explicit dependency of time and to compare the results.

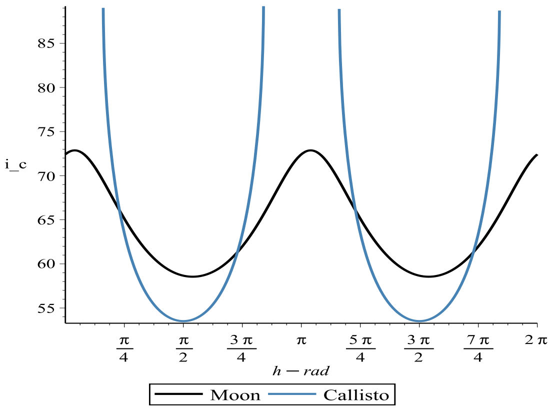

Figure 5: Diagram $i_c \times h$. Perturbations: $J_2 + C_{22}$.

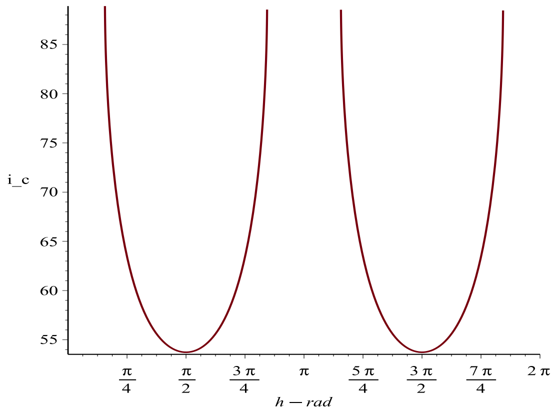

Figure 6: Diagram $i_c \times h$. (Ganymede). Perturbations: $J_2 + C_{22}$.

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