DINCON 2017
CONFERÊNCIA BRASILEIRA DE DINÂMICA, CONTROLE E APLICAÇÕES
30 de outubro a 01 de novembro de 2017 – São José do Rio Preto/SP

Optimal two-impulse Earth-to-Moon trajectories based on elliptic patched-conic approximation

Sandro da Silva Fernandes[1]
Marcus Vinicius Cardoso Macêdo[2]
Luiz Arthur Gagg Filho[3]

Departamento de Matemática, Instituto Tecnológico de Aeronáutica, São José dos Campos – 12228-900 – SP – Brasil

Abstract. In this paper, the problem of transferring a space vehicle from a circular low Earth orbit (LEO) to a circular low Moon orbit (LMO) with minimum fuel consumption is studied. The class of two impulse trajectories is considered: a first accelerating velocity impulse tangential to the space vehicle velocity relative to Earth is applied at a circular LEO and a second braking velocity impulse tangential to the space vehicle velocity relative to Moon is applied at a circular LMO. The dynamical model used to describe the motion of the space vehicle is based on a new version of the patched-conic approximation which includes the eccentricity of the orbit of the Moon. The optimization problem has been solved by means of a gradient algorithm in conjunction with Newton-Raphson method. Numerical results are compared to other ones that use the classical patched-conic approximation or the classical circular restricted three-body problem.

Palavras-chave. Earth-to-Moon trajectories, optimal space trajectories, minimum fuel trajectories.

1 Introduction

In this paper, the problem of transferring a space vehicle from a circular low Earth orbit (LEO) to a circular low Moon orbit (LMO) with minimum fuel consumption is studied. The class of two impulse trajectories is considered: a first accelerating velocity impulse is applied at a circular low Earth orbit and a second braking velocity impulse is applied at a circular low Moon orbit; each impulse is tangential to the respective terminal orbit – LEO or LMO. The minimization of fuel consumption is equivalent to the minimization of the total characteristic velocity which is defined by the arithmetic sum of velocity changes [3].

[1] sandro@ita.br
[2] mvmacedo@ita.br
[3] luizarthur.gagg@gmail.com

The dynamical model used to describe the motion of the space vehicle is based on a new version of the patched-conic approximation which includes the eccentricity of the orbit of the Moon. This model provides an approximation to the planar elliptic restricted three-body problem (PER3BP). Since the eccentricity of the orbit of the Moon is included, the optimization problem has two degrees of freedom and has been solved by means of a gradient algorithm [4] in conjunction with Newton-Raphson method [6]. The analysis of optimal trajectories is carried out considering several final altitudes of a clockwise or counterclockwise circular low Moon orbit for a specified altitude of a counterclockwise circular low Earth orbit. Only the outgoing trip with direct ascent is considered in this study. Numerical results are compared to the ones obtained using the classical patched conic approximation and the results obtained using the circular restricted three-body problem, found in the literature.

2 Optimization problem based on patched-conic approximation

In this section, the optimization problem based on a new version of the patched-conic approximation, which includes the eccentricity of the orbit of the Moon, is formulated. The following assumptions are employed:

1. The Earth is fixed in space;
2. The eccentricity of the Moon orbit around Earth is considered;
3. The flight of the space vehicle takes place in the Moon orbital plane;
4. The gravitational fields of Earth and Moon are central and obey the inverse square law;
5. The trajectory has two distinct phases: geocentric and selenocentric trajectories. The geocentric phase corresponds to the portion of the trajectory which begins at the point of application of the first impulse and extends to the point of entering the Moon’s sphere of influence. The selenocentric phase corresponds to the portion of trajectory in the Moon’s sphere of influence and ends at the point of application of the second impulse. In each one of these phases, the space vehicle is under the gravitational attraction of only one body, Earth or Moon;
6. The class of two impulse trajectories is considered. The impulses are applied tangentially to the space vehicle velocity relative to Earth (first impulse) and Moon (second impulse).

For a given initial position of the Moon on its orbit defined by the true anomaly $f_M(t_0)$, an Earth-Moon trajectory is completely specified by four quantities: r_0 - radius of circular LEO; v_0 - velocity of the space vehicle at the point of application of the first impulse after the application of the impulse; ϕ_0 - flight path angle at the point of application of the first impulse and γ_0 - phase angle at departure. These quantities must be determined such that the space vehicle is injected into a LMO with specified altitude after the application of the second impulse. It is particularly convenient to replace γ_0 by the angle λ_1 which specifies the point at which the geocentric trajectory crosses the Moon’s sphere of influence. On the other hand, it should be noted that the initial position of the Moon determines the distance *D* between Earth and Moon at the time at which the space vehicle reaches the sphere of influence of the Moon. So, it is also convenient to specify the eccentric anomaly $E_M(t_1)$, replacing the true anomaly $f_M(t_0)$, which can be determined after solving the two-point boundary value problem of going from LEO to LMO.

Equations describing each phase of an Earth-Moon trajectory are briefly presented in what follows. It is assumed that the geocentric trajectory is direct and that lunar arrival occurs prior to apoapsis of the geocentric orbit. Figure 1 shows, separately, the geometry of the geocentric phase and of the selenocentric phase for a clockwise arrival to LMO.

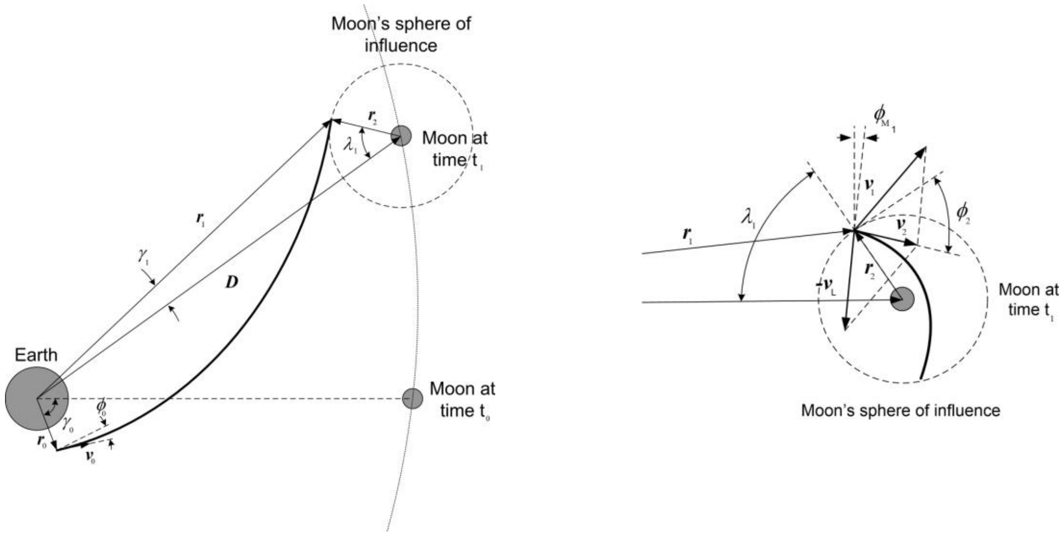

Figure 1 – Geometry of geocentric and selenocentric phases

For a given set of initial conditions (r_0, v_0, ϕ_0) and specified eccentric anomaly $E_M(t_1)$, energy and angular momentum of the geocentric trajectory can be determined from the equations

$$\varepsilon = \frac{1}{2}v_0^2 - \frac{\mu_E}{r_0}\,, \tag{1}$$

$$h = r_0 v_0 \cos\phi_0\,, \tag{2}$$

where μ_E is Earth gravitational parameter.

From the geometry of the geocentric phase (Fig. 1), one finds

$$r_1 = \sqrt{D^2 + R_S^2 - 2DR_S\cos\lambda_1}\,, \tag{3}$$

$$\sin\gamma_1 = R_S\sin\lambda_1/r_1\,, \tag{4}$$

where D is the distance from the Earth to the Moon at time t_1, R_S is the radius of the Moon's sphere of influence. Subscript 1 denotes quantities of the geocentric trajectory calculated at the edge of the Moon's sphere of influence.

From energy and angular momentum of the geocentric trajectory, one finds

$$v_1 = \sqrt{2\left(\varepsilon + \mu_E/r_1\right)}\,, \tag{5}$$

$$\cos\phi_1 = \frac{h}{r_1 v_1}. \tag{6}$$

The selenocentric phase begins at the point at which the geocentric trajectory crosses the Moon's sphere of influence. Thus,

$$r_2 = R_S, \tag{7}$$

$$\mathbf{v_2} = \mathbf{v_1} - \mathbf{v}_{M_1}, \tag{8}$$

where $\mathbf{v}_{M_1}$ is the velocity vector of the Moon relative to the center of the Earth. Subscript 2 denotes quantities of the selenocentric trajectory calculated at the edge of the Moon's sphere of influence.

From Eq. (8), one finds

$$v_2 = \sqrt{v_1^2 + v_{M_1}^2 - 2 v_1 v_{M_1} \cos\left(\phi_1 - \gamma_1 - \phi_{M_1}\right)}, \tag{9}$$

$$\tan\left(\lambda_1 \pm \phi_2\right) = -\frac{v_1 \sin\left(\phi_1 - \gamma_1\right) - v_{M_1} \sin\phi_{M_1}}{v_{M_1} \cos\phi_{M_1} - v_1 \cos\left(\phi_1 - \gamma_1\right)}. \tag{10}$$

The upper sign refers to clockwise arrival to LMO and the lower sign refers to counterclockwise to LMO. ϕ_{M_1} is the flight path angle and v_{M_1} is the velocity of the Moon at time t_1, and they are computed using the well-known equations of the two-body problem.

The semi-major axis a_f and eccentricity e_f of the selenocentric trajectory are given by

$$a_f = \frac{r_2}{2 - Q_2}, \tag{11}$$

$$e_f = \sqrt{1 + Q_2\left(Q_2 - 2\right)\cos^2\phi_2}, \tag{12}$$

where $Q_2 = r_2 v_2^2 / \mu_M$ and μ_M is Moon gravitational parameter.

The second impulse is applied at the periselenium of the selenocentric trajectory such that the terminal conditions, before the impulse, are defined by

$$r_{p_M} = a_f\left(1 - e_f\right), \tag{13}$$

$$v_{p_M} = \sqrt{\frac{\mu_M\left(1 + e_f\right)}{a_f\left(1 - e_f\right)}}. \tag{14}$$

Equations (1) – (14) lead to the following two-point boundary value problem: For specified values of λ_1 and $E_M(t_1)$, and a given set of initial parameters r_0 and $\phi_0 = 0$ (the impulse is applied tangentially to the space vehicle velocity relative to Earth) determine v_0 such that the final condition $r_{p_M} = r_f$ is satisfied, where r_0 is the radius of LEO and r_f is the radius of LMO (both orbits, LEO and LMO, are circular). This boundary value problem can be solved by means of Newton-Raphson method [6].

After computing v_0, the velocity changes at each impulse can be determined

$$\Delta v_1 = v_0 - \sqrt{\frac{\mu_E}{r_0}} \,, \tag{15}$$

$$\Delta v_2 = \sqrt{\frac{\mu_M\left(1+e_f\right)}{a_f\left(1-e_f\right)}} - \sqrt{\frac{\mu_M}{r_f}} \,. \tag{16}$$

The total characteristic velocity is then given by

$$\Delta v_{Total} = \Delta v_1 + \Delta v_2 \,. \tag{17}$$

Note that the total characteristic velocity is function of λ_1 and $E_M(t_1)$ for a given set of parameters $(r_0, \phi_0 = 0, r_f)$. Accordingly, the following optimization problem can be formulated: Determine λ_1 and $E_M(t_1)$ to minimize Δv_{Total}. This minimization problem has been solved by means of a classic gradient method in conjunction with Newton-Raphson method [4].

The total flight time of an Earth-Moon trajectory is given by

$$T = \Delta t_E + \Delta t_M \,, \tag{18}$$

where Δt_E is the flight time of the geocentric trajectory and Δt_M is the flight time of the selenocentric trajectory. These times are calculated from the well-known equations times of flight of two-body dynamics as described in Gagg Filho and da Silva Fernandes (2016).

4 Results

In this section, results are presented for lunar missions using the elliptic patched conic approximation described in the preceding section. The results are compared to the ones found in the literature considering different dynamical models [1, 2, 5] and different optimization methods: [1, 5] uses the sequential gradient restoration algorithm and [2] uses the same algorithm adopted in this work. The following data are used: $\mu_E = 3.986\times10^5\ \text{km}^3/\text{s}^2$, $\mu_M = 4.903\times10^3\ \text{km}^3/\text{s}^2$, a_M (semi-major axis of the orbit of the Moon), $a_E = 6378\,\text{km}$ (Earth radius), $a_M = 1738\,\text{km}$ (Moon's radius), $h_0 =$ 463 km (altitude of LEO), $h_f = 100,\ 200,\ 300$ km (altitude of LMO), $e_M = 0.05490$ (Moon's eccentricity).

Table 1 shows the results for lunar missions with clockwise arrival at LMO and Table 2 shows the results for lunar missions with counterclockwise arrival at LMO. The departure from LEO is counterclockwise for all missions. The major parameters that are presented in these tables are the velocity changes Δv_1 and Δv_2 at each impulse, the total characteristic velocity $\Delta v_{Total} = \Delta v_1 + \Delta v_2$, the flight time T of lunar mission and the initial position of the space vehicle in the inertial reference frame Exy at the initial time $t_0 = 0$ defined by the angle $\theta_{EP}(0)$.

Table 1 – Lunar mission, clockwise LMO arrival, major parameters

LMO altitude *km*	***Model***	Δv_{Total} *km/s*	Δv_1 *km/s*	Δv_2 *km/s*	T *days*	$\theta_{EP}(0)$ *deg*
100	Classic Patched-conic	3.8528	3.0683	0.7845	4.936	–113.681
	Elliptic patched-conic	3.8437	3.0731	0.7706	5.344	–113.931
	Classical PCR3BP	3.8829	3.0686	0.8143	4.763	–113.795
200	Classic Patched-conic	3.8379	3.0683	0.7696	4.941	–113.638
	Elliptic patched-conic	3.8284	3.0731	0.7553	5.349	–113.898
	Classical PCR3BP	3.8688	3.0686	0.8002	4.769	–113.742
300	Classic Patched-conic	3.8243	3.0683	0.7560	4.944	–113.608
	Elliptic patched-conic	3.8144	3.0731	0.7413	5.352	–113.866
	Classical PCR3BP	3.8559	3.0687	0.7872	4.771	–113.716

Table 2 – Lunar mission, counterclockwise LMO arrival, major parameters

LMO altitude *km*	***Model***	Δv_{Total} *km/s*	Δv_1 *km/s*	Δv_2 *km/s*	T *days*	$\theta_{EP}(0)$ *deg*
100	Classic Patched-conic	3.8482	3.0655	0.7827	4.794	–115.723
	Elliptic patched-conic	3.8394	3.0704	0.7690	5.124	–116.473
	Classical PCR3BP	3.8777	3.0658	0.8119	4.573	–116.410
200	Classic Patched-conic	3.8331	3.0654	0.7677	4.794	–115.750
	Elliptic patched-conic	3.8240	3.0704	0.7536	5.125	–116.478
	Classical PCR3BP	3.8634	3.0658	0.7976	4.571	–116.451
300	Classic Patched-conic	3.8194	3.0654	0.7540	4.793	–115.777
	Elliptic patched-conic	3.8099	3.0703	0.7396	5.126	–116.504
	Classical PCR3BP	3.8502	3.0657	0.7845	4.569	–116.491

The results of Table 1 and 2 show that the eccentricity of the Moon’s orbit around Earth has effect on the fuel consumption for a direct ascent maneuver. In fact, the elliptic patched-conic approximation presents a Δv_{Total} smaller than the other models. However, the time of flight for the elliptic patched-conic approximation is larger. Moreover, the transfer with counterclockwise arrival (Table 1) has smaller fuel consumption than the transfer with clockwise arrival (Table 2) considering all the models, including the elliptic patched-conic approximation.

Conclusions

This paper presents a new lunar patched-conic approach where the eccentricity of the Moon's orbit around Earth is included in the model. Only bi-impulsive direct ascent maneuvers are considered for an Earth-Moon transfer. To determine the Earth-Moon trajectory, a two-point boundary value problem is proposed, in which the eccentric anomaly of the Moon at the instant when the space vehicle reaches the SOI of the Moon is set as a parameter. This problem has been solved by means of a gradient algorithm in conjunction with Newton-Raphson method, and the results are compared with other models that do not include the Moon's eccentricity. The data provided by the trajectories show that the elliptic patched-conic approximation saves fuel consumption if the eccentric anomaly of the Moon on its elliptic orbit is chosen properly.

Acknowledgements

This research is supported by CAPES, by CNPq under contract 304913/2013-8 and by grant 2012/25308-5, São Paulo Research Foundation (FAPESP).

References

[1] Arthur Gagg Filho L and Da Silva Fernandes S 2016 Optimal round trip lunar missions based on the patched-conic approximation *Computational and Applied Mathematics* **35**(3): p. 753-787.

[2] Da Silva Fernandes S and Marinho C M P 2012 Optimal two-impulse trajectories with moderate flight time for earth-moon missions *Mathematical Problems in Engineering* **2012**: p. 34-34.

[3] Marec J-P 1979 *Optimal space trajectories*: Elsevier.

[4] Miele A, Huang H Y, and Heideman J C 1969 Sequential gradient-restoration algorithm for the minimization of constrained functions---Ordinary and conjugate gradient versions *Journal of Optimization Theory and Applications* **4**(4): p. 213-243.

[5] Miele A and Mancuso S 2001 Optimal trajectories for Earth-Moon-Earth flight *Acta Astronautica* **49**(2): p. 59-71.

[6] Stoer J and Bulirsch R 2002 *Introduction to Numerical Analysis* ed. T. EditionNew York: Springer.