DYNAMIC PILE-SOIL-PILE INTERACTION BASED ON A LAGRANGIAN FUNCTIONAL APPROACH

Josue Labaki, labaki@fem.unicamp.br[1]
Euclides Mesquita, euclides@fem.unicamp.br[1]

[1]Department of Computational Mechanics, School of Mechanical Engineering, University of Campinas. Mendeleyev St. 200, Cidade Universitária Zeferino Vaz, Campinas SP, Brazil.

Abstract: *This paper presents a semi-analytical formulation of the problem of time-harmonic vertical vibration of two elastic bars interacting with each other through and with their surrounding transversely isotropic half-space. The formulation is presented within the framework of linear theory of elastic wave propagation that is applicable for most cases of inertial response of embedded elastic buries under external excitations. The embedding medium is described by a classical Green's functions corresponding to buried distributed vertical loading in a homogeneous half-space. The motion of the bar is represented by a time-harmonic polynomial function containing a set of generalized coordinates. The body force field in the half-space corresponding to each polynomial shape function of the bar is obtained numerically by solving a flexibility equation system based on the Green's functions corresponding to buried distributed vertical body forces in the half-space. The generalized coordinates of the bar's displacement function are thereafter computed by formulating the Lagrange's equations of motion for the bar. Numerical solution of the equation of motion results in the solutions for generalized coordinates. The bar displacements and axial load profiles are determined from the solutions for generalized coordinates. The influence of the bars on each other is obtained by considering that the bar at the measuring end is a one-dimensional line within the half-space. This is an adequate hypothesis, given that the distance between piles is much larger than their radii in most engineering applications. This technique adequately incorporates the wave propagation through the half-space from one bar to the other. Vertical impedance of the two-bar system is presented for different frequency of excitations and distances between bars. The present formulation can be readily extended for the case of a larger pile group.*

Keywords: *soil-foundation interaction, pile groups.*

1. INTRODUCTION

Several studies are reported in the literature for pile dynamics based on approximate numerical and semi-analytical schemes. Novak (1977) investigated the harmonic response of piles embedded in layered soils by using a thin layer approximation. Apsel (1979) solved the problem of a rigid foundation embedded in an elastic half-space by using the boundary integral equation method and analytically derived Green's functions.

Rajapakse and Shah (1987) presented solutions for longitudinal, rocking, transverse and coupled rocking-transverse responses of an elastic pile embedded in an elastic isotropic soil. They employed a semi-analytical formulation based on variational techniques and Green's functions for internal loading. Models of finite beam elements have been used by Barros (2003, 2004) to study axial and transverse response of a pile. In these two studies, the surrounding medium of a pile are transversely isotropic continua. Zeng and Rajapakse (1999) were the first to formulate a mathematical model for vertical vibration of a pile in a poroelastic soil. Lu et al. (2009) investigated the axial response of piles embedded in multilayered poroelastic media. More recently, Labaki, Mesquita and Rajapakse (2015) presented a model of time-harmonic axial vibrations of an elastic pile embedded in a transversely isotropic elastic soil by extending the work of Rajapakse and Shah (1987). The pile is perfectly bonded to the soil. The formulation is presented within the framework of linear theory of elastic wave propagation that is applicable to most cases of foundation vibration problems under external excitations.

In this article, the model presented by Labaki, Mesquita and Rajapakse (2015) is extended to treat the case of two elastic bars interacting with each other through their surrounding soil. The influence of the bars on each other is obtained by considering that the bar at the measuring end is a one-dimensional line within the half-space. In most engineering applications, the distance between piles is much larger than their radii. This technique adequately incorporates the wave propagation through the half-space from one bar to the other.

2. PROBLEM STATEMENT

Consider two elastic circular piles interacting with a half-space (Fig. 1). The piles have Young's modulus E_P, Poisson ratio ν_P and mass density ρ_P. Upper indices (1) or (2) in these parameters to denote pile (1), which rests on the z-axis, and pile (2), which is at a distance r_0 from pile (1). The distance r_0 is measured between the center lines of the piles. Initially, consider that both piles have length h and radius a. Both piles are under the effect of time-harmonic axisymmetric vertical loads of intensity p_0. The half-space may assume a range of properties, such as be homogeneous or layered, isotropic or not, porous or not. The constitution of the half-space does not affect the present formulation, as shown later on.

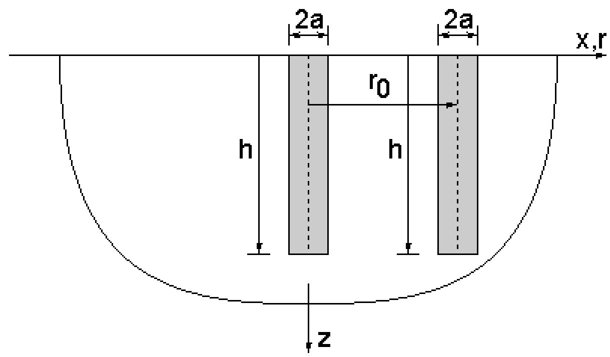

Figure 1. Half-space containing two loaded elastic piles.

3. GOVERNING EQUATIONS AND SOLUTION

Consider time-harmonic wave propagation in a transversely isotropic elastic medium. The problem is governed by the elastodynamic Cauchy-Navier equations expressed in terms of the displacement components $u_i(r,\theta,z,t)=u_i(r,\theta,z)e^{i\omega t}$, $i=r,\theta,z$ and ω denotes the circular frequency of motion. Rajapakse and Wang (1993) proposed a general solution for this class of elastodynamic problems by using Hankel transforms with respect to the radial coordinate. The general solution can be expressed in the Hankel transform space in terms of a set of arbitrary functions. These functions can be determined from the boundary and continuity conditions corresponding to a given problem.

In order to model the problem shown in Fig. 1, we decompose each pile into a fictitious pile and an extended half-space as shown in Fig. 2. The fictitious pile and the extended half-space interact through the vertical contact tractions $p_z(z)$ acting along the cylindrical surfaces S_p and S_h of the pile and half-space respectively. Concentrated load transfer at the base of fictitious pile has been ignored. The magnitude of vertical tractions is such that vertical displacement continuity is ensured along S_p and S_h. It should be noted that the model proposed earlier by Rajapakse and Shah (1987) for similar problems is based on an interacting body force field between the fictitious pile and extended half-space. Although the body force model allows more consistent displacement compatibility between the fictitious pile and half-space, it is computationally expensive.

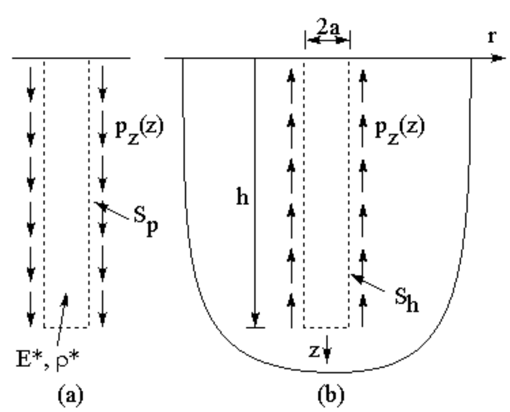

Figure 2. Decomposition of pile-soil system into (a) fictitious pile and (b) extended half-space.

In order to determine the vertical traction field along S_p and S_h, we consider a homogeneous transversely isotropic elastic half-space (identical to the extended half-space in Fig. 2) subjected to a vertical circular ring load of radius a and unit intensity acting at depth z'. the solution to this problem can be derived by considering the half-space as a two-domain problem and expressing the general solution of each domain by using the solutions given by Rajapakse and Wang (1993). Thereafter, a set of linear simultaneous equation system can be solved explicitly for a homogeneous medium to obtain the Green's functional analytically. This ring load solution can be integrated over a segment $z_{j1}<z'<z_{j2}$ to generate the Green's function corresponding to a cylindrical ring load of unit intensity acting in the interior of half-space. Let $u_{ZZ}(r_i,z_i;z_{j1},z_{j2})$ denote the vertical displacement at a point (r_i,z_i) due to unit cylindrical ring load of radius a acting over the vertical segment $z_{j1}<z'<z_{j2}$. A detailed description of the derivation of the Green's functions is presented by Labaki et al. (2014).

Consider the system shown in Fig. 2. The fictitious pile is assumed to behave as a one-dimensional continuum. Its Young's modulus and mass density are denoted by $E^*=E_p-E_S$ (E_p and E_S denote the elastic moduli of the pile and soil in the vertical direction respectively) and $\rho^*=\rho_p-\rho_S$ (ρ_p and ρ_S denote the densities of the pile and soil respectively) respectively. Rajapakse and Shah (1987) discussed the substantial errors that could result in by not using the modified mass density in the formulation of dynamic problems.

The equation of motion of one fictitious pile can be expressed in terms of the vertical displacement w(z,t) and traction $p_z(z)$ as:

$$A_p E^* \frac{d^2}{dz^2} w(z) - \rho^* A_p \ddot{w}(z) = p_z(z) \tag{1}$$

in which A_p is the cross-sectional area of pile.

A trial solution for the longitudinal deflection profile of piles (1) and (2), $w^{(1)}(z)$ and $w^{(2)}(z)$, is:

$$w^{(i)}(z) = \sum_{n=1}^{N} \alpha_n^{(i)} w_n^{(i)}(z) = \sum_{n=1}^{N} \alpha_n^{(i)} e^{-(n-1)z/h} \text{ ; } 0\le z\le h,\ i=1,2 \tag{2}$$

in which $\alpha^{(i)}_n$ are arbitrary generalized coordinates that depend on a certain set of physical conditions.

Without loss of generality, both profiles are approximated by the same number N of generalized coordinates. Based on Eq. (2), the strain (U_p) and kinetic energies (T_p) of the piles can be expressed as:

$$U_p^{(i)} = \sum_{m=1}^{N}\sum_{n=1}^{N} D_{mn}^{(i)} \alpha_m^{(i)} \alpha_n^{(i)} \text{ , } i=1,2 \tag{3}$$

$$T_p^{(i)} = \sum_{m=1}^{N}\sum_{n=1}^{N} C_{mn}^{(i)} \dot{\alpha}_m^{(i)} \dot{\alpha}_n^{(i)} \text{ , } i=1,2 \tag{4}$$

in which

$$D_{mn}^{(i)} = \frac{\pi E^* a^2 (n-1)(m-1)}{2(m+n-2)h}\left(1-e^{-(m+n-2)}\right)\text{, } i=1,2 \text{ for } m+n\neq 2 \tag{5}$$

$$D_{mn}^{(i)} = 0 \text{, } i=1,2\text{, for } m+n=2 \tag{6}$$

$$C_{mn}^{(i)} = \frac{\pi\rho^* h a^2}{2(m+n-2)}\left(1-e^{-(m+n-2)}\right)\text{, } m+n\neq 2\text{, } i=1,2 \tag{7}$$

$$C_{mn}^{(i)} = \frac{\pi\rho^* h a^2}{2}\text{, } m+n=2\text{, } i=1,2 \tag{8}$$

Let $T^{(i)}_{nz}(z)$ denote a vertical traction field acting on the surface of the pile i (i=1,2) that results in a vertical displacement identical to the n^{th} term of the displacement representation in Eq. (2) with $\alpha^{(i)}_n=1$. Then the traction field corresponding to Eq. (2) can be expressed as:

$$T_z^{(i)} = \sum_{n=1}^{N} \alpha_n^{(i)} T_{nz}^{(i)} \text{ , } i=1,2 \tag{9}$$

A solution for $T^{(i)}_{nz}(z)$ by analytical methods is not feasible due to the complexity of the problem. A numerical solution can be obtained by using the Green's function for a buried cylindrical vertical ring load shown above. Consider that the surface of each pile is discretized by M cylindrical surface elements. A solution for $T^{(i)}_{nz}(z)$ is obtained by developing the following flexibility equation system:

$$\begin{bmatrix} A_{ij}^{(1,1)} & A_{ij}^{(1,2)} \\ A_{ij}^{(2,1)} & A_{ij}^{(2,2)} \end{bmatrix} \begin{Bmatrix} T_{nz}^{(1)} \\ T_{nz}^{(2)} \end{Bmatrix} = \begin{Bmatrix} w_{n}^{(1)} \\ w_{n}^{(2)} \end{Bmatrix} \tag{10}$$

in which

$$A_{ij}^{(1,1)}\Big|_{M\times M} = u_{zz}\left(r_i = 0, z_i; r_j = 0, z_{j1}, z_{j2}\right) \tag{11a}$$

$$A_{ij}^{(1,2)}\Big|_{M\times M} = u_{zz}\left(r_i = 0, z_i; r_j = r_0, z_{j1}, z_{j2}\right) \tag{11b}$$

$$A_{ij}^{(2,1)}\Big|_{M\times M} = u_{zz}\left(r_i = r_0, z_i; r_j = 0, z_{j1}, z_{j2}\right) \tag{11c}$$

$$A_{ij}^{(2,2)}\Big|_{M\times M} = u_{zz}\left(r_i = r_0, z_i; r_j = r_0, z_{j1}, z_{j2}\right) \tag{11d}$$

$$\left\{w_n^{(i)}\right\} = \left\langle e^{-(n-1)z_1/h} \quad \cdots \quad e^{-(n-1)z_M/h} \right\rangle^T, \text{ i=1,2} \tag{12}$$

The Lagrangian (L) of the pile-soil system can be expressed as:

$$\begin{aligned} L = {} & U_p^{(1)} + U_p^{(2)} - T_p^{(1)} - T_p^{(2)} + \tfrac{1}{2}\int_{S_h} T_z^{(1)}(z) w^{(1)}(z) ds + \tfrac{1}{2}\int_{S_h} T_z^{(2)}(z) w^{(2)}(z) ds \\ & - p_0^{(1)} w^{(1)}(0) - p_0^{(2)} w^{(2)}(0) \end{aligned} \tag{13}$$

By using the Lagrange's equation of the system, after laborious derivation and simplification, the following set of linear algebraic equations can be established to solve for $\alpha^{(i)}_n$:

$$\sum_{n=1}^{N} \alpha_n^{(i)} \left[-2\omega^2 C_{nm}^{(i)} + 2D_{nm}^{(i)} + \pi \sum_{j=1}^{M} r_j \left(T_{zj}^{(i)m} e_{nj} + T_{zj}^{(i)n} e_{mj} \right) \Delta z_j \right] = p_0 \text{, m=1,N; i=1,2} \tag{14}$$

in which $e_{nj}=e^{-(n-1)zj/h}$ and $e_{mj}=e^{-(m-1)zj/h}$.

The solution of Eq. (14) results in the solutions for generalized coordinates $\alpha^{(i)}_n$ from which the deformation profile of the piles can be obtained (Eq. 2).

4. NUMERICAL RESULTS

This section shows a particular physical consistency test on the present implementation. If the distance r_0 between the two piles is large enough (see Fig. 1), the results should tend to that of a single pile. Figure 3 shows how the present implementation performs in this test. It shows the vertical impedance $K_V=P_0/\mu_S a\cdot w(z=0)$ of one of the piles in the two-pile system, in which μ_S is the shear modulus of the soil, P_0 is the load on the top end of the pile and *a* is the pile radius. The piles are embedded within a homogeneous half-space with properties μ_S, E_S and ρ_S. The distance r_0 between them varies (in the figures, r_0 is shown as "d" because of space constrains). These results consider a short pile with length $h/a=5$, $\mu_S=1$, $\rho_p/\rho_S=1$ and $E_p/E_S=10$, and the sub indices p and s indicate the pile and the soil. The results are presented according to a normalized frequency $a_0=\omega(\mu_S/\rho_S)^{1/2}$.

The stars in Fig. 3 correspond to the vertical impedance of a single pile, as obtained by an indirect-boundary element discretization proposed by Barros (2006). It can be seen that the results do tend to that of a single pile as the distance between the piles increases, and this happens monotonically. However, the response of a single pile should be achieved for distances much smaller than $r_0=200a$. The oscillations in the response also point to some problem in the current integration scheme. Some work still must be invested in this implementation.

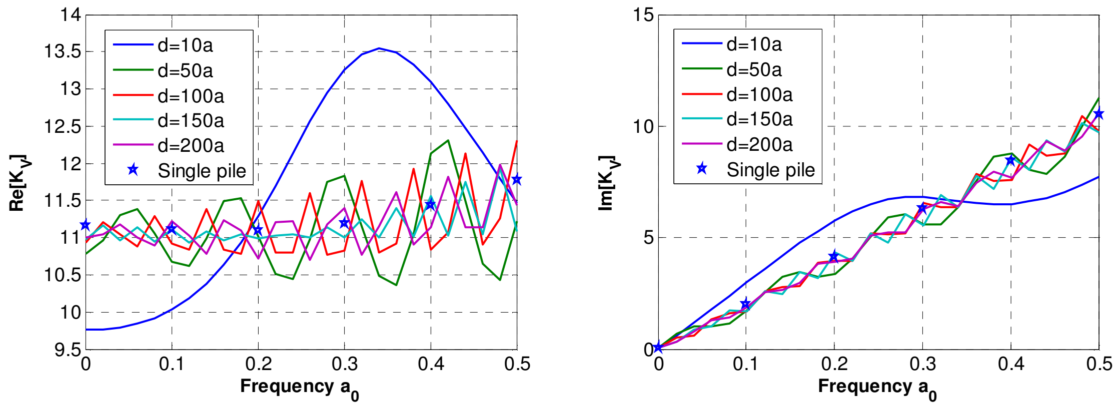

Figure 3. Vertical impedance KV of a pile in a two-pile system.

5. CONCLUDING REMARKS

A semi-analytical method is developed to study the time-harmonic vertical response of two elastic piles embedded in a transversely isotropic soil. The model is based on an interacting contact traction field instead of a body force field that was used in previous studies. An energy functional approach is used to model the pile-soil-interaction, in which the longitudinal deflection of the piles is described by a set of generalized coordinates. The present implementation still requires validation.

6. ACKNOWLEDGMENTS

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8. AUTHORSHIP DISCLOSURE

"The authors are the only responsible for the content of this work."