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DYNAMICS MODELLING OF A ROBOTIC ARM WITH FREE FLYING BASE USING LAGRANGIAN MECHANICS

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Abstract: *The objective of this paper is to present how to use the well known Lagrange Mechanics and the Lagrange Multipliers in a way to derive the dynamic model of a robotic arm assembled over a 6 degrees of freeedom free flying platform which can be simulated in a low friction simulated enviroment inside a laboratory. The dynamic model can also be used to help to design a control system and help the control system during operation knowing the reaction forces during all time.*

Keywords: *Dynamic Modelling, Robotic Arm, Lagrangian Mechanics, Lagrange Multipliers.*

1. INTRODUCTION

In the last decades, we have seen the increased use and application of robots and robotic arms in tasks that have been accomplished in space. Some of these tasks include grab a satellite for maintenance, docking of unmanned spacecraft at the Space Station, using a robotic arm to catch the spacecraft and slowly bring it to dock at the Station. However, these operations are actually a challenge when taken into account the conditions or the space environment where the operations are performed.

Because space is an environment with microgravity, vacuum and almost no friction with the atmosphere, it is important to have a good knowledge of systems and their behavior during the execution of defined tasks. For this, robotic arms are assembled over a six degrees of freedom free flying bases when working in Space.

Considering the above, any operation or movement of any of the joints or moving parts of a robot operating in this situation becomes a complex problem based on the need to keep the base of the system fixed or as closer as possible at its desired position and attitude.

For this reason, it is necessary to derive a dynamic model taking into account the reactions that can appear during execution of needed tasks. However because of the complexity of those dynamic systems, the classical Newtonian Formulation becomes difficult to apply when considered the propagation of forces and movements through the whole manipulator, and for this reason is necessary to apply a different formulation that allows define and calculate the reaction forces.

One manner to derive the dynamic equations considering the above is to use the Lagrangian Mechanics associated to the Lagrange Multipliers. That is, the use of an extended set of coordinates to allow us to derive the constraint equations, which are necessary to find the reaction forces. On the other hand, the constraint equations derived for the system eliminate the unnecessary degrees of freedom that appear when using an extended set of coordinates.

From this, the main goal of this paper is to show one way to apply the Lagrangian Mechanics with the use of the Lagrange Multipliers to derive the dynamic model of a robotic arm assembled over a free flying platform. However, the dynamic model has to be derived in a way that we can simulate the behavior of the system in a simulated low friction environment in a laboratory.

Sections 2 and 3 of this paper will show the main theoretical concepts needed to derive the dynamic equations. In sections 4 and 5, the Lagrange Equation will be developed for our case of study.

2. MANIPULATOR KINEMATICS

In applying the dynamic equations derived from the Lagrangian Mechanics, we will first remember some important concepts used to derive those equations.

One of these concepts is the definition of rotation matrices as found in Craig, (2005):

$$R_X(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & sin(\alpha) \end{bmatrix} \tag{1}$$

$$R_Z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

Where: $R_X(\alpha)$: Represents a rotation of angle α of the reference frame, about the $\hat{X}$ direction;

$R_Z(\theta)$: Represents a rotation of angle θ of the reference frame, about the $\hat{Z}$ direction;

For attaching the reference frames to the robotic manipulator links in study, will be used the Denavit-Hatenberg convention, which can be summarized in four parameters as shown next. More details can be found in Craig (2005).

a_i: distance between $\hat{Z}_i$ and $\hat{Z}_{i+1}$ measured along $\hat{X}_i$

α_i: angle between $\hat{Z}_i$ and $\hat{Z}_{i+1}$ measured about $\hat{X}_i$

d_i: distance between $\hat{X}_{i-1}$ and $\hat{X}_i$ measured along $\hat{Z}_i$

θ_i: angle between $\hat{X}_{i-1}$ and $\hat{X}_i$ measured about $\hat{Z}_i$

From this, is possible to define the reference frames of the whole robotic manipulator, considering that the robot is assembled over a platform. The defined frames are as shown in the Fig. 1.

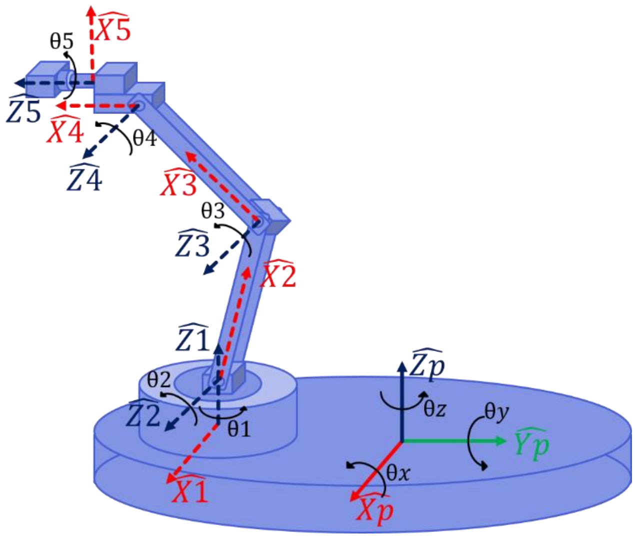

Figure 1: Robotic Manipulator with the defined reference frames for each link.

Craig (2005), defines the general equations for the movement propagation, as shown next.

For the rotational movement, we have:

$${}_{i+1}^{\ \ i}\omega = {}_{i}^{i}\omega + {}_{i+1}^{\ \ i}R\,\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \tag{3}$$

or, rewriting:

$${}_{i+1}^{\ \ i}\omega = {}_{i}^{i}\omega + {}_{i+1}^{\ \ i}R \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} \tag{4}$$

Where: ${}_{i+1}^{\ \ i}\omega$: is the angular speed of frame $i+1$ related to the frame i;

${}_{i}^{i}\omega$: is the angular speed of frame i related to itself;

${}_{i+1}^{\ \ i}R$: is the rotation matrix from frame $i+1$ to frame i;

$\dot{\theta}_{i+1}$: is the angular speed about the direction indicated by ${}^{i+1}\hat{Z}_{i+1}$;

${}^{i+1}\hat{Z}_{i+1}$: is the direction, which about the angular speed appears;

For the translational movement, we have:

$$ {}_{i+1}^{\;\;i}v = {}_{i}^{i}v + {}_{i}^{i}\omega \times {}_{i+1}^{\;\;i}P \tag{5}$$

Where: ${}_{i+1}^{\;\;i}v$: is the translational speed of frame $i+1$ related to frame i;
${}_{i}^{i}v$: is the translational speed of frame i related to itself;
${}_{i+1}^{\;\;i}P$: is the distance vector from frame i to frame $i+1$;

The symbol ($\times$) at Eq. (5) represents a Cross Product.

Having defined the reference frames of the dynamic system, which is the robotic manipulator fixed over a platform with six degrees of freedom, as shown at Fig. 1, and using Eq. (3) and Eq. (5), it is possible to derive all the equations that represent the movement propagation between any required links.

Here, it is also important define the Jacobian matrix for a robotic manipulator. Which is the partial derivative of the desired element of the manipulator with respect to the joint angles.

Thus we have:

$$ J_i(\theta) = \frac{\partial o}{\partial \theta_i} \tag{6}$$

Where: $J(\theta)$: is the Jacobian matrix;
o: is the position of the desired element or part of the robotic arm;
$\frac{\partial}{\partial \theta_i}$: are the partial derivatives of end effector position with respect to all joint angles, represented by θ_i;
θ_i: represents the i joint angle;

From Spong and Vidyasagar (1989), we know that the Jacobian matrix relates the movement of a desired point with the movement of the joint angles, in other words, the Jacobian is a linear transformation from joint angles to angular and translational speeds, which can be written as Eq. (7).

$$ \begin{bmatrix} v \\ \omega \end{bmatrix} = J(\theta)\dot{\theta} \tag{7}$$

Where: $\dot{\theta}$: is the vector of joint angular speeds;

These concepts are important when we are going to derive the dynamic equations of the dynamic system in study, as will be showed further.

3. MANIPULATOR DYNAMICS

Before entering into the Manipulator Dynamics, it is important to have a brief discussion about Celestial Mechanics, which is the manner used to locate the system in Space.

Because of this, the Inertial Reference System is centered at the center of the Earth, with the xy plane parallel to the Ecuador line, and the $\hat{x}$ axis pointing to the Vernal Equinox represented by γ. The Vernal Equinox is the point where the projection of the Ecliptic (Path of the Sun) intercepts the Ecuador line.

Considering that the system is describing a circular orbit, we have:

$$ v = \sqrt{\frac{\mu}{a}} \tag{8}$$

$$ x = rcos(f) \tag{9}$$

$$ y = rsin(f) \tag{10}$$

$$ z = 0 \tag{11}$$

Where: v: orbital velocity;
μ: constant: 3.986x10^5 m^3/s^2;
a: orbit radius;
x: x position of the system in the Inertial Reference Frame;
y: y position in the Inertial Reference Frame;
r: radius vector of the orbit for a circular orbit;
f: angular position of r with respect to the initial position.
z: z position of the system in the Inertial Reference Frame;

Considering $z = 0$ means that the system only moves on the plane of the orbit.

Therefore, from Eqs (9), (10) and (11) we have the three coordinates of the system in space allowing us to derive the dynamic equations of the system, where those coordinates are necessary. Is important to mention that z is considered as a coordinate, so when we derive the constraint equations as will be showed at section 3.2, is possible to ensure that the platform and the manipulator will only move in the $\hat{x}$ and $\hat{y}$ directions, by constraining z.

In the way to simulate our system in a laboratory, we will consider that the laboratory is located on a small section of the orbit line described by the system in space. More information about the Celestial Mechanics used can be found in Kuga, et. al., (2012).

From this, to derive the dynamic equations of the system, we will use the Lagrangian Mechanics, which can be represented by the next general equation (Meirovitch, 1970):

$$L = T - V \tag{12}$$

Where: L: represents the Lagrangian equation;
T: represents the Kinetic Energy of the system;
V: represents the Potential Energy of the system;

To be able to apply Eq. (12), is necessary to define a set of coordinates that can map during all time the position of any desired point of the structure of the manipulator.

The next considerations are made based on the Principles of Virtual Work, which can be found in details at Meirovitch, (1979), Shabana, (2010).

3.1. Coordinates System

By definition, the motion of a system of N particles subject to c Kinematical conditions can be described uniquely by n independent coordinates q_k $(k = 1,2, \ldots, n)$, where n is the number of degrees of freedom of the system. The number of degrees of freedom is the same as the number of minimum coordinates necessary to describe the system uniquely (Meirovitch, 1970). This statement can be represented as the equation:

$$n = 3N - c \tag{13}$$

However to apply the Lagrange Multipliers to the system, it is necessary to consider an expanded set of variables. For the case in study, a good choice is to define the x, y, and z components of the position of the Center of Mass (CM) of each link of the robotic manipulator and of the base, which are added to the original set of variables. Hence, at each CM of the links, a new reference frame is defined on which are mapped the position of the CM with respect to a general reference frame, which is the Inertial Reference frame. At Fig. 2 below, are showed the new reference frames needed to map the CM positions, and at Fig. 3 is showed the inertial reference frame.

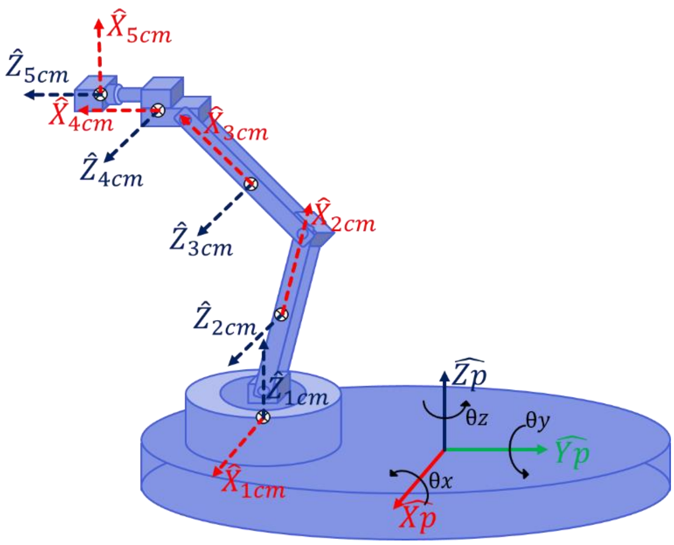

Figure 2: Robotic Manipulator with the defined reference frames for each CM and for clarity, the reference frames at the joints are suppressed.

Because the set of variables now be extended, they cannot be called Generalized Coordinates anymore, when we consider that we actually now have a redundant set of variables caused by the new variables of the CM positions.

3.2. Constraint Equations

In order to constrain the movement of the manipulator, and correctly relate the interaction between different frames, it is necessary to derive the Constraint Equations of the robot. These Constraint Equations reduce the number of degrees of freedom of the manipulator from the number of coordinates to the number of real degrees of freedom as defined by Eq. (13).

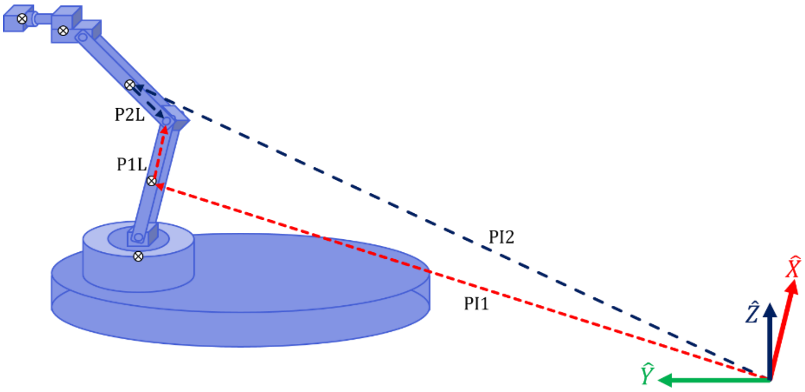

Figure 3: Out of scale example of the constraint equations imposed to one of the joints of the manipulator. Consider that $\hat{x}$, $\hat{y}$ and $\hat{z}$ are centered in the center of Earth.

At Fig. (3) above, it is possible to observe the vectors $PI1$ and $PI2$ which point from the inertial reference frame to the respective CM, and vectors $P1L$ and $P2L$ which point from the CM to the joint 3. Repeating the process for joints one, two, four and five we derive a set with five equations. However, it is important to remember that each of these equations has components in $\hat{x}$, $\hat{y}$ and $\hat{z}$ directions and from this we are actually constraining 15 coordinates, which are the coordinates that give us the reaction forces when associated with the Lagrange Multipliers. Besides of those 15 constraint equations, is necessary to add one more, which constraints the z position of the system in space, as mentioned before.

Deriving the equation for the vectors presented at Fig. 3, we have:

$$C_3(q) = PI1 + P1L - (PI2 + P2L) = 0 \tag{14}$$

Where $C_3(q)$ is the constraint equation related to joint 3, and by definition is equal to zero (Shabana, 2010).

Remembering the concepts from section 3.1, our manipulator has a total of 26 coordinates. Where six are from the platform, three are the x, y and z position components as described at the beginning of section 3 and the other three are the angular position of the platform about $\hat{x}$, $\hat{y}$ and $\hat{z}$ directions of the platform, or in other words, the attitude angles of the platform. The remaining 20 coordinates are one for each angular position of the joints and three for the x, y and z position of each CM, as identified at Fig. 3.

Thus, from Eq. (13), the system has 10 real degrees of freedom instead of 26 as expected from the coordinates.

3.3. Lagrange Multipliers

By definition, the Lagrange Multipliers are undetermined multipliers related to the magnitude of the constraint forces (Meirovitch, 1970). Mathematically they can be described by the Eq. (15) below:

$$Q_c^i = -C_{q^i}^T \lambda \tag{15}$$

Where: Q_c^i: Constraint force related to variable i;

$C_{q^i}^T$: Transposed Jacobian matrix of the constraint equations of the system;

λ: Vector of Lagrange Multipliers with the number of elements equal the number of constrained coordinates;

Before add the Eq. (15) to the Lagrangian Dynamic Equation, is important to define some important dynamic equations for the case of using the generalized coordinates in order to derive the dynamic model of the dynamic system.

3.4. Kinetic Energy

For the case of generalized coordinates, and considering we need to have a model that could be simulated on ground, inside a laboratory, with a real robot, where there is a strong influence of the gravity, is necessary to consider both, translational and rotational movement.

Therefore, for the Kinetic Energy we have:

$$T = \underbrace{\frac{1}{2} m v^T v}_{T_v} + \underbrace{\frac{1}{2} \omega^T I \omega}_{T_\omega} \tag{16}$$

Where: m: mass of the link;
v^T: transposed vector of translational speed in the $\hat{x}$, $\hat{y}$ and $\hat{z}$ directions;
v: vector of translational speed in the $\hat{x}$, $\hat{y}$ and $\hat{z}$ directions;
ω^T: transposed vector of angular speed about the $\hat{x}$, $\hat{y}$ and $\hat{z}$ directions;
ω: vector of angular speed about the $\hat{x}$, $\hat{y}$ and $\hat{z}$ directions;
I: inertia matrix of the link;
T_v: represents the translational component of the total Kinetic Energy;
T_ω: represents the angular component of the total Kinetic Energy;

Replacing Eq. (7) in Eq. (16) and remembering that now we are using a general set of variables, Eq. (16) can be written as:

$$T_i = \frac{1}{2} m_i \big(J_{vi}\,(q)\dot{q}\big)^T \big(J_{vi}\,(q)\dot{q}\big) + \frac{1}{2}\big(J_{\omega i}\,(q)\dot{q}\big)^T I_i J_{\omega i}\,(q)\dot{q} \tag{17}$$

Rearranging:

$$T_i = \frac{1}{2}\dot{q}^T [m_i J_{vi}(q)^T J_{vi}(q) + J_{\omega i}(q)^T I_i J_{\omega i}(q)]\dot{q} \tag{18}$$

Which can be written as:

$$T_i = \frac{1}{2}\dot{q}^T D_i(q)\dot{q} \tag{19}$$

And finally, for the whole system, the Kinetic Energy can be written as:

$$\boldsymbol{T} = \sum_{i=1}^{j} T_i \tag{20}$$

Where: q: general variables;
D_i: Inertia matrix of the element i;
T_i: Kinetic Energy of the element i;
T: Kinetic Energy of the system;
j: number of elements of the system;

It is important to mention that all the equations above are derived with respect to the inertial reference frame, as the potential energy and dissipation equations will be, in the next sections.

3.5. Potential Energy

For the case of in orbit operations, is necessary to consider the gravity gradient, however, for the case of a simulation on ground, the influence of the gravity is considered instead of the gravity gradient, to describe correctly the behavior of the manipulator. Assuming a concentrated mass actuating on the CM for each element, we have:

$$\boldsymbol{V} = \sum_{i=1}^{k} m_i g q_i \tag{21}$$

Where: g: gravity acceleration;
q_i: variable which maps the distance from the CM to the ground;
k: number of elements;

3.6. Dissipation Equations

Because of the friction between the different parts of the joints of the manipulator, is interesting to introduce a Rayleigh's Dissipation Function, which considers the loss of energy because of the friction between the different parts of the joints, so the dynamic model considers the loss of energy that occurs while the manipulator is moving. From Meirovitch, (1970), the dissipative forces, are forces proportional to the velocity of a defined element and resist to the movement as they act in a direction opposite to that of the velocity. Mathematically, it can be described by Eq. (22) below:

$$\boldsymbol{F} = \sum_{i=1}^{N} \frac{1}{2} c_{q_i} \dot{q}_i^2 \tag{22}$$

Where: N: Number of coordinates;

F: dissipation forces (Rayleigh's Dissipative Function);
c_{q_i}: dissipation constant related to variable q_i;

Having defined the energies of the system, we are able to derive the general dynamic equation of system from Eq. (12).

4. DYNAMIC MODEL

From Eq. (12), and considering that the Potential Energy is only position dependent, we have:

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{T}}{\partial \dot{q}_i}\right) - \frac{\partial \boldsymbol{T}}{\partial q_i} + \frac{\partial \boldsymbol{V}}{\partial q_i} = 0 \tag{23}$$

However, in our case, is necessary to consider the dissipations, Eq. (22), and external forces, $\boldsymbol{Qe}$, which are the control forces applied to control the system. Therefore, we have:

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{T}}{\partial \dot{q}_i}\right) - \frac{\partial \boldsymbol{T}}{\partial q_i} + \frac{\partial \boldsymbol{V}}{\partial q_i} - \frac{\partial \boldsymbol{F}}{\partial \dot{q}_i} = \boldsymbol{Q}e \tag{24}$$

4.1. Dynamic Model with Lagrange Multipliers

From the concepts of virtual work, presented by Meirovitch, (1970), is possible to rewrite Eq. (24) as:

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{T}}{\partial \dot{q}_i}\right) - \frac{\partial \boldsymbol{T}}{\partial q_i} + \frac{\partial \boldsymbol{V}}{\partial q_i} - \frac{\partial \boldsymbol{F}}{\partial \dot{q}_i} + \boldsymbol{C}_{q^i}^T \boldsymbol{\lambda} = \boldsymbol{Q}e \tag{25}$$

Where: C_{q^i}: Jacobian matrix of the constraints equations;

However, it is important to mention that it is possible to add the Eq. (15) to Eq. (24) because by definition, as shown at section 3.2, the constraint equations are equal to zero, as are their partial derivatives with respect to the coordinates. This means that the constraint forces are unable to realize work. From this it is possible to say that when adding the Lagrange Multipliers term to Eq. (24), we are actually summing zero, what is necessary whether we do not want to change the response of the system.

Therefore, from Eq. (25) we have a good representation of the real behavior of the system, which is going to be applied to our case of study as shown in the next section.

5. DYNAMIC MODEL OF THE ROBOTIC MANIPULATOR

Having defined Eq. (17), and remembering that all terms need to be written with respect to the inertial reference frame, we have as follows. For the rotation matrices we have:

$${}_P^IR = R_Z(\theta_Z) = \begin{bmatrix} \cos(\theta_Z) & -\sin(\theta_Z) & 0 \\ \sin(\theta_Z) & \cos(\theta_Z) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{26}$$

$${}_1^PR = R_Z(\theta_1) = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{27}$$

$${}_2^1R = R_Z(\theta = 90°) * R_X(\alpha = 90°) * R_Z(\theta_2) = \begin{bmatrix} 0 & 0 & 1 \\ cos(\theta_2) & -\sin(\theta_2) & 0 \\ \sin(\theta_2) & \cos(\theta_2) & 0 \end{bmatrix} \tag{28}$$

$${}_3^2R = R_Z(\theta_3) = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 \\ \sin(\theta_3) & \cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{29}$$

$${}_4^3R = R_Z(\theta_4) = \begin{bmatrix} \cos(\theta_4) & -\sin(\theta_4) & 0 \\ \sin(\theta_4) & \cos(\theta_4) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{30}$$

$${}_5^4R = R_Z(\theta = -90°) * R_X(\alpha = -90°) * R_Z(\theta_5) = \begin{bmatrix} 0 & 0 & 1 \\ -\cos(\theta_5) & sin(\theta_5) & 0 \\ -sin(\theta_5) & -\cos(\theta_5) & 0 \end{bmatrix} \tag{31}$$

Before write Eq. (18) for each part of the system, is necessary to define the Inertia Matrix of each part, which need to be written with respect to the inertial reference frame. For this, from Greenwood, (1987) and from the Parallel Axis Theorem, we have:

$$I_P = {}^I_PR^T I_{Pcm} {}^I_PR + m_P d_P^2 \tag{32}$$

$$I_1 = ({}^I_PR {}^P_1R)^T I_{1cm} {}^I_PR {}^P_1R + m_1 d_P^2 \tag{33}$$

$$I_2 = ({}^I_PR {}^P_1R {}^1_2R)^T I_{2cm} {}^I_PR {}^P_1R {}^1_2R + m_2 d_P^2 \tag{34}$$

$$I_3 = ({}^I_PR {}^P_1R {}^1_2R {}^2_3R)^T I_{3cm} {}^I_PR {}^P_1R {}^1_2R {}^2_3R + m_3 d_P^2 \tag{35}$$

$$I_4 = ({}^I_PR {}^P_1R {}^1_2R {}^2_3R {}^3_4R)^T I_{4cm} {}^I_PR {}^P_1R {}^1_2R {}^2_3R {}^3_4R + m_4 d_P^2 \tag{36}$$

$$I_5 = ({}^I_PR {}^P_1R {}^1_2R {}^2_3R {}^3_4R {}^4_5R)^T I_{5cm} {}^I_PR {}^P_1R {}^1_2R {}^2_3R {}^3_4R {}^4_5R + m_5 d_P^2 \tag{37}$$

Where: I_n: are the translated Inertia matrices;
I_{ncm}: are the Inertia matrices at the center of mass of the element;
d_n^2: distance from the CM to the inertial reference frame;
Writing Eq. (18) for each part of the system, results on:

$$T_P = \frac{1}{2}\dot{q}^T \left[m_P J_{vP}(q)^T J_{vP}(q) + \left({}^I_PRJ_{\omega P}(q)\right)^T I_P({}^I_PRJ_{\omega P}(q)) \right] \dot{q} \tag{38}$$

$$T_1 = \frac{1}{2}\dot{q}^T \left[m_1 J_{v1}(q)^T J_{v1}(q) + \left({}^I_PR {}^P_1RJ_{\omega 1}(q)\right)^T I_1({}^I_PR {}^P_1RJ_{\omega 1}(q)) \right] \dot{q} \tag{39}$$

$$T_2 = \frac{1}{2}\dot{q}^T \left[m_2 J_{v2}(q)^T J_{v2}(q) + \left({}^I_PR {}^P_1R {}^1_2RJ_{\omega 2}(q)\right)^T I_2({}^I_PR {}^P_1R {}^1_2RJ_{\omega 2}(q)) \right] \dot{q} \tag{40}$$

$$T_3 = \frac{1}{2}\dot{q}^T \left[m_3 J_{v3}(q)^T J_{v3}(q) + \left({}^I_PR {}^P_1R {}^1_2R {}^2_3RJ_{\omega 3}(q)\right)^T I_3({}^I_PR {}^P_1R {}^1_2R {}^2_3RJ_{\omega 3}(q)) \right] \dot{q} \tag{41}$$

$$T_4 = \frac{1}{2}\dot{q}^T \left[m_4 J_{v4}(q)^T J_{v4}(q) + \left({}^I_PR {}^P_1R {}^1_2R {}^2_3R {}^3_4RJ_{\omega 4}(q)\right)^T I_4({}^I_PR {}^P_1R {}^1_2R {}^2_3R {}^3_4RJ_{\omega 4}(q)) \right] \dot{q} \tag{42}$$

$$T_5 = \frac{1}{2}\dot{q}^T \left[m_5 J_{v5}(q)^T J_{v5}(q) + \left({}^I_PR {}^P_1R {}^1_2R {}^2_3R {}^3_4R {}^4_5RJ_{\omega 5}(q)\right)^T I_5({}^I_PR {}^P_1R {}^1_2R {}^2_3R {}^3_4R {}^4_5RJ_{\omega 5}(q)) \right] \dot{q} \tag{43}$$

The first part of Eq. (38) through (43) correspond to T_v which are not rotated because they are written using the extended set of coordinates, which means that the coordinates used are the position of the CM written in the Inertial Reference Frame.

Replacing Eq. (32) through (37) into Eq. (38) through (43) and then replacing these equations into Eq. (20) we have the total Kinetic Energy of the system. The resulting equation is not presented because of the size of it. However, it can be obtained applying the equations demonstrated before.

Replacing the obtained Eq. described before into Eq. (25) and solving the partial and time derivatives of the kinetic energy of Eq. (21) for the system that we are studying, we have:

$$\frac{1}{2}(\boldsymbol{D}(q) + \boldsymbol{D}^T(q))\ddot{\boldsymbol{q}} + \boldsymbol{C}_q^T \ \boldsymbol{\lambda} = \boldsymbol{Q}_e + \boldsymbol{Q}_v + \boldsymbol{Q}_p + \boldsymbol{Q}_f \tag{44}$$

Where:

$$\boldsymbol{Q}_v = -\frac{1}{2}(\dot{\boldsymbol{D}}(q) + \dot{\boldsymbol{D}}(q)^T)\dot{\boldsymbol{q}} + \underbrace{\frac{1}{2}\dot{\boldsymbol{q}}^T \frac{\partial \boldsymbol{D}(q)}{\partial q_i}\dot{\boldsymbol{q}}}_{CL} \tag{45}$$

$$\boldsymbol{Q}_p = \frac{\partial V}{\partial q_i} \tag{46}$$

$$\boldsymbol{Q}_f = \frac{\partial \boldsymbol{F}}{\partial \dot{q}_i} \tag{47}$$

Going back to Eq. (45) and analyzing the CL part, it is possible to say that this part of the equation gives us the Centrifugal and the Coriolis forces, acting during the movement of the system.

However having the Eq. (44), which is a set of differential equations, we are not able to solve them directly. Analyzing Eq. (44) carefully, it is possible to say that we have 26 differential equations and $26 + 16$ variables, where 26 is the number of coordinates and 16 is the number of constraint equations or the number of Lagrange Multipliers. From this, the system is undetermined.

As a way to solve Eq. (44), which involves actually differential equations and algebraic equations, is possible to use the Augmented Formulation presented by Shabana, (2010).

By this method, we can use the 16 constraint equations as the additional equations needed to solve the system of equations represented by Eq. (44).

Adding the constraints to Eq. (44) and writing it in the matrix form:

$$\begin{bmatrix} \frac{1}{2}(\boldsymbol{D}(q) + \boldsymbol{D}^T(q)) & \boldsymbol{C}_q^T \\ \boldsymbol{C}_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q}_e + \boldsymbol{Q}_v + \boldsymbol{Q}_p + \boldsymbol{Q}_f \\ \boldsymbol{Q}_d \end{bmatrix} \tag{48}$$

Solving for $\ddot{\boldsymbol{q}}$ and $\boldsymbol{\lambda}$, we have:

$$\begin{bmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\boldsymbol{D}(q) + \boldsymbol{D}^T(q)) & \boldsymbol{C}_q^T \\ \boldsymbol{C}_q & 0 \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{Q}_e + \boldsymbol{Q}_v + \boldsymbol{Q}_p + \boldsymbol{Q}_f \\ \boldsymbol{Q}_d \end{bmatrix} \tag{49}$$

For our problem, because the constraints do not depend explicitly on time, $\boldsymbol{Q}_d$ can be written as (Shabana, 2010):

$$\boldsymbol{Q}_d = -\boldsymbol{C}_{\boldsymbol{q}}\dot{\boldsymbol{q}}^2 \tag{50}$$

Therefore, from Eq. (49) we have $\ddot{\boldsymbol{q}}$ which can be integrated forward in time once and twice resulting in velocity and position respectively, and yet from Eq. (49) we get the Lagrange Multipliers $\boldsymbol{\lambda}$ that represent the reaction forces.

6. FINAL CONSIDERATIONS

Having derived all the terms of Eq. (49), which is a system of equations, is possible to solve it by the use of any mathematical solver as MATLAB/SIMULINK® to investigate the behavior of the system.

It is also possible, having a good model, associated with the use of the Lagrange Multipliers, anticipate the forces acting and from this design a control system that actuates while the movement is happening or which actuates even before the movement happens, avoiding unwanted displacements of the base or of any other part of the robotic manipulator.

Because of the size of the equations derived by the method explained before, they were not showed, remembering that the main goal of this paper is to show how to use well known methods in a way to get needed information about the system.

7. ACKNOWLEDGMENTS

CAPES and ITA.

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