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MODELING THE STEADY CAVITATING FLOW IN HYDRAULIC PIPELINES BY MEANS OF A CONSISTENT THERMODYNAMIC THEORY

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Abstract. *This work presents a consistent thermodynamic model to describe the cavitating flow in hydraulic pipelines. Although the model is capable to describe the cavitation phenomenon in unsteady as well as steady states, the application presented herein is restricted to slack flow, which take places under steady state. The flow is assumed to be homogeneous and isothermal and the fluid is regarded as a pseudo-mixture, comprising the liquid and the vapor phase along with an inert gas. The constituents are assumed to be compressible and to coexist at every material point and time instant. The balance equations of mass for each constituent are considered in the model, along with one balance equation of momentum for the mixture as a whole within the one-dimensional context. The existence of a small amount of inert gas along with the vapor ensures the necessary thermodynamic condition to describe the opening and closing of the cavity. It also allows the description of the cavitation phenomenon in any region of the fluid flow domain by means of a same set of equations. The phase change transformation is properly accounted for as an irreversible process. The obtained results by means of a simple numerical simulation show that model is capable to coherently describe the continuous growing of the vapor cavity until the flow becomes sonic.*

Keywords: *Cavitation, inert gas, irreversible phase change, thermodynamics of irreversible processes*

1. INTRODUCTION

The vaporous cavitation phenomenon is the formation of the vapor phase in a liquid due to the pressure reduction below the saturation pressure of liquid at given temperature. In hydraulic pipelines that operates in steady state, cavitation can cause a flow pattern known as slack flow in which the liquid flows through the bottom of the pipe whereas the vapor recirculates on its top. The pressure within a pipeline can become equal to or less than the saturation pressure of the liquid depending on the pipe topographic profile as well as on the friction. In such events, part of the liquid transforms into vapor forming a cavity whose extension may vary significantly according to the flow conditions.

The slack flow substantially affects the linefill of the pipeline (Nicholas, 1995). As a result, in oil pipelines particularly, its occurrence may increase significantly the time required to detect a leak or the occurrence of false alarms as compared with a same pipeline operating in tight flow condition. Thus, it becomes particularly important to predict adequately when and where the cavity will be formed as well as its extension, so as to ensure an appropriate performance of the leak detection system.

In spite of its relevance, this subject has received little attention on the literature in the past decade. As a result, there are only a few works in the technical literature about the modeling of cavitating flows under steady conditions (Nicholas, 1995). On the other hand, because of the catastrophic damage caused by unsteady cavitation, there is a great number of models about unsteady cavitating flows, such as those of Wylie (1984); Wylie & Streeter (1993); Kessal and Amaouche (2001); Freitas Rachid (2003); Bergant et al. (2006) and references therein. Although the models previously cited describe nicely the unsteady cavitating flow, most of them are not able to describe the opening and closing of the liquid column in steady state regimes.

The main purpose of this work is to develop a mechanical model that describes the cavitation phenomenon in unsteady as well as in steady state flows in pipelines. To properly describe the wall friction in both slack flow and the tight flow conditions a suitable modification is introduced in the friction factor. Instead of admitting only the existence of liquid and vapor in the mixture, an inert gas is taken as an additional constituent into account. Since it is assumed that the inert gas coexists along with the vapor throughout the flow domain, the fluid is now regarded as a two-constituent two-phase mixture. Therefore, besides the momentum conservation equation for the mixture as a whole, mass conservation equations for the liquid, the vapor and the gas are considered. The temperature is supposed to be the same for the three constituents and the cavitation process is assumed to be an isothermal irreversible transformation. The macroscopic dissipative effects of the liquid-vapor transformation and the cavity growth and collapse are properly accounted for in such a way that the second law of thermodynamics is always satisfied.

To evaluate the model capability in dealing with the phase change transformation in the cavitation phenomenon, a simple numerical simulation is carried out in which a continuous growth of a cavity is described inside a pipeline until the the flow becomes locally sonic, as a result of the decrease of the wave speed of the mixture due to the vapor formation.

2. BALANCE EQUATIONS AND BASIC ASSUMPTIONS

In contrast to general two-phase fluid flows where the phases can assume very different geometrical configurations throughout the flow region (Ishii and Hibiki, 2006), cavitation is a localized phenomenon which takes place at discrete and small regions of the fluid flow. Therefore, it is reasonable to assume that there exists no significant relative motion or slip between the liquid and gaseous phases, what is equivalent to consider that the constituents have the same velocities. In addition, if we assume that both phases have always the same temperature during liquid-vapor phase transformations, it suffices to consider the balance equations of mass for each constituent one balance equation of momentum for the mixture as a whole.

Under suitable regularity assumptions, and also by restricting the analysis to isothermal transformations, only the following classical forms of the balance equations along with the Clausisu-Duhen inequality are required to describe the one-dimensional fluid flow through a compliant pipe that makes an angle θ with the horizontal and has cross-sectional area A and wetted perimeter P (Germain and Muller, 1995):

$$\frac{\partial}{\partial t}\left(\alpha\rho_g\right)+\frac{\partial}{\partial s}\left(\alpha\rho_g v\right)=-\frac{\alpha\rho_g}{A}\frac{dA}{dt}, \tag{1}$$

$$\frac{\partial}{\partial t}\left(\alpha\rho_v\right)+\frac{\partial}{\partial s}\left(\alpha\rho_v v\right)=-\frac{\alpha\rho_v}{A}\frac{dA}{dt}+\Gamma, \tag{2}$$

$$\frac{\partial}{\partial t}\left((1-\alpha)\rho_l\right)+\frac{\partial}{\partial s}\left((1-\alpha)\rho_l v\right)=-\frac{(1-\alpha)\rho_l}{A}\frac{dA}{dt}-\Gamma, \tag{3}$$

$$\frac{\partial}{\partial t}\left(\rho v\right)+\frac{\partial}{\partial s}\left(\rho v^2+p\right)=-\frac{1}{A}\left(\rho v\frac{dA}{dt}+\tau P+\rho g A\sin\theta\right), \tag{4}$$

$$d:=-\frac{d\Psi}{dt}-\left(\Psi+p\right)\left(\frac{\partial v}{\partial s}+\frac{1}{A}\frac{dA}{dt}\right)+\frac{P}{A}\tau v, \tag{5}$$

for all $(s,t)\in(0,L)\times(0,+\infty)$, in which L stands for the pipe length. In the above equations, v is the spatial velocity filed of the mixture whose thermodynamic pressure is denoted by p. Based on the previous assumptions, the fluid can be regarded a pseudo-mixture of a two-phase, two-constituent homogeneous mixture with average properties. The gaseous phase is formed by the vapor of a liquid and an inert gas, whereas the liquid phase is constituted of liquid solely. The liquid and gaseous phases are assumed to coexist at every material point and time instant. To take it into account, it is considered an internal variable α, $\alpha\in(0,1)$,usually called gaseous volume fraction, which is defined as being the ratio of the volume occupied by the vapor along with the inert gas and the total volume of the mixture. As a result, the mass density of the mixture can be expressed as:

$$\rho=\left(1-\alpha\right)\rho_l+\alpha\left(\rho_v+\rho_g\right), \tag{6}$$

in which ρ_l, ρ_v and ρ_g stand for the mass densities of the liquid, vapor and inert gas, respectively.

As usual, $d\chi/dt$ stands for the time material derivative of the variable χ and Ψ denotes the Helmholtz free energy per unit volume. Since the pipe is assumed to have a circular cross-section, then $A=(\pi D^2)/4$ and $P=\pi D$; with D being the internal diameter of the pipe. All dependent variables are functions of the axial coordinate s measured along the pipe centerline and the time instant t. The rate of mass exchange between the gaseous and liquid phases as a result of the vaporous cavitation is represented by Γ. If the vapor is transformed into liquid, then $\Gamma>0$. On the other hand, if the liquid becomes vapor, then $\Gamma<0$. Finally, if there is no mass transfer at all, then $\Gamma=0$ and the mass conservation principles associated with the liquid and the vapor become independent from each other.

Equations (1), (2) and (3) express the mass conservation principle for the inert gas, the vapor and for the liquid. Equation (4) expresses the momentum conservation principle for the mixture as a whole and Eq. (5) is a local version of

the second law of the thermodynamics (SLT), which states that the rate of energy dissipation d must be non-negative. It distinguishes the possible processes ($d \geq 0$) from the impossible ones ($d < 0$). The possible processes are classed as reversible as long as $d = 0$ and irreversible when $d > 0$.

To complete the problem description, constitutive relationships must be provided. They encompass expressions for A, τ, p and Γ in such a way that Eq. (5) be satisfied no matter the external actions, the initial and the boundary conditions.

3. CONSTITUTIVE RELATIONSHIPS

The constitutive relationships necessary to complete the model encompass additional information regarding both the pipe and the fluid. In the following sections, we restricted the analysis by presenting specific relationships: i) between the cross-sectional area of the pipe and the fluid pressure; ii) for the wall shear stress and; iii) for the mixture pressure and the partial pressures of its components along with the rate of mass exchange between the gaseous and liquid phases. As it will be clear ahead, the last two steps will be carried out on the basis of a consistent thermodynamic theory.

3.1 Relationship between cross-sectional area and pressure

By assuming that the pipe is a thin-walled tube subjected to small deformations and that the pipe material exhibits a linear elastic mechanical behavior, then it is possible to prove that the cross-sectional area of the pipe and the fluid pressure are related according to the following expression:

$$A = A_o \left(1 + \xi \frac{pD_o}{eE}\right), \tag{7}$$

in which A_o is the unperturbed cross-sectional area of the pipe whose diameter is D_o. In the past expression the factor ξ may assume different values according to the pipe's anchorage. If the pipe is anchored with expansion joints, then $\xi = 1$. Otherwise, $\xi = 1 - \nu^2$, if the pipe is supported against axial deformations.

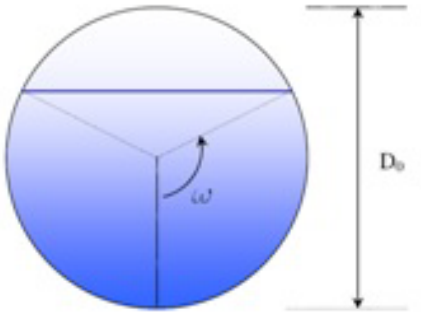


Figure 1. Cross section of pipe operating in slack flow condition

3.2 Wall shear stress and the friction factor in slack flow

In steady state, the wall shear stress τ is usually expressed in terms of the flow velocity v and of the Darcy-Weisbach friction factor f by the expression (Wylie & Streeter, 1993):

$$\tau = f \frac{\rho v |v|}{8}. \tag{8}$$

The friction factor in thigh flow conditions is commonly expressed as a function of Reynolds number and the relative roughness of the pipe wall. Aiming to properly extend this approach capable to handle not only thigh but also slack flow conditions (Nicholas, 1995) proposed the following expression which will be used herein:

$$\tau = f_m \frac{\rho v |v|}{8}, \quad \text{with} \quad f_m = \frac{f}{\left(\dfrac{1 - sin\,(2\omega)}{2\omega}\right)^{4/3}}, \tag{9}$$

in which f_m stands for the modified friction factor. In the above equation, the angle ω characterizes the the way the liquid and gaseous phases fill the cross-sectional area of the pipe, as illustrated in Fig. 1. It should be noticed that ω is a monotonic function of α.

3.3 Thermodynamic Theory

The constitutive relations describing the macroscopic mechanical behavior of the liquid-gas mixture are derived in the framework of the thermodynamics of irreversible processes. In this theory, once the local state of the material has been characterized by means of an appropriate choice of the set of state variables, two thermodynamic potentials - the Helmholtz free energy and a pseudo-potential of dissipation - are sufficient to derive a complete set of constitutive equations. For this particular problem, we choose as state variables the local mass densities of the liquid ρ_l, the inert gas ρ_g and the vapor ρ_v, the gaseous volume fraction α and the absolute temperature θ. As we shall see, the restriction associated with α is treated in this work as a physical property in the constitutive equations. This approach has already been used by Freitas Rachid (2003) and Freitas Rachid (2006) in the modeling of cavitation in flows of compressible fluids and of surface tension effects in homogeneous liquid-gas flows, respectively.

3.4 Helmholtz free energy and state laws

The free energy per unit volume Ψ is supposed to be a function of the state variables ρ_l, ρ_g, ρ_v, α and θ. Since the fluid is regarded as a mixture, its behavior is supposed to comprise a combination of the liquid, the inert gas and the vapor thermo-mechanical properties, taking α as a weighting factor. Thus, we choose

$$\Psi = \Psi\left(\rho_l, \rho_g, \rho_v, \alpha, \theta\right) := \Psi'\left(\rho_l, \rho_g, \rho_v, \alpha, \theta\right) + \overline{I}(\alpha), \tag{10}$$

in which

$$\Psi'\left(\rho_l, \rho_g, \rho_v, \alpha, \theta\right) := (1-\alpha)\rho_l\Psi_l(\rho_l, \theta) + \alpha\rho_g\Psi_g(\rho_g, \theta) + \alpha\rho_v\Psi_v(\rho_v, \theta), \tag{11}$$

$$\overline{I}(\alpha) := \begin{cases} 0 & \text{, if } \alpha \in (0,1) \\ +\infty & \text{, otherwise.} \end{cases} \tag{12}$$

In the above expression, Ψ' is a smooth function which describes the thermo-mechanical properties of the fluid whereas $\overline{I}(\alpha)$ represents the indicator function of the convex set $(0,1)$ (Moreau et al., 1988). The term $\overline{I}(\alpha)$ is the non-smooth parcel of the free energy and is included to take the internal constraint $\alpha \in (0,1)$ into account as a constitutive assumption. In other words, it prevents α from getting out of its admissible interval since it would be required a infinite amount of energy to do it.

The terms Ψ_l, Ψ_g and Ψ_v represent the free energies per unit mass of liquid, the gas and the vapor, respectively. As suggested by its functional dependence, these free energies are supposed to represent the thermo-mechanical behavior of the liquid, the gas and the vapor as if they were single constituents. According to the theory presented in Assumpção and Freitas Rachid (2011), the following forms for these free energies are assumed:

$$\Psi_l = \Psi_l(\rho_l, \theta) := -C_l\theta\log\theta + a_l^2\left(\frac{\rho_l^o}{\rho_l} + \log\rho_l\right), \tag{13}$$

$$\Psi_g = \Psi_g(\rho_g, \theta) := -C_g\theta\log\theta + a_g^2\log\rho_g, \tag{14}$$

$$\Psi_v = \Psi_v(\rho_g, \theta) := -C_v\theta\log\theta + a_v^2\log\rho_v. \tag{15}$$

In the above expressions, C_l, C_g and C_v are assumed to be constants. They represent the specific heats at constant volume of the liquid, the inert gas and the vapor, respectively. The material parameters ρ_l^o, a_l^2, a_g^2 and a_v^2 are temperature dependent, being the last three the square of the isothermal wavefront propagation velocities in the pure liquid, the inert gas and the vapor, respectively.

The state laws for the fluid, relating the reversible components of the thermodynamic forces to the state variables, are obtained from the free energy potential and defined as follows:

$$B^{\varrho_l} := \frac{\partial\Psi'}{\partial\rho_l} = (1-\alpha)\big(\Psi_l + \frac{p_l}{\rho_l}\big) = (1-\alpha)g_l, \tag{16}$$

$$B^{\varrho_g} := \frac{\partial\Psi'}{\partial\varrho_g} = \alpha\big(\Psi_g + \frac{p_g}{\rho_g}\big) = \alpha g_g, \tag{17}$$

$$B^{\varrho_v} := \frac{\partial\Psi'}{\partial\varrho_v} = \alpha\big(\Psi_v + \frac{p_v}{\rho_v}\big) = \alpha g_v, \tag{18}$$

$$B^{\alpha} := \frac{\partial\Psi'}{\partial\alpha} + h, \quad \text{with} \quad h \in \partial_\alpha\overline{I}(\alpha), \tag{19}$$

in which;

$$\frac{\partial\Psi'}{\partial\alpha} = \rho_g\Psi_g + \rho_v\Psi_v - \rho_l\Psi_l, \tag{20}$$

$$p_l := \rho_l^2\frac{\partial\Psi_l}{\partial\rho_l} = a_l^2\left(\rho_l - \rho_l^o\right) \quad \text{and} \quad p_g := \rho_g^2\frac{\partial\Psi_g}{\partial\rho_g} = a_g^2\rho_g \quad \text{and} \quad p_v := \rho_v^2\frac{\partial\Psi_v}{\partial\rho_v} = a_v^2\rho_v. \tag{21}$$

In the above equations, p_l, p_g and p_v stand for the liquid, the gas and the vapor partial pressures, respectively. The term h in Eq. (19) is an element of the sub-differential set $\partial_\alpha \overline{I}(\alpha)$ (also called generalized derivative) with respect to α of the convex function $\overline{I}(\alpha)$. The sub-differential of the indicator function $\overline{I}(\alpha)$ at α is given by the set (Ekeland & Teman, 1976; Moreau et al., 1988);

$$\partial_\alpha \overline{I}(\alpha) := \{h \in \mathbb{R} | \overline{I}(\alpha^\star) - \overline{I}(\alpha) \geq h(\alpha^\star - \alpha); \quad \forall \alpha^\star \in (0,1)\}. \tag{22}$$

A straightforward calculation shows that $\partial_\alpha \overline{I}(0 < \alpha < 1) = \{0\}$ and $\partial_\alpha \overline{I}(\alpha) = \emptyset$ if $\alpha \notin (0,1)$. The restriction $\alpha \in (0,1)$ is effectively taken into account through the constitutive law Eq. (19), since this relation implies that the sub-differential $\partial_\alpha \overline{I}(\alpha)$ is not empty.

Keeping in mind that time derivatives must be left derivatives in order to cope with the principle of determinism, it comes out that the material time derivative of Ψ which appears in Eq. (5) can be written for $\alpha \in (0,1)$ as:

$$\frac{d\Psi}{dt} = \frac{\partial \Psi'}{\partial \rho_l}\frac{d\rho_l}{dt} + \frac{\partial \Psi'}{\partial \rho_g}\frac{d\rho_g}{dt} + \frac{\partial \Psi'}{\partial \rho_v}\frac{d\rho_v}{dt} + \frac{\partial \Psi'}{\partial \alpha}\frac{d\alpha}{dt}. \tag{23}$$

When the above result is used in Eq. (5) along with the state laws Eqs. (16-19), Eq. (5) can be rewritten as:

$$d := -(\Psi + p)\left(\frac{\partial v}{\partial s} + \frac{1}{A}\frac{dA}{dt}\right) - B^{\rho_l}\frac{d\rho_l}{dt} - B^{\rho_g}\frac{d\rho_g}{dt} - B^{\rho_v}\frac{d\rho_v}{dt} - B^{\alpha}\frac{d\alpha}{dt} + \frac{P}{A}\tau v \geq 0. \tag{24}$$

To obtain a complete set of constitutive equations, it suffices to specify a pseudo-potential of dissipation from which complementary laws are derived in such a way that the local version of the SLT given by Eq. (5) or (24) is always verified.

3.5 Pseudo-potential of dissipation and complementary laws

To introduce the irreversible behavior of the gas-liquid mixture, and also to ensure that the SLT is always satisfied, we assume that there exists a pseudo-potential of dissipation Φ, which is an objective, convex and differentiable function of B^Γ and v. Moreover, the pseudo-potential $\Phi = \Phi(B^\Gamma, v; \alpha, \theta)$ is assumed to have the following properties:

$$\Phi(B^\Gamma, v; \rho_l, \rho_g, \rho_v, \alpha, \theta) \geq 0 \quad \text{and} \quad \Phi(0,0; \rho_l, \rho_g, \rho_v, \alpha, \theta) = 0, \tag{25}$$

whatever the values of B^Γ and v.

The additional information associated with the dissipative behavior of the fluid is obtained from Φ through the following complementary laws:

$$\Gamma := \frac{\partial \Phi}{\partial B^\Gamma} \quad \text{and} \quad \tau := \frac{\partial \Phi}{\partial v}. \tag{26}$$

In addition, if the rate of energy dissipation d is supposed to have the following form for $\alpha \in (0,1)$,

$$d := B^\Gamma \Gamma + \frac{P}{A}\tau v, \tag{27}$$

then we get from the convexity property of Φ that, for any actual evolution (Berger, 1977):

$$d := B^\Gamma \Gamma + \frac{P}{A}\tau v \geq \Phi(B^\Gamma, v; \rho_l, \rho_g, \rho_v, \alpha, \theta) - \Phi(0,0; \rho_l, \rho_g, \rho_v, \alpha, \theta). \tag{28}$$

In view of Eq. (25), it is easy to see that $d \geq 0$ for any actual evolution of the fluid and so the SLT given by the inequality Eq. (5) is always satisfied.

From the mechanical viewpoint, Eq. (27) establishes that the dissipated energy rate is due to the phase change process and to the fluid friction. The constitutive assumptions made so far are sufficient to partially characterize the mechanical behavior of the fluid. Since the mass balance equations given by Eqs. (1-3) define a subspace of the linear space spanned by $(\partial v/\partial s + 1/A dA/dt)$, $d\rho_l/dt$, $d\rho_g/dt$, $d\rho_g/dt$ and $d\alpha/dt$, then in order that Eq. (24) be equal to Eq. (27) for any actual evolution it is possible to prove that (Assumpção and Freitas Rachid, 2011), for $\alpha \in (0,1)$, one must have the following relationships among the thermodynamic forces:

$$p = p_l = p_g + p_v, \tag{29}$$

$$B^\Gamma = g_l - g_v. \tag{30}$$

The above relationships establish that the mixture pressure is equal to the partial pressure of the liquid which in turn is equal to the sum of the partial pressures of the vapor and the inert gas and also that the thermodynamic force associated with the rate of mass transfer between the liquid and the vapor is equal to the difference of the Gibbs free energies in these two phases.

In the sequel, we assume the following simple and appropriate choice for Φ:

$$\Phi(B^{\Gamma}, v; \rho_l, \rho_g, \rho_v, \alpha, \theta) := \frac{\beta}{2}\left(B^{\Gamma}\right)^2 + f_m \frac{\rho v^2 |v|}{12}. \tag{31}$$

in which $\beta = \hat{\beta}(\theta)$ is a positive material constant. Since the expression in Eq. (31) satisfies the conditions set forth before, then the SLT given by Eq. (24) is unconditionally satisfied. As a result, based on Eqs. (26), (29-30) and (31), the specific choices expressed by Eq. (13-31) give rise to the following constitutive equations:

$$\Gamma = \beta\left(g_l - g_v\right) \quad \text{and} \quad \tau = f_m \frac{\rho v |v|}{8}. \tag{32}$$

Based on the preceding assumptions, it can be proved that the Gibbs free energy difference between the liquid and the vapor may be expressed as (Assumpção and Freitas Rachid, 2011):

$$g_l - g_v = a_l^2 \log\left(\frac{p_l + a_l^2 \rho_l^o}{p_{sv} + a_l^2 \rho_l^o}\right) + a_v^2 \log\frac{p_{sv}}{p_v} + a_l^4 \rho_l^o \left(\frac{1}{p_{sv} + a_l^2 \rho_l^o} - \frac{1}{p_l + a_l^2 \rho_l^o}\right) \tag{33}$$

Since in the absence of free gas $p_l = p_v$ in equilibrium, it can be seen from Eq. (33) that $g_l - g_v = 0$ if and only if $p_l = p_v = p_{sv}$. However, when a free gas is present $p_l = p_v + p_g$ and it can be verified based on Eq. (33) that the condition $g_l - g_v = 0$ is satisfied for different values of p_l and p_v. In other words, there will exist more than one equilibrium state when a free gas is present in the mixture according to its partial pressure. This condition enables the simulation of the phase change transformation in cavitating flows without the need to consider the extremal situations of pure liquid $\alpha = 0$ and pure vapor $\alpha = 1$, for which different set of equations must be solved when compared to those associated with the cavitating flow region ($\alpha \in (0, 1)$).

4. THERMO-MECHANICAL MODEL IN STEADY STATE

Since the main purpose of this work is to analyze the capability of the present model in describing coherently the slack flow in hydraulic pipelines, we hereafter restrict the analysis to steady state regimes. Thus, taking into account the basic assumptions, the governing equations given by Eqs. (1) to (4) can be reduced to and written in following matrix form:

$$\mathbf{M}\,\frac{d\mathbf{U}}{ds} = \mathbf{B}, \quad \text{for} \quad s \in (0, L), \tag{34}$$

in which

$$\mathbf{U} = \begin{bmatrix} \alpha & p & p_g & v \end{bmatrix}^T, \tag{35}$$

$$\mathbf{B} = \begin{bmatrix} -\Gamma & \Gamma & 0 & -\dfrac{\tau P}{A} - \rho g \sin\theta \end{bmatrix}^T, \tag{36}$$

$$\mathbf{M} = \begin{bmatrix} -v\rho_l & \dfrac{(1-\alpha)\,v}{a_l^2} + \dfrac{(1-\alpha)\,v\rho_l}{A}\dfrac{\xi A_o D_o}{eE} & 0 & (1-\alpha)\,\rho_l \\ v\rho_v & \dfrac{\alpha v}{a_v^2} + \dfrac{\alpha v \rho_v}{A}\dfrac{\xi A_o D_o}{eE} & -\dfrac{\alpha v}{a_v^2} & \alpha\rho_v \\ v\rho_g & \dfrac{\alpha v \rho_g}{A}\dfrac{\xi A_0 D_o}{eE} & \dfrac{\alpha v}{a_g^2} & \alpha\rho_g \\ v^2\left(-\rho_l + \rho_v + \rho_g\right) & 1 + v^2\left(\dfrac{\rho}{A}\dfrac{\xi A_o D_o}{eE} + \dfrac{1-\alpha}{a_l^2} + \dfrac{\alpha}{a_v^2}\right) & \dfrac{\alpha v^2}{a_g^2} - \dfrac{\alpha v^2}{a_v^2} & 2\rho v \end{bmatrix}. \tag{37}$$

5. NUMERICAL RESULTS

The main objective of this section is to explore the capability of the model in properly describing the opening of a cavity due to the vaporous cavitation. By prescribing an initial condition at the pipe inlet, we want to know whether the model will be able coherently describe the vaporous cavitation phenomenon in the fluid flow until the onset of a sonic flow. The occurrence of the vaporous cavitation will promote an increase in the gas volume fraction which in turn will result in a decrease of the wavefront speed. As a result, the Mach number will approach the unit value. Under these circumstances, the model described by the system of four ordinary differential equations for the variables $\alpha = \hat{\alpha}(s)$, $p = \hat{p}(s)$, $p_g = \hat{p_g}(s)$ and $v = \hat{v}(s)$ given by Eq . (34) along with initial conditions at the pipe inlet $s = 0$;

$$\mathbf{U_0} = \begin{bmatrix} \alpha_0 & p_0 & p_{g_0} & v_0 \end{bmatrix}^T, \tag{38}$$

form an initial-vale problem for ordinary differential equations. Although there are many numerical methods available to obtain approximated solutions to this problem, herein we have used the well-known Gear's method. It is an implicit, multi-step, adaptive method up to fifth order. This choice has been based on the stiff nature of the system of equations given by Eq. (34) for which the Gear's method has a good performance.

It can be proved that this initial-value problem is well-posed as long as the Mach Number, $\mathrm{M_a}$,

$$\mathrm{Ma} = \frac{v}{a}, \tag{39}$$

in which a stands for the sonic velocity of the mixture inside the compliant pipe, does not reach the unity. It is equivalent to ensure that the flow will not be sonic in the pipe. It can also be proved that the sonic velocity can be evaluated by the following expression:

$$a^2 = \frac{a_m^2}{\left(1 + \dfrac{\rho a_m^2 D_o \xi}{Ee}\right)}, \tag{40}$$

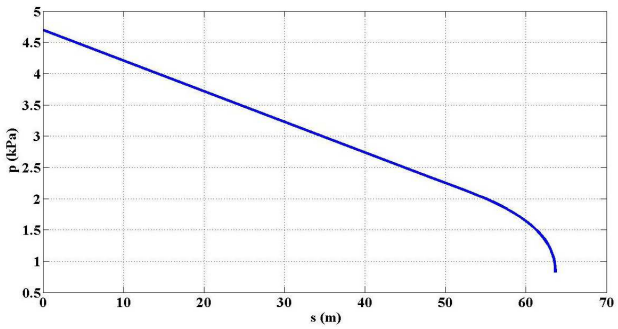
in which D_o is the internal diameter of the pipe, e is the pipe thickness and E is the Young modulus of the pipe material. In the above equation, a_m is the wave speed with which the wavefront of disturbances propagate in the mixture and is given by:

$$a_m^2 = \frac{\rho_l a_l^2 \left(\rho_v a_v^2 + \rho_g a_g^2\right)}{\rho\left(\left(\rho_v a_v^2 + \rho_g a_g^2\right)(1-\alpha) + \alpha \rho_l a_l^2\right)}. \tag{41}$$

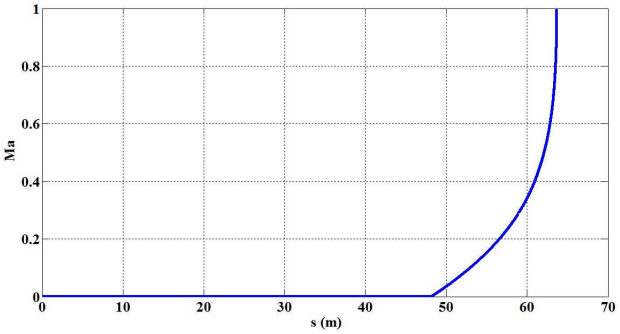
In what follows, the ability and reliability of model are evaluated by means of a numerical example. To do so, consider a pipe of diameter D_o = 203.2 mm and length $L = 150$ m through which a water-vapor-air mixture is flowing at 20°C. At this temperature, the values of p_{sv}, ρ_l^o, a_l, a_v and a_g are known and equal to 2.34 kPa, 998.154 kg/m^3, 1477 m/s, 361 m/s and 290 m/s, respectively. The pipe is made of steel (E = 207 GPa and $\nu = 0.27$) and is anchored with expansion joints so that ξ = 1. For the sake of simplicity we assume that topographic profile of pipe is flat. The initial conditions at the pipeline entrance are characterized by the vector given by Eq. (38), whose components assume the following values: $\alpha_0 = 10^{-8}$, $p_0 = 4.70$ kPa and $v_0 = 1.0$ m/s. The initial partial pressure of the inert gas, p_{g_0}, is evaluated in terms of p_0 by means of the relationship $g_l - g_v = 0$, which states that at the pipeline entrance the liquid is at equilibrium with the vapor. Thus, for $p_0 = 4.70$ kPa it comes out that $p_{g_0} = 2.36$ kPa. Under these initial conditions the mixture is essentially made of liquid water whose saturation vapor pressure is slightly greater than the saturation pressure at the same temperature (p_{vs} =2.34 kPa at 20°C), without the presence of air. To assign an irreversible character to the phase change transformation we have run the simulation by adopting $\beta = 10^{-8}$ kg^2/m^{3}Js (Freitas Rachid, 2003; Assumpção and Freitas Rachid, 2011)

As the water flows downstream the pipe, the pressure of the mixture falls linearly due to the friction until the vaporous cavitation phenomenon begins, what happens at approximately $s = 48$ m as it can be observed in Fig. 2(a). As it can be seen in Fig. 2(b), from that point until $s \approx 63$ m, the Mach number increases significantly from 0.7×10^{-4} to one and the flow becomes sonic. As a result of the phase change process, a stringent increase of the gas volume fraction from 10^{-8} to 0.45 (see Fig. 2(c)) is observed due to the transformation of liquid into vapor. The above statements can be corroborated through the graphic of the rate of vapor mass production as a function of the position as illustrated in Fig. 2(d). From the pipe entrance to $s = 48$ m there is no vapor generation. However, from that point on, vapor is generated causing an appreciable increase of the gaseous volume fraction which in turn reduces drastically the sonic speed. When approximately one-half of the pipe's cross-section is filled by the vapor, the flow becomes sonic.

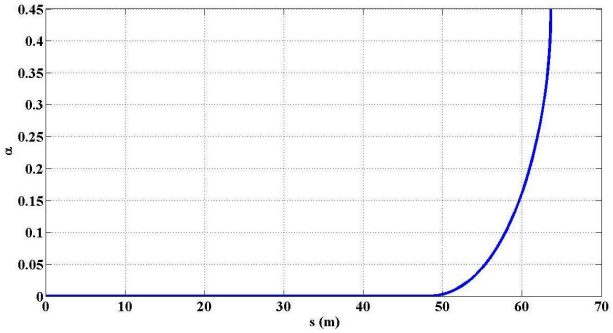
Although the lowering of the partial pressures of the vapor and of the air induces the sonic speed reduction of the mixture, the main contribution responsible for such an effect is due to the evolution of the gaseous volume fraction. While the gaseous volume fraction experiences an overall increase of 10^8 within a extension of 15 m only, the variations of the partial pressures are restricted to one order of magnitude of the saturated vapor pressure, as it can be noticed in Fig. 3(a) and 3(b). Note that while the partial pressure of the air varies linearly with the position until the onset of the cavitation, the vapor partial pressure remains practically unaltered within that extension. However, from that point on the partial pressure of vapor falls well below the saturated vapor pressure by promoting this way the increase of the Gibbs free energy difference between the liquid and the vapor (see Fig. 3(c)); the driving force responsible for the phase change transformation. Although the process of phase change takes place from $s = 48$ m to $s \approx 63$ m (where the flow becomes sonic, or equivalently, the Mach number becomes equal to one), the wave speed in the mixture jumps down sharply at the beginning of this process at $s = 48$ m, as it can be observed in Fig. 3(d) which exhibits the mixture-to-liquid wave speed ratio as a function of the spatial position along the pipe. This abrupt change in the wave speed of the mixture as a result of the phase change process is one of the major difficulties when numerical techniques are applied to obtain approximated solutions in both steady and unsteady regimes. In the case of steady state, this example clearly demonstrates that Gear's method is a good choice since it has captured this sharp discontinuity with success.



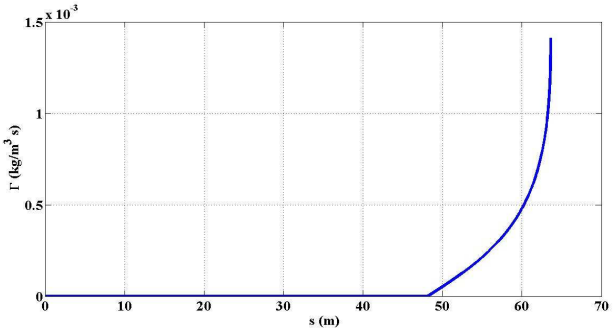
(a) Pressure against spatial position along the pipe



(b) Mach number against spatial position along the pipe

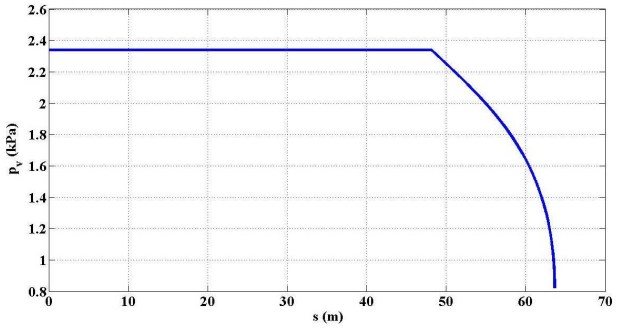


(c) Gas volume fraction against spatial position along the pipe

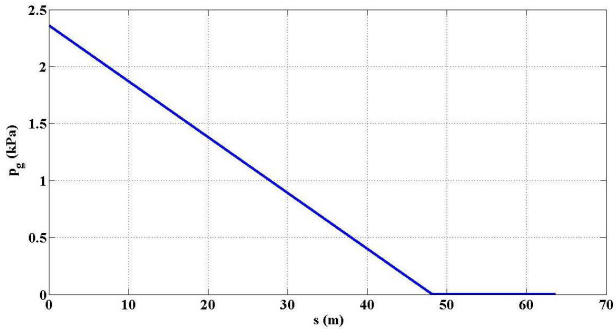


(d) Rate of vapor mass production against spatial position along the pipe

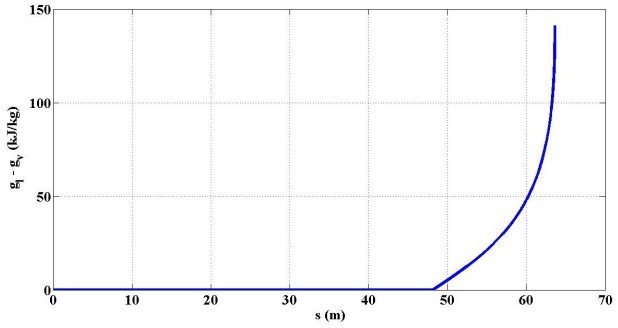
Figure 2. Mechanical response of a water-vapor-air mixture at room temperature flowing in a horizontal pipe.



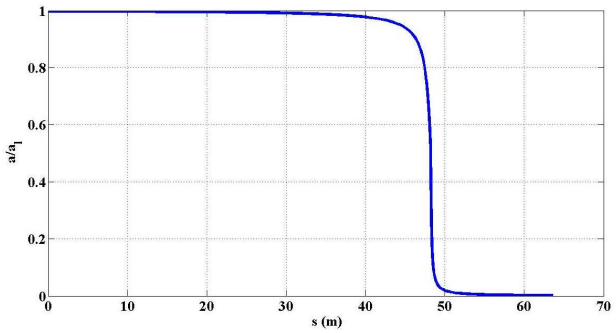
(a) Partial pressure of vapor against spatial position along the pipe



(b) Partial pressure of air against spatial position along the pipe



(c) Gibbs free energy difference against spatial position along the pipe



(d) Mixture-to-liquid wave speed ratio against spatial position along the pipe

Figure 3. Mechanical response of a water-vapor-air mixture at room temperature flowing in a horizontal pipe.

Finally, is presented in Fig. 4 the plot of the rate of energy dissipation associated with the phase change process as a function of pipe extension. As expected, from the pipe entrance until $s = 48$ m, this rate of energy dissipation remains equal to to zero, since there is no liquid-vapor transformation. However, apart from the section $s = 48$ m, the liquid is progressively converted into vapor. As the phase change transformation is modeled as an irreversible process - as it should

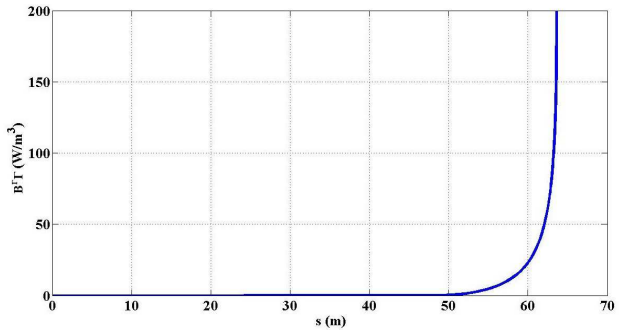


Figure 4. Rate of energy dissipation associated with the phase change process against the spatial position for a water-vapor-air mixture at room temperature flowing in a horizontal pipe.

be from the correct physical viewpoint - the rate of energy dissipation associated with the phase change also increases reaching the mark of approximately 200 W/m^3 at $s \approx 63$ m, where the flow becomes sonic.

6. CONCLUDING REMARKS

This work has presented a thermodynamically consistent model to describe cavitating flows in hydraulic pipelines. Although the model is capable to deal with the cavitation phenomenon in unsteady flows, the application presented herein is restricted to slack flow, which takes place in steady state regimes. To ensure that the same set of equations is used throughout the whole fluid domain under cavitation or not, a small amount of inert gas is assumed to coexist along with the vapor. Moreover, it is shown that the presence of an inert gas ensures the necessary thermodynamic conditions to properly describe the opening as well as the closing of the vapor cavity. The numerical example presented for a air-water mixture flowing in a horizontal pipe has shown that the cavitation phenomenon is properly described until the local Mach number equals the unity.

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