HYBRID SOLUTIONS FOR ELECTROOSMOTIC FLOWS IN ELLIPTICAL MICROCHANNELS

Fernanda Gabrielle Cavalcante dos Santos
Diana Patrícia Santos de Souza Braga
Emanuel Negrão Macêdo
João Nazareno Nonato Quaresma

School of Chemical Engineering, ITEC, Universidade Federal do Pará, Campus Universitário do Guamá, Rua Augusto Corrêa, 01, 66075-110, Belém, PA, Brazil.
fernandagabrielle @hotmail.com; dianaps15@hotmail.com; enegrao@ufpa.br; quaresma @ufpa.br

Abstract. *In this work, the effects of the electric double layer (EDL) near the solid-liquid interface and the electroosmotic flow combined with pressure gradient of Newtonian fluids through an elliptical microchannel are analyzed. Nonlinear two-dimensional Poisson-Boltzmann and momentum equations that govern the electrical double layer field in the cross section of elliptical channel are solved with a hybrid solution through the Generalized Integral Transform Technique (GITT). The behavior of the velocity profile was analyzed for governing parameters of the flow, such as: the ratio of the pressure force to the electric force, aspect ratio of the cross-section and the electrokinetic diameter. The results clearly show that the presence of a pressure gradient strongly modifies the flow behavior of a purely electroosmotic and they highlights that the flow rate is strongly influenced by the aspect ratio of the cross-section, while the pressure head is more sensible to the electrokinetic diameter. Besides, a convergence analysis for different values of governing parameters was performed to demonstrate the effectiveness of the GITT approach.*

Keywords: *Electroosmotic flow, pressure gradient, elliptical microchannel, GITT approach.*

1. INTRODUCTION

In the literature, the electroosmotic flow has been deeply studied as replacement of the flow induced by pressure gradient, since this requires specialized control to the application, while the electroosmotic flow is an alternative with a control more viable. This type of flow is extremely important in various branches of science such as medicine and engineering because of their different applications. Furthermore, the study of microchannels offer many advantages over conventional equipment, since they are more efficient, have higher portability and reduced analysis time and costs used in research.

The electrokinetics effects are caused by an electric double layer at the solid-liquid interface. The electric double layer is the result of the interaction between an electrolyte and a charged surface. Counter ions from the electrolyte are attracted by the surface and form a thin charge layer. This charge layer is also called the Stern layer. The Stern layer is immobilized on the surface by electrostatic force between opposite charges. The Stern layer builds up a thicker charge layer inside the electrolyte. The second layer is called the diffuse layer. Under an electric field, the second layer can move relative to the solid surface. The immobile Stern layer and the mobile diffuse layer form the so-called electric double layer (EDL). The interface between these two layers is called the shear layer. The electric potential of the wall and of the shear layer are called the wall potential and the zeta potential, respectively.

Previous discussions reveal that most of the relevant studies in the literature are limited to some simple geometry or trapezoidal cross section. Also, drastic assumptions are made so that mathematical analysis can be simplified, and an analytical result can be derived. Unfortunately, the cross section of microchannels used in practice may assume various shapes. Blood capillary and lymphatic microchannels, for instance, will deform when they experience pressure from surrounding muscle. This suggests that a geometry more flexible than those assumed previously should be considered. To this end, a channel with an elliptical cross section seems to be an appropriate choice since it can simulate various geometries by adjusting its shape parameters (Hsu *et al.*, 2002).

Flow within a microchannel can also be generated by a combination of flow induced by a pressure gradient and electroosmotic flow. The flow rate induced by electroosmotic force is usually small and therefore even a small pressure gradient applied along a microchannel may cause velocity distributions and corresponding flow rates that deviate from a purely electroosmotic flow. The pressure gradient can result from various reasons such as the presence of alternative pumping mechanism and the existence of changes in the zeta potential of the wall. A study in the literature about this combination is done in the work of Sphaier (2012). The author presents solutions for heat transfer problems that occur in microtubes flow driven by the combined effect of electro-osmosis and a pressure gradient using the Generalized Integral Transform Technique (GITT).

Some research studies have been conducted in order to examine the flow characteristics combined when the pressure forces and electro-osmosis are present simultaneously, the resulting velocity profile is a superposition of the electroosmotic flow induced by a pressure gradient. The velocity distributions of the combined flow distributions are very different from those purely electroosmotic flow or traditional flow induced by a pressure gradient. Chen (2009) studied this combined effect considering the effects of Joule heating for constant and variable properties of the fluid, an extension of this work was performed by Sadeghi and Saidi (2010) and Dey *et al*. (2010), by considering the viscous heating effects and constant fluid properties.

The GITT, present in this work, is known as a powerful method in solving of certain classes of problems of heat and mass diffusion. The GITT allows solution of problems in a hybrid form to problems with a degree of complexity involved that cannot be treated by usual analytical techniques. The basic idea is to transform a system of partial differential equations on an infinite system of ordinary equations, by eliminating spatial dependencies, where these can be solved more simply, with the advantage of producing a more accurate and more economical solution and to allow the control over the relative error results. The GITT is applied even in works where electroosmotic effects are observed, Soares *et al*. (2008) present a study about the electroosmotic flow and the transient forced convection in a parallel plate microchannel, with 10 μm between the plates using this technique.

The objective of the present paper is the use of the Generalized Integral Transform Technique in solving two-dimensional Poisson-Boltzmann and momentum equations related to electroosmotic flow combined with pressure gradient of Newtonian fluids through an elliptical microchannel. Numerical results are produced for analyzing the behavior of the velocity profile with the governing parameters of the flow, such as: the ratio of the pressure force to the electric force, aspect ratio of the cross-section and the electrokinetic diameter. Also, the numerical results are confronted with those results from the literature in order to assess the present GITT approach.

2. ANALYSIS

This section shows the general mathematical formulation of the flow induced by a pressure gradient and an electrical field in microchannels having elliptical geometry, which can be mapped through a function to describe the physical problem and the simplifying assumptions of the problem studied.

The diagram in Fig. 1 shows the cross section of the microchannel considered in this work, with aspect ratio β defined as the ratio of the semi-minor axis (b) and the semi-major axis (a).

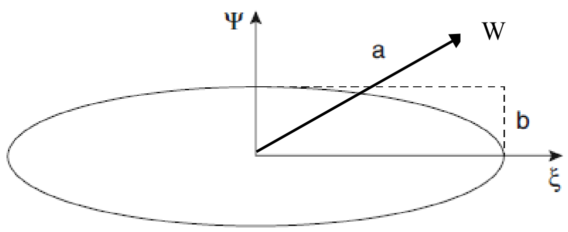


Figure 1. Sketch of the microchannel cross-sectional geometry.

Because of the influence of the channel wall, the flow behavior in the microchannel deviates from predictions of the traditional form of the Navier Stokes equations. The velocity field induced by electro-osmosis is determined by the modified Navier-Stokes relating the electrolyte movement with the distribution of electric potential in the fluid; it was considered that the physical properties are constant, the flow is also considered to be steady, laminar and two-dimensional fully developed subjected to an external pressure gradient in an isothermal system. Therefore, The Poisson-Boltzman and the momentum equations can be written as:

$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}=D_e^2 \sinh(\Phi(x,y)) \qquad \frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=\frac{D_e^2}{Z}G_{EO}-\frac{1}{Z}D_e^2 \sinh(\Phi(x,y)) \tag{1,2}$$

Where, in Eqs. (1) and (2) above the following dimensionless groups were employed:

$$x=\frac{\xi}{D_h}; \quad y=\frac{\psi}{D_h}; \quad \Phi=\frac{e\varphi}{K_B T_R}; \quad U=\frac{u^*}{u_{EO}}=u^*\left(\frac{\mu}{\varepsilon\zeta E_{ext}}\right) \tag{3-6}$$

When the electric potential is much smaller than the ion thermal energy, i.e., $ze\varphi << K_B T_R$ ($\Phi \le 1$), $\sinh(\Phi(x,y))$ present in the above equations can be approximated by $\Phi(x,y)$.

$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}=(kD_h)^2 \Phi(x,y) \qquad \frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=\frac{(kD_h)^2}{Z}G_{EO}-\frac{1}{Z}(kD_h)^2 \Phi(x,y) \tag{7,8}$$

Where k is the Debye-Hückel parameter, Z is the dimensionless zeta potential, G_{EO} is the dimensionless value that expresses the ratio between the forces of pressure and electric field forces acting in the fluid and kD_h defines the electrokinetic diameter (D_e).

$$G_{EO} = \frac{dp/d\omega}{eE_{ext}\sum_{i=1}^{N} z_i^2 n_{i\infty}}; \quad k = \left(\frac{e^2}{\varepsilon K_B T_R}\sum_{i=1}^{N} z_i^2 n_{i\infty}\right)^{1/2}; \quad Z = \frac{e\zeta}{K_B T_R} \tag{9-11}$$

Due to the geometrical symmetry, it is better to reduce the domain to a quarter of the elliptical cross-section. Therefore, Eqs. (7) and (8) are subjected to the following boundary conditions on the inner edges:

$$\frac{\partial \Phi}{\partial x} = 0; \quad \frac{\partial U}{\partial x} = 0 \quad at\ x = 0 \quad \frac{\partial \Phi}{\partial y} = 0; \quad \frac{\partial U}{\partial y} = 0 \quad at\ y = 0 \tag{12-15}$$

While at the wall it holds that:

$$\Phi|_w = Z; \qquad U|_w = 0 \tag{16,17}$$

The problem to be studied is defined by considering the fluid flow in a microchannel of elliptical geometry. Regarding the boundary conditions, one difficulty is due to two-dimensional irregularity characteristic of elliptical cross section of the duct. An appropriate change of variables is designed to transform the elliptical profile in a new geometry, which simplifies the application of boundary conditions (Maia *et al*., 2005).

The orthogonal system of elliptic coordinates (u, v) is used to transform the original domain, with elliptical contour in the coordinates (x, y), into a domain with rectangular contour in the transformed system (u,v):

$$\begin{cases} x = c\cosh(v)\cos(u) & 0 \le u \le \pi/2 \\ y = c\sinh(v)\sin(u) & 0 \le v \le v_0 \end{cases}; \quad h^2(u,v) = c^2\left[\sin^2(u) + \sinh^2(v)\right] \tag{18-20}$$

Since c is the dimensionless focal length given by Eq. (21)

$$c^2 = a^2 - b^2 \qquad v_0 = \arctan h(\beta) \tag{21,22}$$

Making the transformation of the coordinate system using the above equations and implementing the homogenization of the boundary conditions, applying the change of variables given by $\Phi = Z + \phi$, to increase the convergence rate of the series that appear when the solution through the application of the GITT approach, the mathematical formulation of the problem is now written as:

$$\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2} = h^2(u,v)De^2(Z + \phi(u,v)) \qquad \frac{\partial^2 U}{\partial u^2} + \frac{\partial^2 U}{\partial v^2} = \frac{h^2(u,v)De^2}{Z}\left(G_{EO} - Z - \phi(u,v)\right) \tag{23,24}$$

The boundary conditions are redefined as:

$$\frac{\partial\phi}{\partial u} = 0; \quad \frac{\partial U}{\partial u} = 0 \quad at \quad u = 0; \qquad \frac{\partial\phi}{\partial u} = 0; \quad \frac{\partial U}{\partial u} = 0 \quad at \quad u = \pi/2 \tag{25-28}$$

$$\frac{\partial\phi}{\partial v} = 0; \quad \frac{\partial U}{\partial v} = 0 \quad at \quad v = 0; \qquad \phi = 0; \quad U = 0 \quad at \quad v = v_0 \tag{29-32}$$

2.1 Solution Methodology

The Generalized Integral Transform Technique (GITT) is then employed in the hybrid numerical-analytical solution of the problem. For this purpose, the following auxiliary eigenvalue problems are chosen for the electrical potential and velocity fields, respectively:

$$\frac{d^2\Omega_i(v)}{dv^2} + \alpha_i^2\Omega_i(v) = 0 \tag{33}$$

$$\frac{d\Omega_i(0)}{dv} = 0; \quad \Omega_i(v_0) = 0 \tag{34,35}$$

$$\frac{d^2\psi_i(v)}{dv^2} + \mu_i^2\,\psi_i(v) = 0 \tag{36}$$

$$\frac{d\psi_i(0)}{dv} = 0; \quad \psi_i(v_0) = 0 \tag{37,38}$$

Equations above can be analytically solved, to yield the eigenfunctions and eigenvalues, to the electrical potential and velocity profile, respectively, as:

$$\Omega_i(v) = \cos(\alpha_i v) \qquad \alpha_i = \frac{(2i-1)\pi}{2v_0}, \quad i = 1,2,3,... \tag{39,40}$$

$$\psi_i(v) = \cos(\mu_i v) \qquad \mu_i = \frac{(2i-1)\pi}{2v_0}, \quad i = 1,2,3,... \tag{41,42}$$

It can be shown that the eigenfunctions $\Omega_i(v)$ and $\psi_i(v)$ obey the following orthogonality properties, respectively, where L_i and M_i are the normalization integrals:

$$\int_0^{v_0} \Omega_i(v)\Omega_j(v)dv = \begin{cases} 0, & i \neq j \\ L_i, & i = j \end{cases}; \quad L_i = \int_0^{v_0} \Omega_i^2(v)\, dv = \frac{v_0}{2} \tag{43,44}$$

$$\int_0^{v_0} \psi_i(v)\psi_j(v)\,dv = \begin{cases} 0, & i \neq j \\ M_i, & i = j \end{cases}; \quad M_i = \int_0^{v_0} \psi_i^2(v)\, dv = \frac{v_0}{2} \tag{45,46}$$

Equations (33) to (42) together with the respective orthogonality properties allow the definition of the integral transform pairs for the electrical potential and velocity fields, respectively, as:

$$\bar{\phi}_i(u) = \int_0^{v_0} \tilde{\Omega}_i(v)\phi(u,v)\,dv, \quad \text{transform;} \quad \phi(u,v) = \sum_{i=1}^{\infty} \tilde{\Omega}_i(v)\bar{\phi}_i(u), \quad \text{inverse} \tag{47,48}$$

$$\bar{U}_i(u) = \int_0^{v_0} \tilde{\psi}_i(v)U(u,v)\,dv, \quad \text{transform;} \quad U(u,v) = \sum_{i=1}^{\infty} \tilde{\psi}_i(v)\bar{U}_i(u), \quad \text{inverse} \tag{49,50}$$

Where,

$$\tilde{\Omega}_i(v) = \Omega_i(v)/\sqrt{L_i}; \quad \tilde{\psi}_i(v) = \psi_i(v)/\sqrt{M_i} \tag{51,52}$$

To obtain the resulting system of differential equations for the transformed potentials, Eqs. (23) and (24) are multiplied by the normalized eigenfunctions given by Eqs. (51) and (52), respectively, and integrated over domain $[0,v_0]$ in the v-direction after that the inverse formulae given by Eqs. (48) and (50) are employed. After the appropriate manipulations, the following coupled ordinary differential systems are obtained for calculating the transformed potentials:

$$\frac{d^2\bar{\phi}_i(u)}{du^2} = \alpha_i^2\bar{\phi}_i(u) + A_i(u); \quad \frac{d^2\bar{U}_i(u)}{du^2} = \beta_i^2\bar{U}_i(u) + B_i(u) \tag{53,54}$$

Subjected to the following transformed boundary conditions:

$$\frac{d\bar{\phi}_i(0)}{du} = 0; \quad \frac{d\bar{U}_i(0)}{du} = 0; \quad \frac{d\bar{\phi}_i(\pi/2)}{du} = 0; \quad \frac{d\bar{U}_i(\pi/2)}{du} = 0 \tag{55-58}$$

The coefficients being determined by:

$$A_i(u) = \left\{ Z\left[C_i \sin^2(u) + E_i\right] + \sin^2(u)\bar{\phi}_i(u) + \sum_{j=1}^{\infty} D_{ij}\bar{\phi}_j(u) \right\} \tag{59}$$

$$B_i(u) = \frac{De^2c^2}{Z}\left\{ (G_{EO} - Z)\left[\sin^2(u)C_i + E_i\right] - \sin^2(u)\bar{\phi}_i(u) - \sum_{j=1}^{\infty} D_{ij}\bar{\phi}_j(u) \right\} \tag{60}$$

Where,

$$C_i = \int_0^{v_0} \tilde{\Omega}_i(v)dv \quad E_i = \int_0^{v_0} \sinh^2(v)\tilde{\Omega}_i(v)\,dv \quad D_{ij} = \int_0^{v_0} \sinh^2(v)\tilde{\Omega}_i(v)\tilde{\Omega}_j(v)dv \tag{61-63}$$

The system formed by Eqs. (53) to (63) form an infinite system of non-linear ordinary differential equations, which has to be truncated in a sufficiently high orders NE and NV, with the objective of obtaining the transformed potentials for the electrical potential and velocity fields, with maximum accuracy. For the solution of such system, due to the expected stiff characteristics, specialized subroutines have to be employed such as the DVPFD from the IMSL Library (1991).

It is necessary to calculate the average velocity (W), and then from the introduction of the inverse formula, into its usual definition, one obtains:

$$W = \frac{\int_S U\,dA}{\int_S dA} \qquad W = \frac{4(1-\beta^2)}{\pi\beta}\sum_{i=1}^{\infty}\left[C_i F_i + E_i G_i\right]; \quad F_i = \int_0^{\pi/2} \sin^2(u)\bar{U}_i(u)du\,; \quad G_i = \int_0^{\pi/2} \bar{U}_i(u)du \qquad (64\text{-}67)$$

3. RESULTS AND DISCUSSION

3.1 Convergence analysis

To solve the system given by Eqs. (53) to (63), it has been developed a computer code in the FORTRAN 90/95 programming language by using the routine DBVPFD from the IMSL Library (1991). To carry out the computer simulations, it was prescribed a relative error of 10^{-4} in the solution of the infinite system of ordinary differential equations.

In order to make a verification of the present solution methodology, it is necessary to make a convergence analysis of the results, which it is performed in the average velocity. Table 1 shows the convergence analysis of the results for D_e = 93 and D_e = 200, different values of aspect ratio (β = 0.2, β = 0.4 and β = 1.0), dimensionless pressure gradient (G_{EO} = 0 and G_{EO} = 1e-03) and Z = 1, with increasing number of terms in the series (N=NE=NV). For D_e = 93 and considering purely electroosmotic flow (G_{EO} = 0), the results converge fast with low numbers of terms for the cases of β = 0.2 and β = 0.4. For the case of circular geometry microchannels (β = 1.0), the convergence takes place between 73 and 89 terms with three significant digits. For De = 200, β = 0.2 and β = 0.4, the convergence is reached between 89 and 97 terms with four significant digits, whereas for β = 1.0 the convergence is slow, requiring more numbers of terms.

Table 1. Convergence behavior of the average velocity for De=93 and De=200 with Z=1 for β = 0.2, 0.4 and 1.0 and G_{EO} =0 and 1e-03.

N	D_e = 93					
	G_{EO} = 0			G_{EO} = 1e-03		
	β=0.2	β=0.4	β=1.0	β=0.2	β=0.4	β=1.0
17	0.9563	0.9687	0.9316	0.7238	0.7059	0.6441
33	0.9571	0.9571	0.9497	0.7246	0.7071	0.6716
49	0.9572	0.9573	0.9543	0.7247	0.7072	0.6799
57	0.9572	0.9573	0.9552	0.7247	0.7072	0.6819
73	0.9572	0.9573	0.9563	0.7247	0.7073	0.6842
89	0.9572	0.9573	0.9567	0.7247	0.7073	0.6854
97	0.9572	0.9573	0.9569	0.7247	0.7073	0.6857
NV	D_e = 200					
	G_{EO} = 0			G_{EO} = 1e-03		
	β=0.2	β=0.4	β=1.0	β=0.2	β=0.4	β=1.0
17	0.9771	0.9762	0.9418	-0.0979	-0.1803	-0.3076
33	0.9795	0.9793	0.9647	-0.0956	-0.1772	-0.2852
49	0.9798	0.9798	0.9722	-0.0953	-0.1766	-0.2778
57	0.9799	0.9799	0.9741	-0.0952	-0.1765	-0.2744
73	0.9799	0.9799	0.9765	-0.0951	-0.1765	-0.2735
89	0.9800	0.9800	0.9778	-0.0951	-0.1764	-0.2722
97	0.9800	0.9800	0.9782	-0.0951	-0.1764	-0.2717

3.2 Effects of electrokinetic diameter and aspect ratio

In the figures shown in this sub-section, we will analyze the behavior of electrokinetic diameter (De) and of aspect ratio for Z = 1 and G_{EO} = 0 (purely electroosmotic flow) and G_{EO} = 1e-03 and G_{EO} = 2.83e-03 (flows combined with pressure gradient). Figure 2 shows the characteristic curves for an elliptical cross section of β = 0.4 with De = 93 and Z = 1. It is carried out a comparison between the results obtained in this work with the work of Vocale *et al.* (2013) for different dimensionless pressure gradient values (G_{EO}) where it is verified an excellent agreement between the results. Furthermore, the graph shows that a purely electroosmotic flow (G_{EO} = 0) is characterized by a plan velocity profile. Since the pressure field emerges and opposes to the electroosmotic flow, the velocity profile tends to have a curved shape as seen for G_{EO} = 1e-03 and G_{EO} = 2.83e-03.

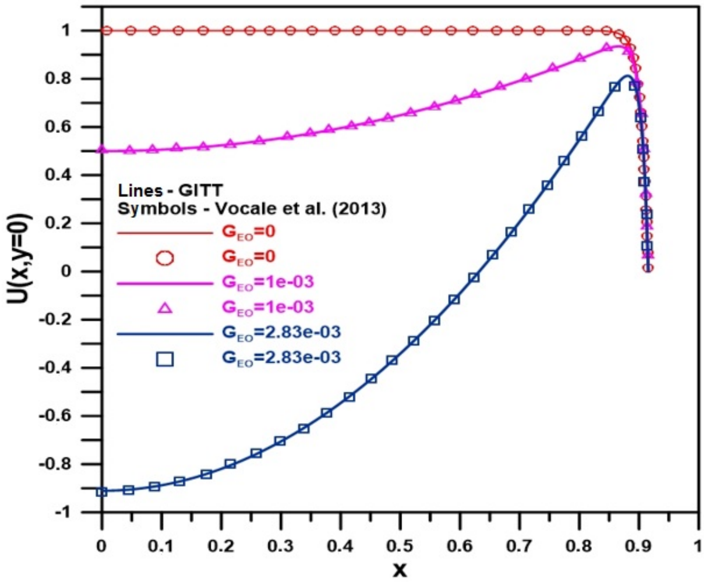
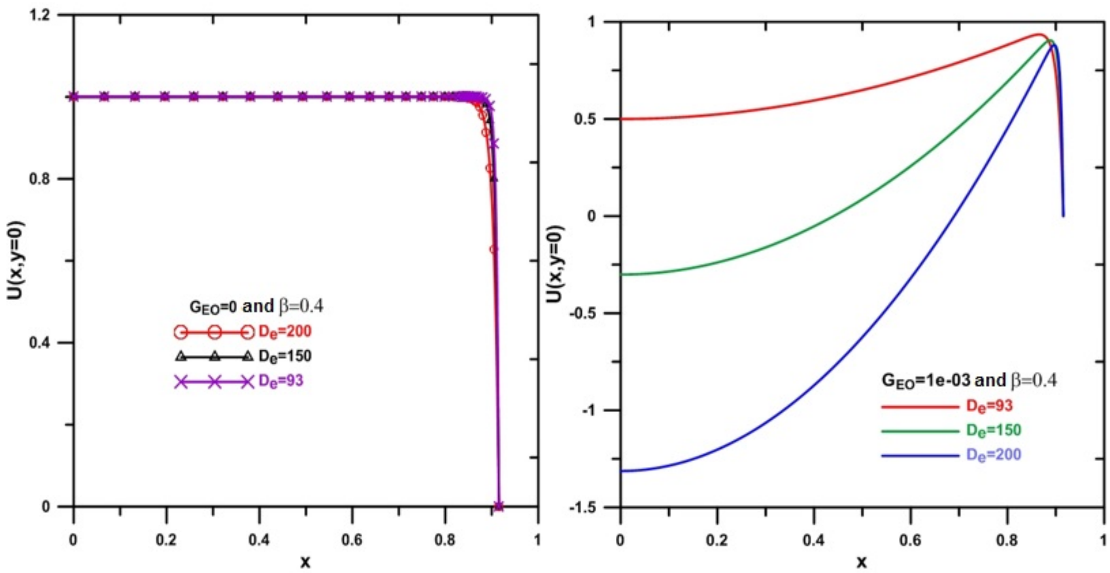


Figure 2. Velocity profile along the axis of the elliptical microchannel as function of GEO.

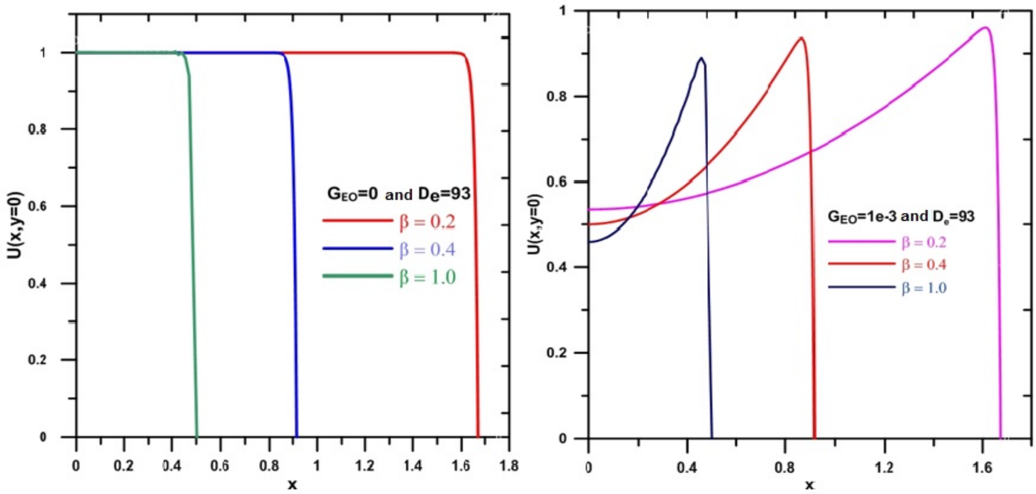
In Figure 3(a), it is verified that the electrokinetic diameter does not cause significant changes in velocity profile of purely electroosmotic flow. The velocity profile has a linear behavior, plan for purely electroosmotic flow, resembling a slug flow. Noting the Fig. 3(a) and 3(b), we observe the difference in behavior of the curves for different G_{EO} values, proving that the electrokinetic diameter is critical to pump performance in terms of pressure. Indeed, the potential to treat the pressure forces acting evenly over the cross section is much higher if the electric force, that is, the driving force acting on most of the region occupied by the fluid. This happens when the EDL is longer, that is, when the electrokinetic diameter is smaller.

Figures 4(a) and 4(b) show that the aspect ratio causes significant changes in the characteristics of velocity profile curves. There is also the linear behavior of the characteristic velocity profile of purely electroosmotic flow. In Figure 4 (b), it is analyzed the flow behavior with dimensionless pressure gradient G_{EO} = 1e-03 and D_e = 93 for different values of the aspect ratio. It is observed that the curve behaviors are basically the same, proving that the aspect ratio does not influence significantly the flow with pressure gradient, unlike of electrokinetic diameter that acts directly on the changes that occur in this type of flow.



(a) (b)

Figure 3. Velocity profile along the axis of the microchannel as function of De for: (a) G_{EO}=0 and β = 0.4; (b) G_{EO}=1e-03 and β = 0.4.



(a) (b)

Figure 4. Velocity profile along the axis of the microchannel as function of β for: (a) G_{EO}=0 and D_e = 93; (b) G_{EO}=1e-03 and D_e = 93.

4. CONCLUSIONS

In this study, the GITT was successfully applied to the problem of electroosmotic Newtonian flow with pressure gradient in microchannel of elliptical geometry. The inherent difficulty of this technique application in problems with this geometry was overcome by performing a coordinate transformation by making the change of Cartesian coordinates to a system of elliptic coordinates that transform the shape of the elliptical section of the duct in a new domain with rectangular shape.

The results converged with appropriate number of terms showing the efficiency of this technique, the values for circular geometry had a slower convergence when compared with the values obtained for other aspect ratios.

It was also discussed the flow behavior due to the electrokinetic diameter, aspect ratio and dimensionless pressure gradient. It was verified that the aspect ratio has fundamental importance in the behavior of the velocity profile, while the diameter electrokinetic acts directly on the pressure gradient, causing no significant changes in the velocity profile of purely electroosmotic flow and in flows with pressure gradient. A linear behavior was observed for curves representing the velocity profile in purely electroosmotic flow, also it was made a comparison with data from the literature resulting in excellent agreement among the results.

5. REFERENCES

CHEN, C. H. Thermal transport characteristics of mixed pressure and electroosmotically driven flow in micro and nanochannels with joule heating. **ASME Journal Heat Transfer**, v. 131, 022401, 0022-1481, 2009.

DEY, R.; CHAKRABORTY D.; CHAKRABORTY, S. Analytical solution for thermally fully developed combined electroosmotic and pressure-driven flows in narrow confinements with thick electrical double layers. **ASME Journal Heat Transfer**, v. 133, pp. 024503(1-5), 2010.

HSU, J.; KAO, C.; TSENG, S.; CHEN, C. Electrokinetic flow through an elliptical microchannel: Effects of aspect ratio and electrical boundary conditions. **Journal of Colloid and Interface Science**, v. 248, pp. 176-184, 2002.

IMSL Library, MATH/LIB, Houston, TX, 1991.

MAIA, C. R. M.; RAMOS, R. A. V.; PELEGRINI, M. F.; ALVES, T. A. Heat Transfer Study in Slug Flow on Elliptical Ducts Cross Sections by Generalized Integral Transform Technique. **Thermal Engineering**, v. 4, pp. 154-163, 2005.

SADEGHI, M. H.; SAIDI, M. H. Viscous dissipation effects on thermal transport characteristics of combined pressure and electroosmotically driven flow in microchannels. **International Journal of Heat and Mass Transfer**, v. 53, pp. 3782–3791, 2010.

SOARES, P. O.; COTTA, R. M.; NETO, A. J. S. Convection in Microchannels with Electroosmotic Flow: Integral Transforms in Pseudo-Transient Formulation. Proceedings of 12th Brazilian Congress of Thermal Sciences and Engineering. Rio de Janeiro, RJ.

SPHAIER, L. A. Analysis of Joule and Viscous Heating in Combined Pressure and Electroosmotic Driven Flows in Isothermal Microtubes. Proceedings of 14th Brazilian Congress of Thermal Sciences and Engineering. Rio de Janeiro, RJ.

VOCALE, P. et al. Numerical analysis of electroosmotic flows through elliptic microchannels. **La Houille Blanche**, v. 3, pp. 42-49, 2013.

6. RESPONSABILITY NOTICE

The authors are the only responsible for the printed material included in this paper.