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HEAT CONDUCTION WITH LINEAR TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITY: AN UNCONSTRAINED MATHEMATICAL DESCRIPTION

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Abstract. *When dealing with nonlinear heat transfer problems, it is very important is to avoid a physically inadmissible solution. Provided the physical solution exists and is unique, an unrestricted mathematical modeling – without inequalities, is a very valuable tool for computational simulations, minimizing costs and effort. In this work a conduction heat transfer problem in which the thermal conductivity decreases linearly with the temperature in a given interval is considered. A physically equivalent alternative form is proposed – valid for any absolute temperature T, giving rise to an unrestricted mathematical modeling and circumventing the need of choosing the solution with physical meaning.*

Keywords: *Nonlinear heat transfer; unconstrained mathematical description, temperature-dependent thermal conductivity; Kirchhoff transformation.*

1. INTRODUCTION

Nonlinear heat transfer problems may give rise to mathematical descriptions with more than one solution, but, in general, there is at maximum, one solution with physical sense. When the problems are numerically simulated, additional work is required to choose the solution with physical sense, avoiding those without physical sense. Assuming that the problem admits at maximum one solution with physical meaning, costs and effort can be minimized if a mathematical modeling without inequalities is chosen, provided the solution exists and is unique.

Physically realistic problems generally present temperature-dependent thermal conductivity (sometimes strongly dependent), leading to nonlinear heat transfer problems, and a moderate temperature variation may lead to thermal conductivity with different orders of magnitude; which is particularly important when the thermal conductivity decreases with a temperature increase (Glassbrenner and Slack, 1964; Gusev et al, 2002; Osman and Srivastava, 2001; Mehri (2015).

An a priori temperature estimate for thermal conductivity with piecewise constant temperature dependency that allows establishing a value greater than (or equal to) the temperature in any point of the body was proposed by Saldanha da Gama et al. (2013), while in the present work there is a linear relationship between the thermal conductivity and the temperature.

2. A LINEAR RELATIONSHIP BETWEEN THERMAL CONDUCTIVITY AND TEMPERATURE

Considering the thermal conductivity as a decreasing linear function of the temperature, such as

$$k = k_1 + \left(k_2 - k_1\right)\left(\frac{T - T_1}{T_2 - T_1}\right) \quad T_1 \le T \le T_2, \quad k_1 \ge k \ge k_2 \tag{1}$$

in which k_1, k_2, T_1 and T_2 are always positive constants, since all temperatures are assumed absolute throughout this work, and in which it is expected temperatures within the interval (T_1, T_2).

The objective of this work may be illustrated by an infinite solid represented in Figure 1, bounded by the planes $x=-2L$ and $x=2L$ (Carslaw and Jaeger, 1959) with a constant internal heat generation per unit time and unit volume $\dot{q}$ for $-L<x<L$ and with no heat generation for $-2L<x<-L$ and $L<x<2L$, assuming a thermal conductivity given by equation (1).

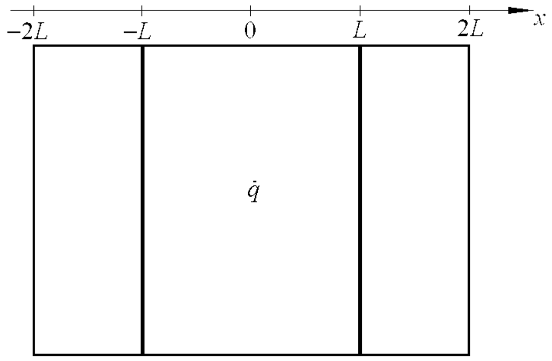

Figure 1. Problem Scheme.

Taking into account the symmetry, assuming steady-state conduction heat transfer, in the one-dimensional geometry presented in Fig. 1, the phenomenon is mathematically described by

$$\begin{aligned}&\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0 \quad L<x<2L\\&k\frac{dT}{dx}=-\dot{q}L \quad \text{at } x=L\end{aligned} \tag{2}$$

This problem requires a boundary condition at $x=2L$: the (known) temperature at $x=2L$ will be denoted by T_{2L}. It is important to note that the second equation of Eq. (2) comes from the symmetry of the problem.

Integrating equation (2), it comes that

$$k\frac{dT}{dx}=-\dot{q}L=\text{constant}, \quad L<x<2L \tag{3}$$

in which $T_{\bar{x}}$ denotes the temperature at any $x=\bar{x}$. Thus.

$$\int_{T_{2L}}^{T_L} k\,dT=-\dot{q}L\left(L-2L\right)\Rightarrow \dot{q}L^2=k_1\left(T_L-T_{2L}\right)+\frac{1}{2}\left(\frac{k_2-k_1}{T_2-T_1}\right)\left(\left(T_L-T_1\right)^2-\left(T_{2L}-T_1\right)^2\right) \tag{4}$$

and the following relation can be obtained

$$T_L=T_1+\alpha k_1\pm\sqrt{k_1^2-2\alpha\dot{q}L^2-2\alpha k_1\left(T_{2L}-T_1\right)+\left(T_{2L}-T_1\right)^2}, \quad \alpha=\left(\frac{T_2-T_1}{k_1-k_2}\right) \tag{5}$$

Therefore, one of the four situations below is possible:

$$
\begin{aligned}
&(\mathrm{a}) \rightarrow k_1^2-2\alpha\dot{q}L^2-2\alpha k_1\left(T_{2L}-T_1\right)+\left(T_{2L}-T_1\right)^2<0 \\
&(\mathrm{b}) \rightarrow k_1^2-2\alpha\dot{q}L^2-2\alpha k_1\left(T_{2L}-T_1\right)+\left(T_{2L}-T_1\right)^2=0 \Rightarrow T_L=T_1+\alpha k_1 \\
&(\mathrm{c}) \rightarrow k_1^2-2\alpha\dot{q}L^2-2\alpha k_1\left(T_{2L}-T_1\right)+\left(T_{2L}-T_1\right)^2>0 \Rightarrow \\
&\begin{cases}
(\mathrm{c1}) \rightarrow T_L=T_1+\alpha k_1+\sqrt{k_1^2-2\alpha\dot{q}L^2-2\alpha k_1\left(T_{2L}-T_1\right)+\left(T_{2L}-T_1\right)^2}>T_1+\alpha k_1 \\
(\mathrm{c2}) \rightarrow T_L=T_1+\alpha k_1-\sqrt{k_1^2-2\alpha\dot{q}L^2-2\alpha k_1\left(T_{2L}-T_1\right)+\left(T_{2L}-T_1\right)^2}<T_1+\alpha k_1
\end{cases}
\end{aligned}
\tag{6}
$$

Situation (a) indicates that the problem has no solution; situation (b) indicates that the conductivity at $x = L$ is zero so that the boundary condition cannot be satisfied.

If situation (c1) holds, then the thermal conductivity would be negative. So, there is only one physically admissible situation, which is (c2). In this simple case the analysis can be easily performed, but in a numerical simulation of a multidimensional problem, it would not be simple to ensure that the desired solution (if it exists) would be reached.

3. PROPOSITION OF AN EQUIVALENT EQUATION THAT ELIMINATES THE SHORTCOMING

In order to preserve the physical meaning of the solution, an alternative form for the equation describing the thermal conductivity as a decreasing linear function of the temperature (Eq. (1)) is now proposed. This equation is physically equivalent to the original one within the original range and physically admissible for any temperature T, being given by

$$
k=\frac{k_1+k_2}{2}+\frac{k_2-k_1}{2}\left(\left|\frac{T-T_1}{T_2-T_1}\right|-\left|\frac{T-T_2}{T_2-T_1}\right|\right) \tag{7}
$$

It is important to be notice that Eq. (7) holds even if $k_1 \le k \le k_2$. The next step is to show the physical and mathematical equivalence between Eq. (7) – the equivalent equation – and Eq. (1) – the original equation.

4. PHYSICAL EQUIVALENCE

First, the physical equivalence can be ensured by equalizing both definitions of the thermal conductivity *k*:

$$
k_1+\left(k_2-k_1\right)\left(\frac{T-T_1}{T_2-T_1}\right)=\frac{k_1+k_2}{2}+\frac{k_2-k_1}{2}\left(\left|\frac{T-T_1}{T_2-T_1}\right|-\left|\frac{T-T_2}{T_2-T_1}\right|\right), \quad \text{for} \quad T_1 \le T \le T_2 \tag{8}
$$

Since k_1 and k_2 are positive and the following inequality holds

$$
-1 \le \left|\frac{T-T_1}{T_2-T_1}\right|-\left|\frac{T-T_2}{T_2-T_1}\right| \le 1, \quad \mathit{for} \quad -\infty < T < \infty \tag{9}
$$

the left hand side of (9) is, for any temperature T, a positive value between k_1 and k_2. This ensures the physical meaning for any T, even for $T < T_1$ and for $T > T_2$.

5. MATHEMATICAL EQUIVALENCE

At this point the Kirchhoff transformation in terms of the variable ω is introduced (Arpaci, 1966) as

$$
\omega=\int_0^T \hat{k}(\theta)\,d\theta, \quad k=\hat{k}(T) \tag{10}
$$

so that

$$\frac{d\omega}{dT} = k \tag{11}$$

It is important to note that if there exists a positive lower bound for *k*, the inverse of the Kirchhoff transformation always exits. From equation (7) it comes that $k \geq \min(k_1, k_2) > 0$ so, the inverse is ensured. When the alternative expression for the thermal conductivity proposed in Eq. (7) is employed, the Kirchhoff transformation is given by

$$\omega = \hat{\omega}(T) = \int_0^T \left(\frac{k_1 + k_2}{2} + \frac{k_2 - k_1}{2} \left(\left| \frac{\theta - T_1}{T_2 - T_1} \right| - \left| \frac{\theta - T_2}{T_2 - T_1} \right| \right) \right) d\theta \tag{12}$$

Eq. (12) is integrated and, after some calculations, the Kirchhoff transformation, valid for any temperature, $-\infty < T < \infty$, is given by

$$\omega = \hat{\omega}(T) = \left(\frac{k_2 + k_1}{2} \right) T + \left(\frac{k_2 - k_1}{4} \right) \left\{ \left(\frac{T - T_1}{T_2 - T_1} \right) |T - T_1| - \left(\frac{T - T_2}{T_2 - T_1} \right) |T - T_2| - T_2 - T_1 \right\} \tag{13}$$

At this point the inverse of the Kirchhoff transformation, $T = \hat{T}(\omega)$, for any $-\infty < \omega < \infty$, which always exists and is unique, may be computed, being given by

$$\begin{aligned} T = {} & \frac{(\omega_2 - \omega_1)}{2k_2} \left(\left(\frac{\omega - \omega_1}{\omega_2 - \omega_1} - 1 \right) + \left| \frac{\omega - \omega_1}{\omega_2 - \omega_1} - 1 \right| \right) + \frac{(\omega_2 - \omega_1)}{2k_1} \left(\left(\frac{\omega - \omega_1}{\omega_2 - \omega_1} \right) - \left| \frac{\omega - \omega_1}{\omega_2 - \omega_1} \right| \right) \\ & + \frac{k_2 T_1 - k_1 T_2}{k_2 - k_1} + \left(\frac{T_2 - T_1}{k_2 - k_1} \right) \sqrt{\frac{k_1^2 + k_2^2}{2} + \left(\frac{k_2^2 - k_1^2}{2} \right) \left(\left| \frac{\omega - \omega_1}{\omega_2 - \omega_1} \right| - \left| \frac{\omega - \omega_1}{\omega_2 - \omega_1} - 1 \right| \right)} \end{aligned} \tag{14}$$

in Eq. (14) the constants ω_1 and ω_2 are given by

$$\omega_1 = k_1 T_1 \quad \text{and} \quad \omega_2 = \omega_1 + \frac{1}{2}(k_2 + k_1)(T_2 - T_1) \tag{15}$$

Now, using Eq. (13), the first derivative of ω with respect to T, may be calculated

$$\begin{aligned} \frac{d\omega}{dT} = {} & \frac{d}{dT} \left\{ \left(\frac{k_2 + k_1}{2} \right) T + \left(\frac{k_2 - k_1}{4} \right) \left(\frac{T - T_1}{T_2 - T_1} |T - T_1| - \frac{T - T_2}{T_2 - T_1} |T - T_2| - T_2 - T_1 \right) \right\} = \\ = {} & \left(\frac{k_2 + k_1}{2} \right) + \left(\frac{k_2 - k_1}{4} \right) \left(\left(\frac{|T - T_1|}{T_2 - T_1} \right) + \left(\frac{T - T_1}{T_2 - T_1} \right) \frac{T - T_1}{|T - T_1|} \right) - \\ & - \left(\frac{k_2 - k_1}{4} \right) \left(\left(\frac{|T - T_2|}{T_2 - T_1} \right) - \left(\frac{T - T_2}{T_2 - T_1} \right) \frac{T - T_2}{|T - T_2|} \right) = \\ = {} & \left(\frac{k_2 + k_1}{2} \right) + \left(\frac{k_2 - k_1}{2} \right) \left(\left| \frac{T - T_1}{T_2 - T_1} \right| - \left| \frac{T - T_2}{T_2 - T_1} \right| \right) \end{aligned} \tag{16}$$

It is important to note that the last expression obtained in Eq. (16) corresponds exactly to Eq. (7), the alternative form proposed in this work for describing the thermal conductivity as a decreasing linear function of the temperature. So, the first derivative of ω with respect to T (left hand side of Eq. (16)) is always positive, being given by

$$\frac{d\omega}{dT} = \begin{cases} k_1 & \text{if} \quad -\infty < T \le T_1 \\ k_1 + \left(\dfrac{k_2 - k_1}{T_2 - T_1}\right)\left(T - T_1\right) & \text{if} \quad T_1 < T < T_2 \\ k_2 & \text{if} \quad T_2 \le T < \infty \end{cases} \quad \Rightarrow \quad \frac{d\omega}{dT} \ge \min\left(k_1, k_2\right) > 0 \tag{17}$$

6. FINAL REMARKS

As an illustration, a graph of thermal conductivity versus temperature is presented for two distinct situations in Figure 2 for the thermal conductivity: a piecewise constant temperature-dependent (left-hand side) and a linear dependency on temperature (right-hand side) – the former is based on an a priori upper bound estimate for the steady-state temperature distribution in bodies performed in a previous work (Saldanha da Gama et al., 2013), while the latter represents the problem discussed in this work.

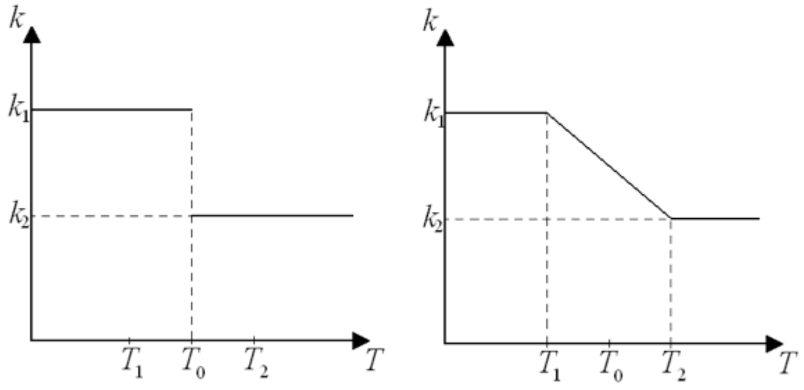

Figure 2. Thermal conductivity versus temperature: (a) Saldanha da Gama et al. (2013); (b) Present work.

When both graphs are compared, the present work thermal conductivity behavior may be considered as an improvement of the one considered by Saldanha da Gama et al, (2013), since it is more realistic to assume a linear relationship between the thermal conductivity and the temperature in the range with available experimental data and constant values outside this range.

7. ACKNOWLEDGEMENTS

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