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STUDY AND COMPARISON OF REDUCTION METHODS OF MECHANICAL SYSTEMS

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Abstract. *Computational numerical analysis of engineering problems, decrease the computational cost is of paramount importance. Dynamic systems are, for the most part, discretized in finite element models that generate a high number of degrees of freedom. In this context, the use of templates reduction techniques is a useful tool in decreasing order and system consequently more computational time and high processing capacity without compromising the structural integrity. This work presents a numerical analysis of a structure beam type of Euler-Bernoulli discretized by finite procedures and submitted to the inclusion of springs with linear characteristics and point loads, based in reduced models using some reduction strategies, such as Static reduction, improved Iterative reduced system (IIRS), System equivalent reduction expansion process (SEREP) and Modal Base (BM). Analyses are performed in the fields of frequency and time by numerical integration method of Newmark.*

Keywords: *mechanical systems, finite element, static reduction, iterative improved reduced system, system equivalent reduction expansion process, modal base, Newmark method.*

1. INTRODUCTION

Numerical analysis of mechanical structures discretized by finite procedures have been highlighted over the years because of the efficiency in analytical approaches as solutions in closed form, they are impractical and flexibility and adaptation. However, of industrial interest structures with higher quality and low cost production requires appropriate mathematical modeling. Models of higher reliability are endowed with large number of degrees of freedom, a fact that leads to a large computational cost, encouraging thus the use of reduced models (Guedri, 2006; Rade, 2011).

For greater time savings in solving structural mechanical problems, it is increasing the development and improvement of model reduction methods. The first reduction technique and probably most popular, directly applied in structural mechanic was proposed by Guyan (1965), known as static reduction or Guyan method. In this method, the inertia terms contributing to the model’s dynamics and damping are not considered. That method is exact only for static problems. In the latter years, to take into account the dynamic effects model, have been developed several variations this method besides the implementation of more efficient methods.

O’Callahan (1989a) suggested the improve method known as the Improved Reduced System (IRS) method where an additional term is added to the static reduction transformation to take some effect of the inertia terms, but it depends on the Guyan reduced model. Thus, O’Callahan *et al.* (1989b) proposed another reduction approach based on modal matrix dependent of arbitrary selection of mode of interested known as the system equivalent reduction and expansion process (SEREP) that is extensively used in structural mechanics. Some iterative schemes have been developed Friswell *et al.* (1995), proposed in the iterative IRS approach, known as IIRS, but the convergence speed of this method is time consuming. Recently, Xia and Lin (2004) suggested an improvement on this dynamic condensation technique to modify the iterative transformation matrix and achieve faster convergence.

In literature, another method commonly used by researchers in different applications is the modal superposition and enrichment method consisting of the truncation of the matrix of eigenvectors excluding modes that contribute little to the dynamic system. Among many works are the Guedri *et al.* (2006).

In this paper, will be presented a comparative study of the behavior and efficiency of some models of reduction methods for analysis in the frequency domain and time of dynamics structures modeled by finite elements of type beam subjected procedures inserting springs and the point loads.

2. REDUCTION OF STANDARD MODELS

The differential equation of second order, which describes the dynamics of a structure discretized by finite element procedures, is of the form:

$$M\ddot{x}(t)+D\dot{x}(t)+Kx(t)=f(t) \tag{1}$$

with M, D, K, order $N\times N$, are the matrices mass, damping and stiffness, respectively; $f(t)$ and x, order $N\times 1$, are the respectively load vector and the unknown state vector with N degree of freedom (DoFs) that, in many cases, leads to large dimension system matrices.

The standard concept of model reduction is to find a low dimension subspace T, order $N\times R$, with R is less than N in order to approximate the state vector x, i.e.:

$$x\approx Tx_R \tag{2}$$

By substituting Eq. (2) in Eq. (1) and then projecting Eq. (1) on the subspace T, a new differential equation of second order and lower dimension is obtained:

$$M_R\ddot{x}_R(t)+D_R\dot{x}_R(t)+K_R x_R(t)=F_R(t) \tag{3}$$

where $M_R=T^T MT$, $D_R=T^T DT$, $K_R=T^T KT$, order $R\times R$, are the reduced matrices mass, damping and stiffness, respectively; $F_R=T^T f$ and $x_R=T^T x$, order $R\times 1$, are the load vector reduced and the unknown state vector reduced.

In any reduction method, the DoF set that will provide the reduced model is of great importance. Therefore, the theory of master (p) and slave (s) DoFs, is applied, Koutsovasilis *et al.* (2007).

Based on the choice of T, various methods and techniques have been developed, some created from the static condensation, sub structuring and modal truncation. Here, emphasis is given to the following methods: static condensation, IIRS, SEREP and BM.

2.1 STATIC CONDENSATION

The static condensation is one of the almost popular known reduction methods. According Guyan (1965), considering the undamped system, the analytical expression of which is given below:

$$\begin{pmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss}\end{pmatrix}\begin{pmatrix}\ddot{x}_m \\ \ddot{x}_s\end{pmatrix}+\begin{pmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss}\end{pmatrix}\begin{pmatrix} x_m \\ x_s\end{pmatrix}=\begin{pmatrix} F_m \\ F_s\end{pmatrix} \tag{4}$$

It is assumed that there is no force applied on the slave DoFs, and then Eq. (4) is solved in x_s. Here, the inertia terms is omitted by to obtain the transformation matrix for the static reduction:

$$x_s=-K_{ss}^{-1}(M_{sm}\ddot{x}_m+M_{ss}\ddot{x}_s+K_{sm}x_m) \tag{5}$$

$$M_{sm}\ddot{x}_m+M_{ss}\ddot{x}_s=0 \tag{6}$$

$$\begin{pmatrix} x_m \\ x_s\end{pmatrix}=\begin{pmatrix} I \\ -K_{ss}^{-1}K_{sm}\end{pmatrix}x_m=T_{static}x_m \tag{7}$$

The T_{static} and I, order $R \times R$, are matrices that represent respectively the low-dimension subspace and the unitary matrix.

2.2 ITERATIVE IMPROVED REDUCTION SYSTEM (IIRS)

Based on the standard IRS method proposed by O'Callahan (1989a), Friswell *et al.* (1995) suggested the IIRS technique in which the iterative scheme is given by:

$$t^{(k)} = T_{static} + K_{ss}^{-1}(M_{ms}^{T} + M_{ss}t^{(k-1)})\left[M_{R}^{(k-1)}\right]^{-1} K_{R}^{(k-1)} \tag{8}$$

$$T^{(k)} = \begin{bmatrix} I_{mm} \\ t^{(k)} \end{bmatrix} \tag{9}$$

$$M_{R}^{(k)} = \left[T^{(k)}\right]^{T} M T^{(k)} \tag{10a}$$

$$K_{R}^{(k)} = \left[T^{(k)}\right]^{T} K T^{(k)} \tag{10b}$$

where the superscript k denotes the kth $(k \geq 2)$ iteration. When $k = 1$, $t^{(1)} = T_{static}$, which is right Guyan method; when $k = 2$, it is equivalent to the standard IRS method. He lowest m eigenvalues and the associated eigenvectors after kth iteration are estimated by solving the generalized eigen problem of the reduced system $\left(M_{R}^{(k)}, K_{R}^{(k)}\right)$:

$$K_{R}^{(k)}\Phi_{m}^{(k)} = M_{R}^{(k)}\Phi_{m}^{(k)}\Lambda_{m}^{(k)} \tag{11}$$

where Φ consists of the mass-normalized eigenvectors and Λ is a diagonal matrix containing corresponding eigenvalues, $\lambda_i (i = 1,2,...,m)$, on the diagonal.

The convergence of this iterative scheme was proved later by Friwell *et al.* (1998), but that some popular technique in the reduction model of mechanical structures. Unfortunately, the convergence process of this method is time consuming and the results depend on the selection masters of the degree of freedom.

By algebraic manipulations, Xia and Lin (2004) suggested an improvement on the IIRS by modifying the iterative formula of the transformation matrix in Eq. (8). This modification of the iterative scheme is given by:

$$t^{(k)} = T_{static} + \left[M_{ms}^{T} + M_{ss}t^{(k-1)}\right]\left[M_{d}^{(k-1)}\right]^{-1}\left[T_{static}^{T} K T_{static}\right] \tag{12a}$$

$$T^{(k)} = \begin{bmatrix} I_{mm} \\ t^{(k)} \end{bmatrix}, \tag{12b}$$

$$M_{d}^{(k-1)} = \left[M_{mm} + M_{ms}t^{(k-1)}\right] + T_{static}^{T}\left[M_{ms}^{T} + M_{ss}t^{(k-1)}\right]. \tag{13}$$

Here the Eq. (10) and Eq. (11) are still used to estimate the lowest m eigenvalues and the associated eigenvectors.

2.3 SYSTEM EQUIVALENT REDUCTION EXPANSION PROCESS (SEREP)

SEREP was introduced by O'Callahan *et al.* (1989b), and is an excellent of reduction model approach and has been used in various applications of structural mechanics.

The methodology basis of SEREP is the modal matrix structure discretized by finite element procedures, containing the eigenvectors organized in a specific column format. For this method, being Φ the $N \times R$ modal matrix, where the terms N and R denote the dimension of the model and the number of the computed eigenvectors, respectively, the relationship between the master degree of freedom can be written in general form as:

$$x_n = \begin{pmatrix} x_m \\ x_s \end{pmatrix} = \Phi q = \begin{pmatrix} \Phi_{m\times r} \\ \Phi_{m\times r} \end{pmatrix} q \tag{14}$$

Based in the pseudo-inverse definition for non-square, usually accomplished by Single Value Decomposition (SVD), the modal coordinates can be expressed in terms of the master DoF coordinates:

$$q = \Phi^{+}_{r\times m} x_m = (\Phi^{T}_{r\times m}\Phi_{m\times r})^{-1}\Phi^{T}_{r\times m} x_m \tag{15}$$

Now, substituting the Eq. (15) in Eq. (14), the T_{serep} coordinate transformation matrix is defined:

$$x_n = \begin{pmatrix} x_m \\ x_s \end{pmatrix} = \begin{pmatrix} \Phi_{m\times r}\Phi^{+}_{r\times m} \\ \Phi_{s\times r}\Phi^{+}_{r\times m} \end{pmatrix} x_m = T_{serep} x_m \tag{16}$$

The SEREP is a method that relies heavily on the choice of degrees of freedom to be masters and slaves. In this work, the system matrices were organized by permutations of lines and columns to choice of master's degrees and preferably to the singular values.

2.4 BASE MODAL (BM) AND BASE MODAL ENRICHED (BME)

The method of base modal, Ritz basis reduction, are often used for the calculation of dynamic responses with a great advantage in saving time. Considering the Eq. (1) without the damping, the solution is sought in the form of Eq. (2), where T is the base modal truncated to the order r. Here, the truncation will keep only modes of the structure in a particular frequency band. Thus, making the appropriate substitutions, one comes to the Eq. (3). Despite the computational gain, modal truncation is, in most cases, insufficient and can affect the quality of answers because the static contributions are not taken into account. Thus, recourse to the enrichment strategies to fill the missing information.

According which the suggested by Masson *et al.* (2006), a new base static residual vectors R is constructed to complete the initial transformation matrix such that,

$$R = (K)^{-1} f \tag{17}$$

being f the load vector, the improved transformation matrix is given by:

$$T = [T_0 \quad R] \tag{18}$$

where T_0 is an initial the transformation matrix.

3. NUMERICAL SIMULATIONS

In this section, the proposed analyze the efficiency and accuracy of the presented reduction models, considered a structure of Euler-Bernoulli beam, clamped at the ends and discretized in 10 elements in the direction of its length (L), in accordance with the Hamilton Principle, proposed in Zienkiewicz *et al.* (2005). The linear spring of the stiffness equal to $100N/m^2$ was placed in the central node, as well as, the external force type $F(t) = F_0 \sin(\omega t)$, with the frequency of excitation coincides with the first natural frequency of the beam and being excited by load of $1N$, scheme shown in Fig. 1.

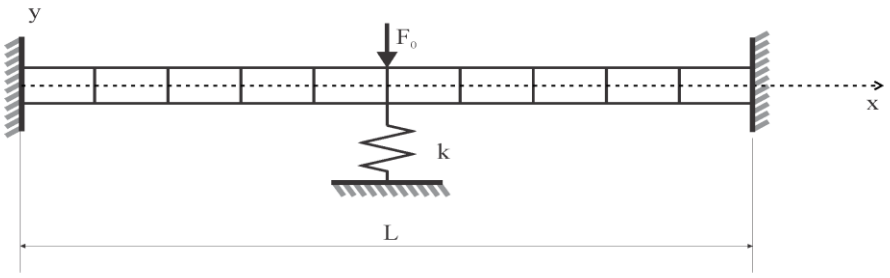

Figure 1. Scheme of the beam modeled.

The beam's physical and geometrical properties are shown in Tab. 1.

Table 1. Beam's physical and geometrical properties.

	ALUMINUM BEAM	
	Length [m]	0.5
Geometrical properties	Width [m]	0.010
	Thickness [m]	0.003
Physical properties	Volume density [Kg/m^3]	2710
	Modulus of elasticity [MPa]	7e10

The Figure 2 (a), (b), (c) and (d) illustrate the amplitudes of function frequency response (FFR) computed for the four bases reduction considering only nine Degree of Freedom, compared with the amplitudes in FFR of the full system that considers all modes, considering a frequency band 0-2000 Hz.

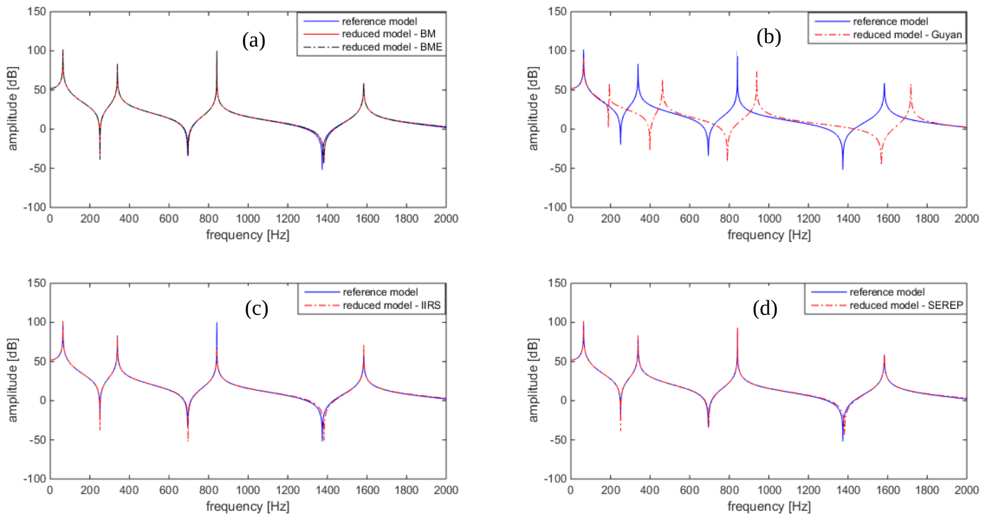

Figure 2. FFRs the full system and reduced utilizing BM and BME (a), Guyan (b), IIRS (c) and SEREP (d).

In the Fig. 2(a) was used two condensation's basis, namely: $T_0 = [\Phi_0]$ with 9 eigenvectors and $T_1 = [\Phi_0 \quad R]$ with 9 eigenvectors and residual vector calculated according Eq. (17). As the addressed system is linear, the two bases showed similar results, the creation of the residual vector did not influence the responses. The reduced models studied in this article effectively represented the dynamic model of reference or full system, with detail to Guyan method that is effective at a lower frequency band, as illustrated in Fig. 2(b).

The Table. 2 presents the first nine frequency values between the full system, Guyan, IIRS, SEREP, BM and BME reduction techniques. Only for the lowest modes, natural frequencies obtained from the full model are approximate compared to the reduction of Guyan. In general, the IIRS, SEREP, BM and BME reduction techniques, provided results equivalent to the complete system for its eigenvalues considered in the analysis. The IIRS reduction technique presented frequencies exact values for the first three considered modes. For other modes, the frequency values were not totally accurate but very close with an error percentage tolerated.

Table 2. Frequency values for full system and BM, BME, Guyan, IIRS and SEREP reduction techniques.

Mode	Full system (rad/s)	BM (rad/s)	BME (rad/s)	Guyan (rad/s)	IIRS (rad/s)	SEREP (rad/s)
1	63.93	63.93	63.93	64.66	63.93	63.93
2	172.86	172.86	172.86	194.36	172.86	172.86
3	339.29	339.29	339.29	462.63	339.31	339.29
4	561.49	561.49	561.49	938.57	561.67	561.49
5	841.43	841.43	841.43	1717.59	841.66	841.43
6	1180.94	1180.94	1180.94	2893.74	1181.30	1180.94
7	1583.32	1583.32	1583.32	4725.04	1583.90	1583.32
8	2049.40	2049.40	2049.40	7442.93	2049.87	2049.40
9	2549.43	2549.43	2549.43	11343.61	2550.14	2549.43

In transient analysis is considered a structural damping and used Newmark's the numerical integration method, implemented in Matlab® environment in accordance with RAO (2008). In the simulations, we estimated a time interval of 0 at 0.5 seconds with time step equal to 1e-4, considering the system actually be at rest at the beginning of the process.

The Figure 3 (a), (b), (c) and (d) illustrated the transient analysis of the reduced models and full system. In this transient analysis, any reduction techniques provided a result equivalent to that of the complete model including the Guyan reduction technique that did not get a good representation in frequencies values.

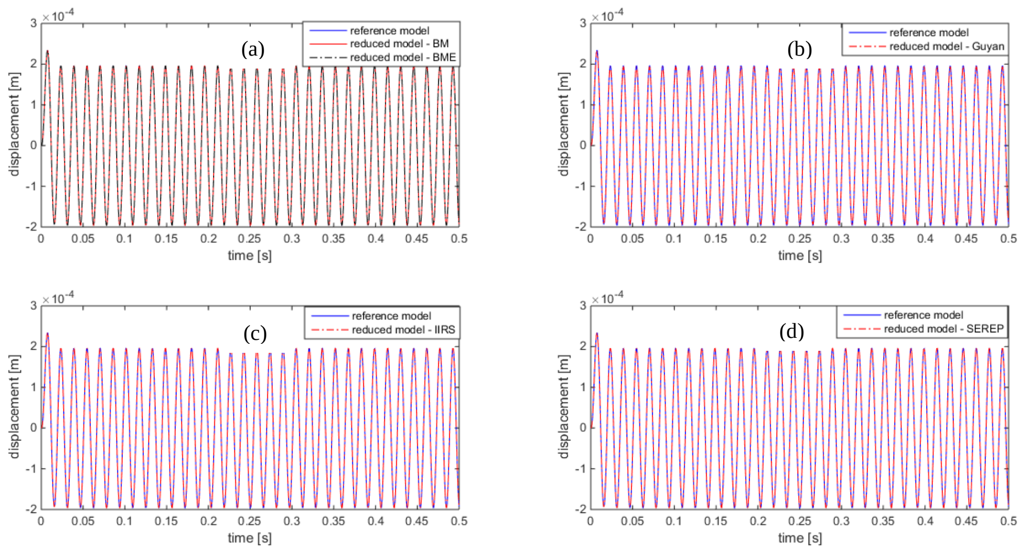

Figure 3. Transient Analysis the full and reduced system, utilizing BM and BME (a), Guyan (b), IIRS (c), and SEREP (d).

With the use of such reduction techniques, the dimensions of the matrices are smaller and consequently the number of degrees of freedom is reduced by simplifying the mathematical calculations, significantly reducing the computational cost throughout the analysis process.

The Figure. 4 slows the computational time during the process of analysis of the complete system, BM, BME, GUYAN, IIRS and SEREP. Different analysis reduction techniques, have provided considerable gain computational, with a time economy ranging from approximately 25% to 48%. In particular, for SEREP reduction technique, the need for a global organization of the system matrix increased the percentage of time during the analysis process, yet, computational cost was satisfactory.

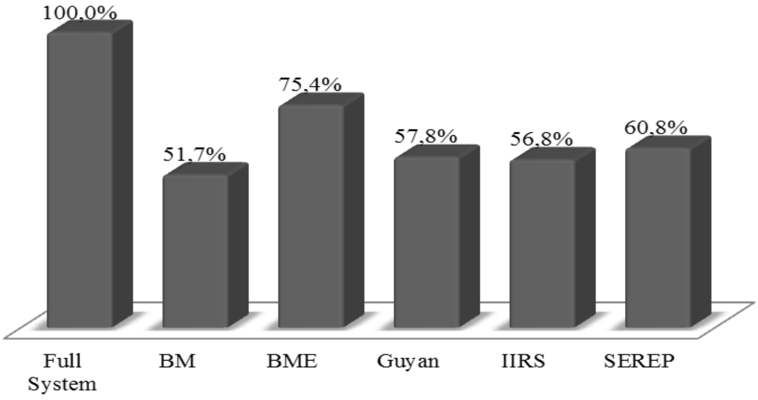

Figure 4. Computational time comparison between full system, BM, BME, Guyan, IIRS and SEREP reduction techniques.

4. CONCLUSION

In this work were described some reduction methods in the literature applied to dynamic systems chevron type discrete finite element and subjected to different boundary conditions as well as the inclusion of springs in some important points of the structure.

In the analysis in the domain of frequency and time, the most accurate methods have been reduced in the representation of the complete model with fewer degrees of freedom. The projection of the entire system in a subspace means the reduction in the size of the order of the matrices contributing to a substantial computational gain and increased storage capacity.

Despite having been instrumental in the development of other reduction techniques, the Guyan method compared with other methods discussed here, did not show good results since it was initially developed for static problems. In dynamic structures are neglected important factors such as moment of inertia, of the external forces and damping.

5. ACKNOWLEDGEMENTS

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