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DISCUSSION ON POSSIBLE INCONSISTENCIES WHEN CONSIDERING SHEAR DEFORMATION IN AXIALLY LOADED BEAMS

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Abstract. *In the classical Timoshenko's beam theory, shear deformation is usually considered as an additional cross-section rotation. Nonetheless, original texts of Timoshenko and Gere had already pointed out another form of considering shear deformation, although the great majority of literature articles neglect this fact. Although the alternative model does not lead to big differences in most results, in some specific situations some discrepancies may be observed. It is important to point out that these differences appear only if the beam is axially loaded, i.e., both approaches lead to the same results for transverse loading only. The objective of this paper is to discuss, in the context of a second-order geometric non-linear analysis, the differences in results considering these two alternatives for the modeling of shear deformation. These differences are shown in numerical examples of known models, such as beam-columns with various end conditions. Some classical expressions in Elasticity Theory can also be modified when using the alternative approach.*

Keywords: *Timoshenko's beam theory, axially loaded beams, shear deformation.*

1. INTRODUCTION

In Timoshenko's classical beam theory, shear deformation is usually considered as an additional cross-section rotation (Timoshenko, 1921). In this formulation, cross-section rotation and transversal displacement are considered as independent variables.

Nonetheless, original texts of Timoshenko and Gere (1961) had already pointed out an alternative manner of considering shear deformation, although the great majority of literature articles neglect this fact. This alternative formulation considers shear deformation not as an additional rotation but as a distortion in the infinitesimal beam element.

In a first order analysis bending moment is not influenced by axial loading, thus both formulations (classical and alternative) lead to the same results. However, as will be shown in this paper, the classical formulation presents some inconsistencies regarding axial strain and, moreover, leads to very different results when considering second-order effects, in terms of bending moments and critical loads.

The objective of this paper is to discuss, in the context of a second-order geometric non-linear analysis, the differences in results considering the two alternatives for shear deformation. For the case of an axially loaded beam, the two alternatives lead to different values for critical loads. This is shown in numerical examples of known models, such as beam-columns with various end conditions.

2. DIFFERENTIAL EQUATIONS

Figure 1 shows the deformed configuration of an infinitesimal beam element subjected to transversal (q) and axial (P) loads. Considering all forces involved, equilibrium imposition leads to the following equations:

$$-dV + q(x)dx = 0 \;\rightarrow\; \frac{dV(x)}{dx} = q(x) \tag{1}$$

$$dM - \left(V + dV\right)dx - Pdv + q(x)\frac{dx^2}{2} = 0 \rightarrow \frac{dM(x)}{dx} = V(x) + P\frac{dv(x)}{dx} \tag{2}$$

Considering the relation between bending moment and beam curvature, after derivation of Eq. (2) and substitution of the expression for the transversal loading's derivative in Eq. (1), the following can be written:

$$EI \cdot \frac{d^3\theta(x)}{dx^3} - P \cdot \frac{d^2v(x)}{dx^2} = q(x) \tag{3}$$

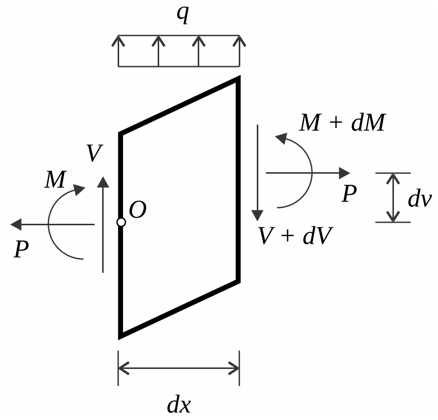

Figure 1. Infinitesimal beam element in equilibrium

2.1 Euler-Bernoulli beam theory

If Navier (Euler-Bernoulli) theory is employed, the rotation can be written as the derivative of the lateral displacement, leading, to the following:

$$\frac{d^4v(x)}{dx^4} - \frac{P}{EI} \cdot \frac{d^2v(x)}{dx^2} = \frac{q(x)}{EI} \tag{4}$$

The differential equation obtained represents the Euler-Bernoulli formulation considering constant axial load, but neglecting shear deformation. Its solution results in the lateral displacement $v(x)$ and the rotation can be obtained using $\theta = dv/dx$.

2.2 Classical Timoshekno theory

When using Timoshenko beam theory (Timoshenko, 1921), the rotation is no longer written as the derivative of the lateral displacement, as seen in Fig. 2.

This new assumption modifies the previous differential equations, since there is a need to include the shear strain in the definition of the shear force (notice the negative sign that comes from the difference between shear force and shear stress conventions):

$$\gamma(x) = \frac{dv(x)}{dx} - \theta(x) \rightarrow Q(x) = -\chi GA \cdot \gamma(x) = \chi GA \cdot \left(\theta(x) - \frac{dv(x)}{dx} \right) \tag{5}$$

The classical Timoshenko beam theory considers shear deformation as an additional cross-section rotatoin (Reddy et al., 1997), as shown in Fig. 3. Notice that the projections are obtained using the total cross-section ($\gamma+\theta$) and not only the bending part (θ). This consideration leads to the following cross-sectional forces:

$$N(x) = P - V(x)(\theta + \gamma) = P - V(x)\frac{dv(x)}{dx} \tag{6}$$

$$Q(x) = P(\theta + \gamma) + V(x) = P\frac{dv(x)}{dx} + V(x) \tag{7}$$

According to the Theory of Elasticity, axial displacement (and, as a consequence, axial strain) depend only on the bending rotation (Boresi et al., 2010):

$$u(x,y) = u_0(x) - y\theta(x) \rightarrow \varepsilon_x = \frac{du_0(x)}{dx} - y\frac{d\theta(x)}{dx} \tag{8}$$

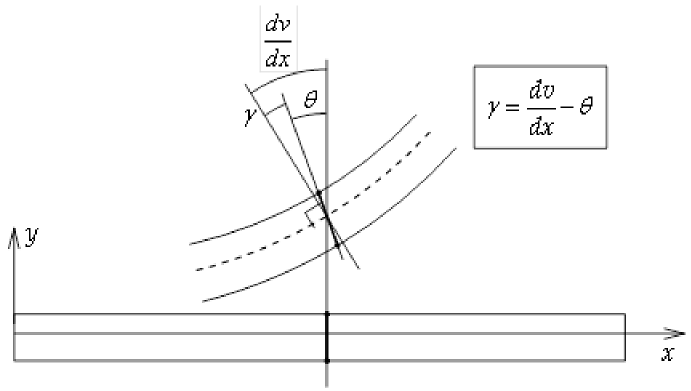

Figure 2. Cross-section rotation and shear strain in classical Timoshenko beam theory

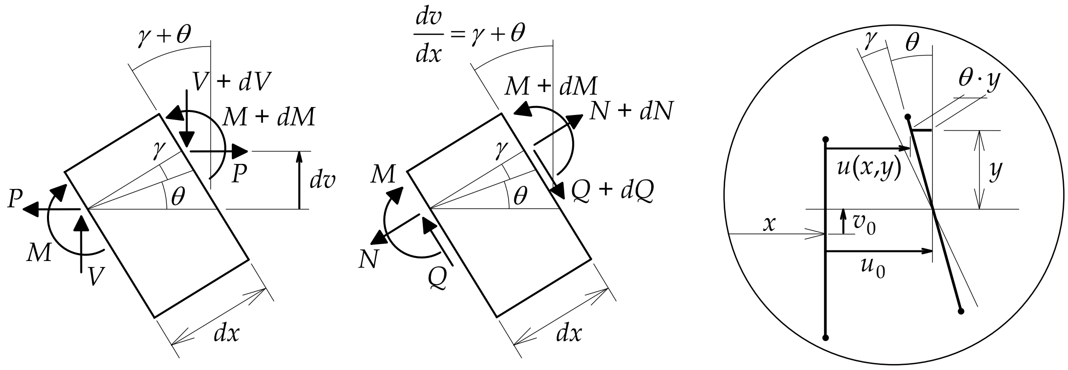

Figure 3. Forces and displacements in classical Timoshenko theory

However, a look at Fig. 3 shows that axial displacement depends also on the shear deformation:

$$u(x,y) = u_0(x) - y\left[\theta(x) + \gamma(x)\right] \rightarrow \varepsilon_x = \frac{du_0(x)}{dx} - y\left[\frac{d\theta(x)}{dx} + \frac{d\gamma(x)}{dx}\right] \tag{9}$$

The difference between Eqs. (8) and (9) reveals a possible inconsistency in the classical Timoshenko formulation. However, if the shear deformation is constant along the beam axis this difference vanishes.

After substitution using the shear force $Q(x)$ and its differential relation with the bending moment $M(x)$, the derivative of the lateral displacement can be obtained:

$$\frac{dM(x)}{dx} = \chi GA \cdot \left(\theta(x) - \frac{dv(x)}{dx}\right) \tag{10}$$

$$\frac{dv(x)}{dx} = \theta(x) - \frac{EI}{\chi GA}\frac{d^2\theta(x)}{dx^2} \tag{11}$$

Substituting in Eq. (3) the expression for the displacement derivative in Eq. (11), the following DE is obtained:

$$EI\left(1 + \frac{P}{\chi GA}\right)\frac{d^3\theta(x)}{dx^3} - P\frac{d\theta(x)}{dx} = q(x) \tag{12}$$

2.3 Alternative Timoshenko theory

Timoshenko and Gere (1961) also present an alternative formulation for taking shear deformation into account, although it's not usually adopted. This approach considers shear deformation not as an additional rotation but as a distortion in the beam infinitesimal element, as shown in Fig. 4. The projection of forces is then performed using only the bending rotation:

$$N(x) = P - V(x)\theta(x) \tag{13}$$

$$Q(x) = P\theta(x) + V(x) \tag{14}$$

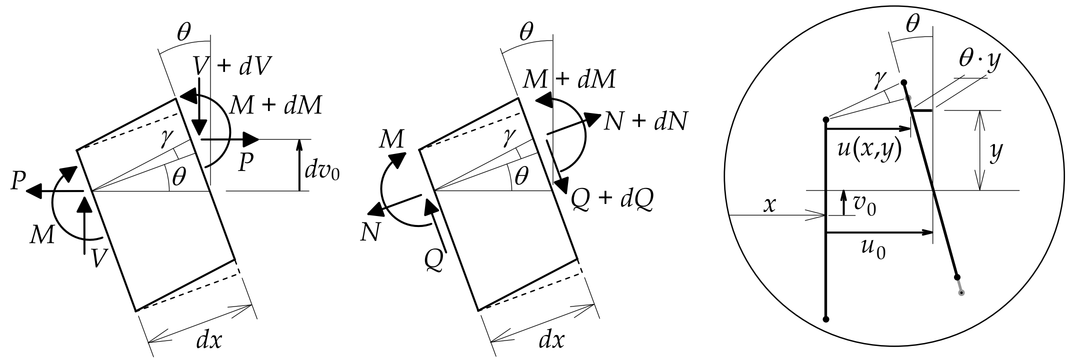

Figure 4. Forces and displacements in alternative Timoshenko theory

A look at Fig. 4 shows that this formulation is in accordance with the Theory of Elasticity in terms of axial displacement. Proceeding in a similar form as in Eqs. (10) and (11), the derivative of the lateral displacement can be obtained:

$$\frac{dM(x)}{dx} = \left(\chi GA - P\right)\left(\theta(x) - \frac{dv(x)}{dx}\right) \tag{15}$$

$$\frac{dv(x)}{dx} = \theta(x) - \frac{EI}{\left(\chi GA - P\right)}\frac{d^2\theta(x)}{dx^2} \tag{16}$$

Looking at Eqs. (10) and (15), it’s evident that differences appear only when the beam is subjected to axial loading. Substituting in Eq. (3) the expression for the displacement derivative in Eq. (16), the following DE is obtained:

$$\frac{EI}{\left(1 - \dfrac{P}{\chi GA}\right)}\frac{d^3\theta(x)}{dx^3} - P\frac{d\theta(x)}{dx} = q(x) \tag{17}$$

2.4 DE Solutions

Both classical and alternative forms of Timoshenko’s beam differential equation present the same solution pattern. When the axial load is compressive, usually the trigonometric solution is adopted for its simplicity. In this case, the load has a negative sign that alters some parameters of the equation. Using the load parameters λ and μ, and the constant Ω for the consideration of shear stiffness (Reddy, 1997), the solution is given by:

$$\theta(x) = A\sin\left(\lambda x\right) + B\cos\left(\lambda x\right) + C \tag{18}$$

$$v(x) = \frac{\left(1 + \Omega\lambda^2 L^2\right)}{\lambda}\left[B\sin\left(\lambda x\right) - A\cos\left(\lambda x\right)\right] + Cx + D \tag{19}$$

where:

$\lambda = \mu\sqrt{1/\left(1 - \Omega\mu^2 L^2\right)}$ = axial load parameter in classical formulation

$\lambda = \mu\sqrt{\left(1 + \Omega\mu^2 L^2\right)}$ = axial load parameter in alternative formulation

$\mu = \sqrt{P/EI}$ = axial load parameter neglecting shear deformation

$\Omega = EI/\chi GAL^2$ = bending to shear stiffness ratio.

3. EXAMPLES

In order to study examples with known analytical solution, critical loads for beam-columns where obtained for two different cases of boundary conditions: simply supported (classical Euler column) and fully clamped. Next, the P-Δ effect was obtained in a cantilevered beam.

3.1 Euler column

Figure 5 shows a simply supported beam-column subjected to a compressive axial load. Its lowest critical load neglecting shear deformation is given by Euler's classical result: $P_{cr} = \pi^2 EI/L^2$.

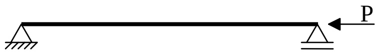

Figure 5. Euler column

After imposing boundary conditions (null displacements and bending moments at support coordinates), the non-trivial solution depends on the load parameter. The same equation describes this condition, regardless of the formulation used (classical or alternative):

$$\sin(\lambda L) = 0 \tag{20}$$

The smallest non-zero value of λ which satisfies Eq. (20) is $\lambda = \pi/L$. Substituting this value for λ in the expression for the classical Timoshenko theory leads to a critical load $P_{cr,c}$ given by:

$$P_{cr,c} = \frac{\pi^2 EI}{L^2\left(1+\Omega\pi^2\right)} \tag{21}$$

Substituting in the expression for the alternative Timoshenko theory leads to a critical load $P_{cr,a}$ given by:

$$P_{cr,a} = \frac{\chi GA}{2}\left(\sqrt{1+4\pi^2\Omega}-1\right) \tag{22}$$

The results shown in Eqs. (21) and (22) are identical to the ones obtained by Timoshenko and Gere (1961). Rewriting the expressions in terms of Euler's classical critical load P_e gives:

$$\frac{P_{cr,c}}{P_e} = \frac{1}{\left(1+\Omega\pi^2\right)} \tag{23}$$

$$\frac{P_{cr,a}}{P_e} = \frac{1}{2\pi^2\Omega}\left(\sqrt{1+4\pi^2\Omega}-1\right) \tag{24}$$

Figure 7 shows a comparison between critical loads given by both formulations for usual values of Ω.

3.2 Clamped column

Figure 6 shows a clamped column subjected to a compressive load. Its critical load is given by $P_{cr,0} = 4\pi^2 EI/L^2$.

Figure 6. Doubly clamped column

For this column, the classical ($P_{cr,c}$) and alternative ($P_{cr,a}$) critical loads in terms of $P_{cr,0}$ are given by:

$$\frac{P_{cr,c}}{P_{cr,0}} = \frac{1}{\left(1+4\Omega\pi^2\right)} \tag{25}$$

$$\frac{P_{cr,a}}{P_{cr,0}} = \frac{1}{8\pi^2\Omega}\left(\sqrt{1+16\pi^2\Omega}-1\right) \tag{26}$$

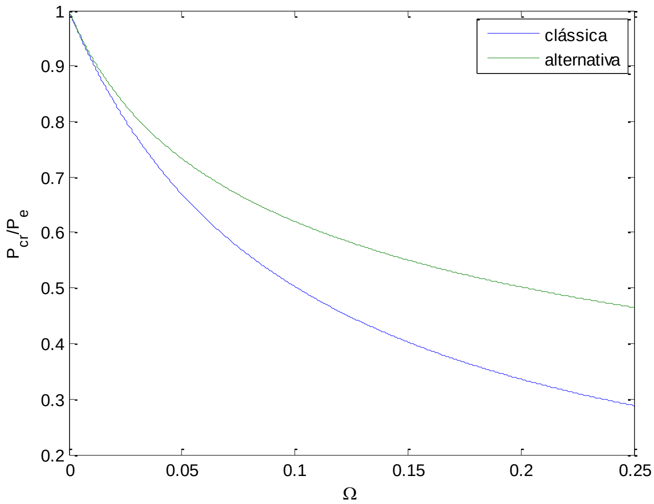

Figure 7. Euler column critical load using classical and alternative Timoshenko theories

Figure 8 shows a comparison between critical loads given by both formulations for usual values of Ω. Its evident that the consideration of shear deformation has a bigger influence in the clamped column than in the supported one.

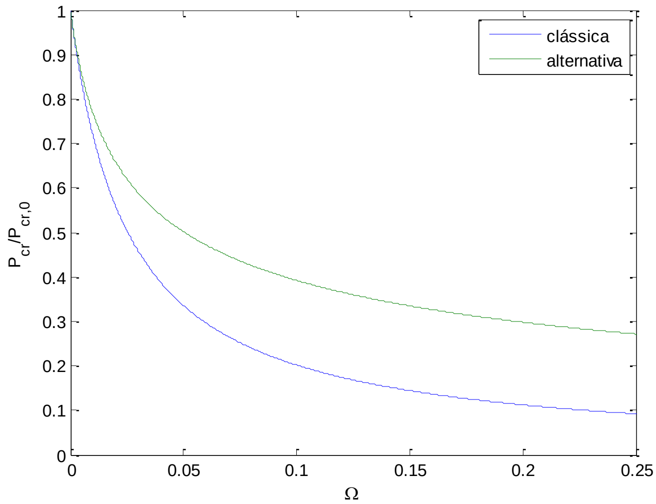

Figure 8. Clamped column critical load using classical and alternative Timoshenko theories

3.3 P-Δ Effect

A cantilever column subjected to an axial load *P* and a lateral load α*P* is shown in Fig. 9. It presents a Bending Amplification Factor (BAF) given by (neglecting shear deformation):

$$\text{BAF} = \frac{\tan(\mu L)}{\mu L} \tag{27}$$

Figure 9. Cantilever column subjected to axial and lateral loads

Using the classical and alternative Timoshenko formulations, the BAF is given, respectively, by:

$$\mathrm{BAF}_c = \frac{\lambda^2}{\mu^2}\left(1-\Omega\mu^2L^2\right)\frac{\tan(\lambda L)}{\lambda L} = \frac{\tan(\lambda L)}{\lambda L} \tag{28}$$

$$\mathrm{BAF}_a = \frac{\tan(\lambda L)}{\lambda L}\left(1+\Omega\mu^2L^2\right) \tag{29}$$

Figure 10 shows a comparison between BAFs given by both formulations for usual values of Ω using $P = P_{cr,0}/2$. Notice that when neglecting shear deformation (Ω = 0), the BAF is given by 1.8168.

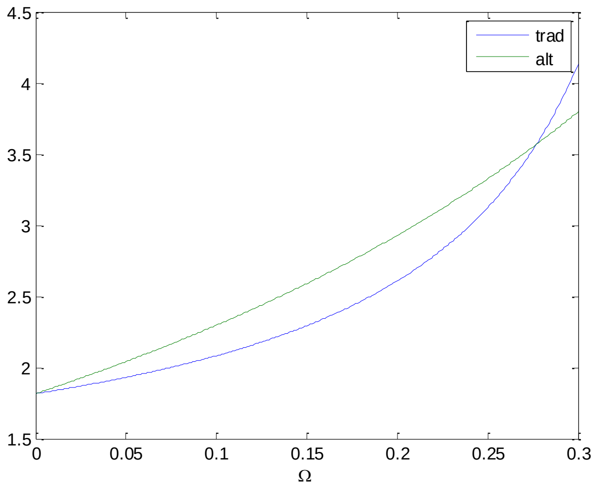

Figure 10. Bending Amplification Factor (BAF) using classical and alternative Timoshenko theories

4. CONCLUSIONS

Standards in structural design usually recommend the consideration of geometrically nonlinearity effects in structural analysis. Usually, these effects are taken into account using second order analysis. Moreover, modern structural analysis softwares take into account not only axial and bending strains but also shear strains, using usually a classical Timoshenko approach, even though for most cases the bars are considered slender, which allows neglecting this contribution.

Stability analysis in slender columns considering second order effects is a well-known problem, with analytical solutions available. On the other hand, stability analysis taking into account shear deformation has very few solutions available in literature (Onu, 2008; Areiza-Hurtado et al., 2005), since generally the contribution of shear deformation is negligible for most beam-columns. However, there are some situations when even for a small length to hight ratio the combination of second order effects with shear deformation become important, requiring a deeper investigation. This was the main goal of this paper.

In this context, two forms of considering shear deformation for axially loaded beam were obtained. The classical Timoshenko beam theory considers shear deformation as an additional cross-section rotation and a less used theory (here called alternative and also presented by Timoshenko) considers shear deformation as a distortion in the infinitesimal beam element. Besides, it has been shown that the classical Timoshenko theory presents an inconsistency in axial strains compatibility. This inconsistency does not appear in beams with constant shear force.

Results show that when a beam is subjected to a compressive axial load, the different interpretations for the shear distortion lead to very different values for critical loads when comparing borth formulations. The second order P-Δ effect was also studied for a cantilever column leading to very different results in both formulations. For the classical theory, bending moments at the base of the column increase significantly as shear influence becomes greater (short columns).

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