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ANALYSIS OF THIN PLATES USING DESLAURIERS-DUBUC SCALING FUNCTIONS

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Abstract. *The use of multiresolution techniques and wavelets has become increasingly popular in the development of numerical schemes for the solution of partial differential equations. In this work, the widely used Galerkin Method has been adapted for the direct solution of differential equations in a meshless formulation using Deslauriers-Dubuc scaling functions, also known as Interpolets (interpolating wavelets). This approach has an advantage of obtaining interpolation coefficients instead of nodal displacements, allowing the solution representation to be in a different resolution than the system of equations. Another possible advantage is the fact that the calculation of integrals of inner products of wavelet scaling functions and their derivatives can be made by solving an eigenvalue problem, thus avoiding the issue of approximating the integral by some numerical method. These inner products are known as connection coefficients and they are employed in the calculation of stiffness matrices and load vectors. Several examples based on the typical differential equation for thin plates were studied successfully.*

Keywords: *wavelets, Deslauriers-Dubuc, Galerkin Method, thin plates*

1. INTRODUCTION

The use of wavelet-based numerical schemes has become popular in the last three decades. Wavelet scaling functions have several properties that are quite useful for representing solutions of differential equations (DE's), such as orthogonality, compact support and exact representation of polynomials of a certain degree. Their capability of representing data at different levels of resolution allows the efficient and stable calculation of functions with high gradients or singularities, which would require a dense mesh or higher order elements in a Finite Element analysis (Qian and Weiss, 1992).

A complete basis of wavelets can be generated through dilation and translation of a mother scaling function. Although many applications use only the wavelet filter coefficients of the multiresolution analysis, there are some which explicitly require the values of the basis functions and their derivatives, such as the Wavelet Finite Element Method (WFEM) (Ma et al., 2003).

Compactly supported wavelets have a finite number of derivatives which can be highly oscillatory, which makes the numerical evaluation of integrals of their inner products difficult and unstable. Those integrals are called connection coefficients and they appear naturally when applying numerical integral methods for the solution of a DE. Due to some properties of wavelet functions, these coefficients can be obtained by solving an eigenvalue problem using filter coefficients.

The most commonly used wavelet family is the one developed by Ingrid Daubechies (1988). All the mathematical foundation for the wavelet theory was formulated for Daubechies wavelets and then extended to other families.

Working with dyadically refined grids, Deslauriers and Dubuc (1989) obtained a new family of wavelets with interpolating properties, later called Interpolets. Unlike Daubechies wavelets, Interpolets are symmetric, which is especially interesting in numerical analysis. The use of Interpolets instead of Daubechies wavelets considerably improves the method's accuracy (Burgos et al., 2015).

The use of wavelets as interpolating functions in numerical schemes such as the Galerkin Method holds some promise due to their multiresolution properties. Accuracy can be improved by increasing either the level of resolution or the order of the wavelet used. For most purposes and in accordance with other methods, the increase in the level of resolution seems to work better.

The method was applied for the static analysis of thin plates showing excellent performance in rectangular models. Results are presented and compared with analytical values, when available.

2. WAVELET SCALING FUNCTIONS

Multiresolution analysis using orthogonal, compactly supported wavelets has been successfully applied in numerical simulation. Wavelet basis are composed of two kinds of functions: scaling functions (φ) and wavelet functions (ψ). The two combined form a complete Hilbert space of square integrable functions. The spaces generated by scaling and wavelet functions are complementary and both are based on the same mother function (Burgos et al., 2013).

In the following expressions, known as the two-scale relation, a_k are the scaling function filter coefficients and N is the wavelet order.

$$\varphi(x) = \sum_{k=1-N}^{N-1} a_k \varphi(2x-k) = \sum_{k=1-N}^{N-1} a_k \varphi_k(2x) \tag{1}$$

In general, there are no analytical expressions for wavelet functions, which can be obtained using iterative procedures like Eq. (1). In order to comply with the requirements of orthogonality and compact support, wavelets present, in general, an irregular fractal-like shape. An example of this irregular shape is shown in Fig. 1, which presents Daubechies' wavelet of order N = 4.

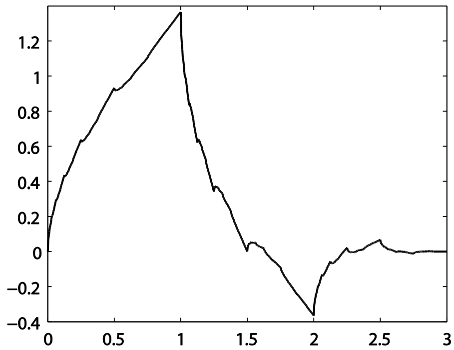

Figure 1. Daubechies scaling function of order N = 4

2.1 Wavelet and Scaling Functions Properties

The set of properties summarized in Eq. (2) is valid for Daubechies wavelets but can be adapted to other wavelet families, such as Deslauriers-Dubuc Interpolets. Some of these properties, like compact support and unit integral, are required for the use of the wavelet family in numerical methods. Others, like orthogonality, are desirable but not necessary.

$$\begin{aligned}
&\text{supp}(\varphi) = [0 \;\; N-1] \\
&\int_{-\infty}^{+\infty} \varphi(x)dx = 1 \\
&\int_{-\infty}^{+\infty} \varphi(x-i)\varphi(x-j)dx = \delta_{i,j} \\
&x^m = \sum_{-\infty}^{+\infty} c_k^m \varphi(x-k), \quad m \le N/2-1
\end{aligned} \tag{2}$$

The last expression in Eq. (2) derives from the vanishing moments property, which states that a set of shifted and scaled Daubechies wavelets of order N is capable of representing exactly an N/2−1 degree polynomial.

2.2 Wavelet derivatives

In the process of solving a DE using numerical methods, derivatives of the basis functions tend to appear. As there are no analytical expressions for wavelets, derivatives are obtained in dyadic grid points and the refinement of the solution depends on the level of resolution needed (Lin et al., 2005). The scale relation can be differentiated d times, generating the following expression:

$$\varphi^{(d)}(x) = 2^d \sum_{i=0}^{N-1} a_i \varphi^{(d)}(2x - i) \tag{3}$$

Applying Eq. (3) to integer points results in the following system of equations shown in matrix form. In Eq. (4), **A** represents the filter coefficients matrix, **I** is the identity matrix and $\mathbf{\Lambda^{(d)}}$ is the vector containing derivative values at integer points of the grid.

$$\begin{aligned} &(2^d \mathbf{A} - \mathbf{I})\mathbf{\Lambda^{(d)}} = 0 \\ &\mathbf{A} = [a_{2i-k}]_{0 \le i,k \le N-1} \end{aligned} \tag{4}$$

Equation (4) is an eigenvalue problem which, for unique solution, has to be normalized using the so-called moment equation, derived from the wavelet property of exact polynomial representation. This equation is given by Latto et al. (1992) and provides a relation between derivative values at integer points. The coefficient M_i^d is the moment of the d^{th} derivative of the scaling function translation i.

$$\begin{aligned} &d! = \sum_{i=0}^{N-1} M_i^d \varphi^{(d)}(x - i) \\ &M_i^j = \frac{1}{2^{j+1} - 2} \sum_{k=0}^{j} \binom{j}{k} i^{j-k} \sum_{l=0}^{k-1} \binom{k}{l} M_0^l \left(\sum_{i=0}^{N-1} a_i i^{k-l} \right) \end{aligned} \tag{5}$$

Once the derivative is obtained at integer values, the scale relation can be applied for any $x = k/2^j$.

$$\varphi^{(d)}\left(\frac{k}{2^j}\right) = 2^d \sum_{i=0}^{N-1} a_i \varphi^{(d)}\left(\frac{k}{2^{j-1}} - i\right), \quad k = 1, 3, 5, \ldots, N-2 \tag{6}$$

2.3 Connection Coefficients

Assuming that a function $f(x)$ is approximated by a series of interpolating scale functions, the following may be written:

$$f(x) = \sum_k \alpha_k \varphi(x - k) \tag{7}$$

The process of solving a differential equation (DE) requires the calculation of the inner products of the basis functions and their derivatives (d_1, d_2). These inner products are defined as connection coefficients and are given by:

$$\Gamma_{i,j}^{d_1,d_2} = \int \varphi^{(d_1)}(\xi - i)\varphi^{(d_2)}(\xi - j) d\xi \tag{8}$$

The values for the limits of the integral in Eq. (8) depend on which method is used to impose boundary conditions. In this work, the limits are given by [0 2^m], where m is the wavelet level of resolution. This method allows the use of Lagrange multipliers to deal with boundary conditions, similarly to what is usually done in a meshless scheme (Nguyen et al., 2008). Connection coefficients at level m can be obtained through the calculation at the lowest level (j = 0), thus avoiding its recalculation while increasing the level of resolution. Wavelet dilation and translation properties allow the calculation of connection coefficients within the interval [0; 1] to be summarized by the solution of an eigenvalue problem based only on filter coefficients (Zhou and Zhang, 1998).

$$\begin{aligned} &\left(\mathbf{P} - \frac{1}{2^{d_1+d_2-1}} \mathbf{I} \right) \mathbf{\Gamma^{d_1,d_2}} = 0 \\ &P_{i,j:k,l} = a_{k-2i} a_{l-2j} + a_{k-2i+1} a_{l-2j+1} \end{aligned} \tag{9}$$

Since Eq. (9) leads to an infinite number of solutions, there is the need for a normalization rule that provides a unique eigenvector. This unique solution comes with the inclusion of an adapted version of the moment equation mentioned before (Latto et al., 1992).

$$\sum_i \sum_j M_i^k M_j^k \, \Gamma_{i,j}^{d_1,d_2} = \frac{(k!)^2}{(k-d_1)!(k-d_2)!(2k-d_1-d_2+1)} \tag{10}$$

2.4 Deslauriers-Dubuc Interpolating Wavelets

The term interpolet was first used by Donoho (1992) to designate wavelets with interpolating characteristics. The basic characteristics of interpolating wavelets require that the mother scaling function satisfies the following condition (Shi et al., 1999):

$$\varphi(k) = \delta_{0,k} = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}, \quad k \in \mathbb{Z} \tag{11}$$

The filter coefficients c_k for Deslauriers-Dubuc scaling function of order N can be obtained by an autocorrelation of the same order Daubechies' filter coefficients (a_m).

$$\begin{aligned} \varphi(x) &= \sum_{k=1-N}^{N-1} c_k \varphi(2x-k) \\ c_k &= \sum_{m=0}^{N-1} a_m a_{m-k} \end{aligned} \tag{12}$$

Interpolets satisfy the same requirements as other wavelets, specially the two-scale relation, which is fundamental for their use as interpolating functions in numerical methods. Figure 2 shows the Interpolet IN4. Its symmetry and interpolating properties are evident. There is only one integer abscissa which evaluates to a non-zero value.

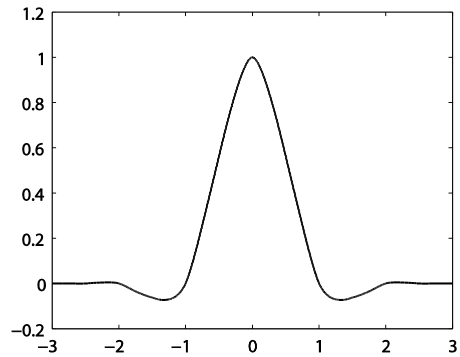

Figure 2. Interpolet IN4 scaling function

All expressions used for the calculation of derivatives, connection coefficients and moments of Daubechies wavelets can be applied to Interpolets. Of course, due to the correlation, the support [0; $N-1$] in the expressions for Daubechies becomes [$1-N$; $N-1$] for Interpolets.

3. WAVELET-GALERKIN METHOD: APPLICATION TO A THIN PLATE

The numerical solution of differential equations is one of the possible applications of the wavelet theory. The Wavelet-Galerkin Method (WGM) results from the use of wavelets scaling functions as interpolating basis in a traditional Galerkin scheme.

The bending of a thin plate with thickness t consisting of a material with Young's modulus E and Poisson's ratio ν is modeled by the following DE:

$$D\left(\frac{\partial^4 w}{\partial \xi^4} + 2\frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 w}{\partial \eta^4} \right) = q(\xi,\eta), \quad D = \frac{Et^3}{12\left(1-\nu^2\right)} \tag{13}$$

Displacement $w(\xi,\eta)$ is modeled using bi-dimensional wavelets, which are products between one-dimensional wavelets:

$$w(\xi,\eta) = \sum_i \sum_j d_{ij}\,\varphi(\xi - i)\,\varphi(\eta - j) \tag{14}$$

As done in a traditional Galerkin approach, Eq. (14) is substituted in the DE and the expression is integrated within the domain [0 ;1], leading to a system of equations which contains the wavelets' connection coefficients (Chen et al., 2004).

$$\begin{aligned} &\mathbf{kd} = \mathbf{f} \\ &\mathbf{k} = D\left[\mathbf{\Gamma^{00}} \otimes \mathbf{\Gamma^{22}} + \nu\left(\mathbf{\Gamma^{20}} \otimes \mathbf{\Gamma^{02}} + \mathbf{\Gamma^{02}} \otimes \mathbf{\Gamma^{20}}\right) + \mathbf{\Gamma^{22}} \otimes \mathbf{\Gamma^{00}} + 2\left(1-\nu^2\right)\mathbf{\Gamma^{11}} \otimes \mathbf{\Gamma^{11}}\right] \\ &\mathbf{f} = \left[\int_0^1 q(\eta)\,\mathbf{\Phi}^{\mathrm{T}} d\eta\right] \otimes \left[\int_0^1 q(\xi)\,\mathbf{\Phi}^{\mathrm{T}} d\xi\right] \end{aligned} \tag{15}$$

The symbol $\otimes$ indicates Kronecker product. The system is solved using the stiffness and load matrices provided by Eq. (15) and imposing essential boundary conditions with Lagrange multipliers:

$$\begin{bmatrix} \mathbf{k} & -\mathbf{G}^T \\ -\mathbf{G} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{0} \end{Bmatrix} \tag{16}$$

In Eq. (16), $\mathbf{G}$ is a matrix associated with boundary conditions and $\boldsymbol{\lambda}$ is a vector of Lagrange multipliers which may or not have physical meaning. The unknowns in vector $\mathbf{d}$ are the interpolating coefficients of the basis functions instead of nodal displacements. This feature allows the representation of the solution to be in a different resolution than the system of equations.

Since degrees of freedom don't have a fixed position, if a concentrated load is applied it must be transformed into wavelet space using the Dirac delta function. For example, for a concentrated load at the center of the plate, the load vector is given by:

$$\begin{aligned} &q(\xi) = P\delta\left(\xi - \frac{1}{2}\right), \quad q(\eta) = P\delta\left(\eta - \frac{1}{2}\right) \\ &\mathbf{f} = P\left[\int_0^1 \delta\left(\eta - \frac{1}{2}\right)\mathbf{\Phi}^{\mathrm{T}} d\eta\right] \otimes \left[\int_0^1 \delta\left(\xi - \frac{1}{2}\right)\mathbf{\Phi}^{\mathrm{T}} d\xi\right] = P\,\mathbf{\Phi}_{1/2}{}^{\mathrm{T}} \otimes \mathbf{\Phi}_{1/2}{}^{\mathrm{T}} \end{aligned} \tag{17}$$

In Eq. (27), the vector $\mathbf{\Phi}_{1/2}$ consists of translations of the scaling functions calculated at ξ = ½, i. e., $\Phi_{1/2,i} = \varphi(1/2 - i)$.

4. EXAMPLES

As a first example, a thin plate was modeled using the equations developed in previous sections. Figure 3 shows a square plate with all edges clamped subjected to a concentrated load applied at its center.

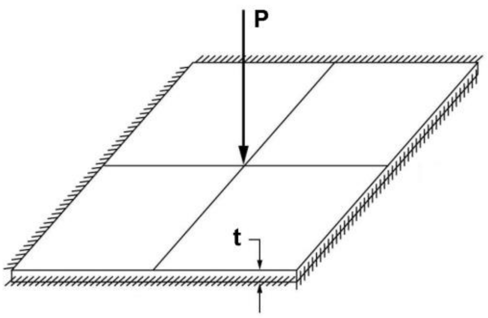

Figure 3. Clamped plate with thickness t subjected to a concentrated load at its center

The plate was modeled using the IN6 Deslauriers-Dubuc scaling functions at level 3, leading to a total number of 289 degrees of freedom. The result for the central displacement was $w = -0.00557\ PL^2/D$ which represents an error of 0.5% when compared to the exact solution $w = -0.00560\ PL^2/D$. Results were extremely good, considering that a finite element mesh using 32x32 Kirchhoff thin plate elements (3267 degrees of freedom) gives an error of 0.7% in the central displacement.

Figure 4 shows the results for displacements, distributed bending moments M_x, M_y and twisting moment M_{xy}. Bending and twisting moment per unit length distributions were obtained using the moment-curvature relation (second derivative of displacement) and the same interpolating coefficients obtained for displacements. Errors in the bending moments M_x and M_y at the center were approximately 4% when compared to analytical values.

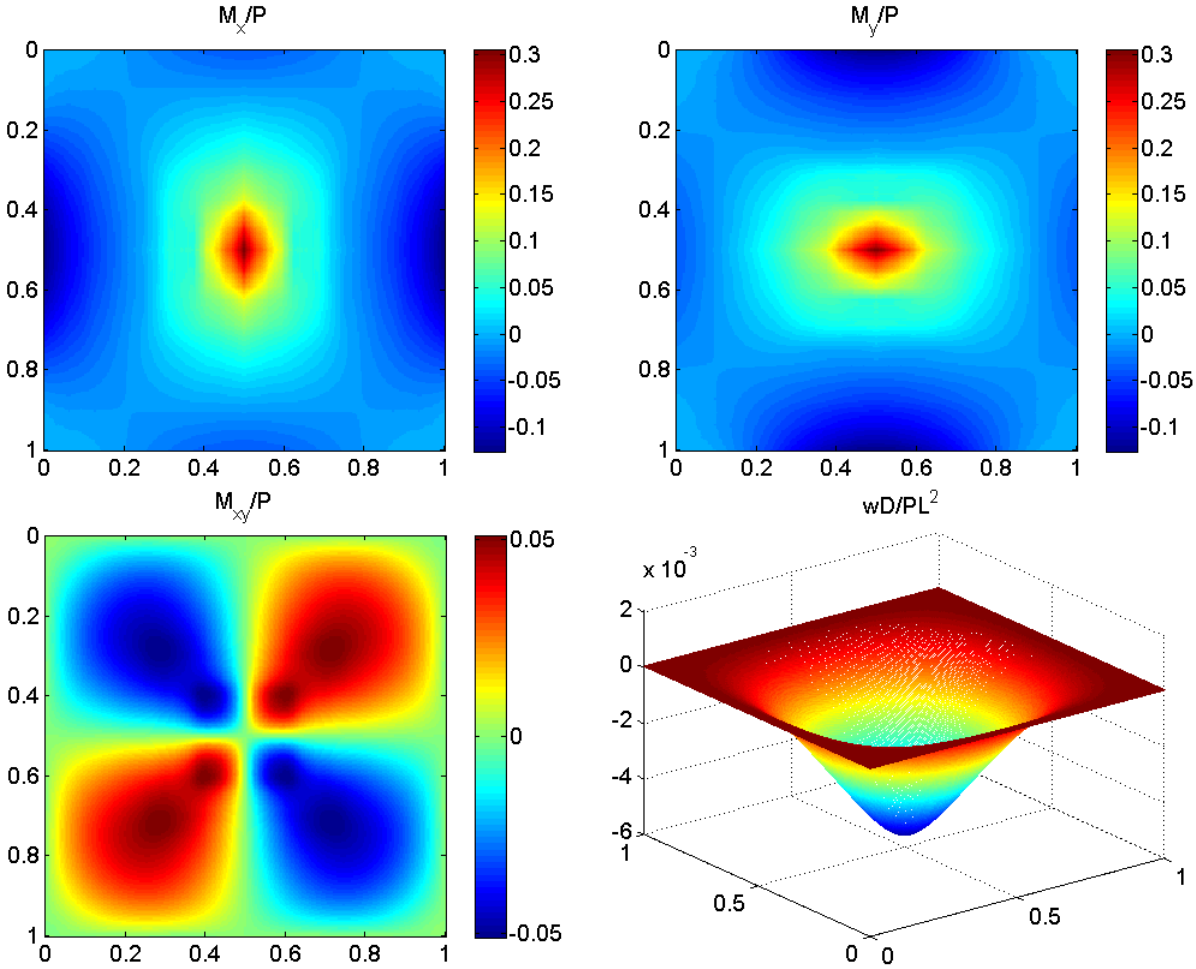

Figure 4. Wavelet-interpolated results for moments M_x, M_y, M_{xy} and displacement w

As a second example, different types of boundary conditions and loadings were tested for a square plate and the values obtained for central displacement are summarized in Tab. 1. Results were compared with exact solutions given by Timoshenko and Woinowsky-Krieger (1959).

Table 1. Thin plate displacement results for different types of boundary conditions and loadings

BOUNDARY CONDITIONS AND LOADING TYPE	EXACT	WGM	ERROR
Clamped / Uniform	$0.00126\,qL^4/D$	$0.00126\,qL^4/D$	0.4 %
Clamped / Concentrated	$0.00560\,PL^2/D$	$0.00557\,PL^2/D$	0.5 %
Simply Supported / Uniform	$0.00406\,qL^4/D$	$0.00406\,qL^4/D$	0.1 %
Simply Supported / Concentrated	$0.01160\,PL^2/D$	$0.01156\,PL^2/D$	0.3 %

5. CONCLUSIONS

This work presented the formulation and validation of the Wavelet-Galerkin Method using Deslauriers-Dubuc scaling functions (Interpolets).

As in the traditional FEM and other numerical methods, the accuracy of the solution can be improved either by increasing the level of resolution or the wavelet order. Sometimes, lower order wavelets at higher resolution can give better results than higher order wavelets at lower resolutions, since the system can become ill conditioned.

For two-dimensional problems, results for displacements and bending moments were extremely good, although only regular geometry problems were studied. The extension of the method to irregular geometries is still a challenge, especially when using multiresolution.

Since the unknowns of the method are interpolation coefficients instead of nodal displacements it is possible to obtain a smooth representation even with a reduced number of degrees of freedom. Moreover, the representation of the solution can be obtained for levels of resolution that are independent of the number of degrees of freedom of the systems obtained.

All matrices involved can be stored and operated in a sparse form, since most of their components are null, thus saving computer resources. Due to the compact support of wavelets, the sparseness of matrices increases along with the level of resolution.

In future works, models with greater complexity will be analyzed and different families of wavelets using various levels of resolution will be explored. Another interesting possibility is the mixing of scaling functions at different levels of resolution in the same finite element model.

6. ACKNOWLEDGMENTS

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