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Semi-Analytical Pressure-Based Solution for Incompressible Navier-Stokes Equations

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Abstract. *The present work developed a novel numerical method for solving the unsteady incompressible Navier-Stokes equations with primitive variables in two dimensions that can be easily extendable to three dimensions. The methodology is based on Projection Methods using a mixed approach through the Classical Integral Transform Technique (CITT). Knowing that the main bottleneck of the classical pressure-correction approach is the solution of the pressure Poisson equation (PPE), this work proposes a different approach using the classic Projection Method for advancing in time the Navier-Stokes equations and CITT to find pressure dependence on the discrete velocity field in a semi-analytical manner, using the previous time-step pressure field as a filter. For comparison purposes, the Finite Volume Method (FVM) was also used, where the PPE was solved with a Gauss-Seidel iterative procedure. Three methods were computed and compared: CITT using single transformation with filtering scheme (CITT-ST-F), CITT using single transformation without filtering scheme (CITT-ST) and Finite Volumes Methods with Gauss-Seidel solver (FVM). The obtained results were compared with the literature showing a very good agreement for both test cases. One could see that for coarse meshes, CITT-ST-F and FVM had similar performances. However for more refined meshes, CITT-ST-F overcame FVM performance.*

Keywords: *Incompressible Navier-Stokes, Hybrid Analytic-Numerical Methods, Integral Transform Technique, Projection Methods*

1. INTRODUCTION

Incompressible flows are an approximation of flows where the flow speed is negligible everywhere compared to the speed of sound of the medium. This approximation results in the variation of the density everywhere in the flow being also negligible. The majority of the fluid and associated flow we find in our daily lives belong to the incompressible category. For example, most of the flow associated with air, water and biological flows (such as blood) are all in the incompressible flow domain.

The major difficulty for the numerical simulation of incompressible flows is that the velocity and pressure are coupled by the incompressibility constraint. To overcome this difficulty in time dependent viscous incompressible flows, fractional step methods, which are also referred in the literature as projection methods, were developed. Projection methods can be classified into three classes (Guermond *et al.*, 2006), namely pressure-correction methods (Thomadakis and Leschziner, 1996), velocity-correction methods (Guermond and Shen, 2003), and consistent splitting methods (Karniadakis *et al.*, 1991). The most attractive feature of projection methods is that, at each time step, one only needs to solve a sequence of decoupled elliptic equations for velocity and pressure, making it very efficient for large scale numerical simulations. Although projection methods are widely used, many authors have already drawn attention to the fact that the decomposition used in this method is intrinsically second order accurate (Munz *et al.*, 2003; Guermond *et al.*, 2006).

Pressure-correction schemes are time-marching techniques composed of two sub-steps for each time step: the pressure is treated explicitly or ignored in the first sub-step then is corrected in the second one. The linear momentum equations play the major role in determining the velocity components. Thus, it is left to the continuity equation to determine pressure, even if this variable does not appear explicitly in this equation. The most common methodology to determine an equation for the pressure-correction is to combine both equations by taking the divergence of the momentum equations and substituting the continuity equation where necessary, effectively generating a Poisson type equation for the pressure.

This makes it possible to obtain an equation for the pressure-correction, using the continuity equation.

The major computational time associated with this procedure is the solution of the Poisson equation at each step, which consumes large amounts of computer time. One reason for this is that the convergence rate of iterative methods that are commonly used for this purpose, such as the Jacobi and Gauss-Seidel, rapidly decreases as the mesh is refined (Versteeg and Malalasekera, 2007).

On the other hand, in the realm of analytical methods, the Integral Transform Technique (Mikhailov and Özişik, 1984) has been playing a big role. It deals with expansions of the sought solution in terms of infinite orthogonal basis of eigenfunctions, keeping the solution process always within a continuous domain. The resulting system is generally composed of a set of uncoupled differential equations which can be solved analytically. However, a truncation error is involved since the infinite series must be truncated to obtain numerical results. This error decreases as the number of summation terms (truncation order) is increased, and the solution converges to a final value. Due to the series representation nature of the Integral Transform Technique, the estimated error can be easily obtained, which results in better global error control of the solution. The disadvantage associated with this approach is the need for more elaborate analytical manipulation. This effort can be greatly minimized with the use of symbolical computation (Wolfram, 2003).

Recent works of Chalhub *et al.* (2014b,a) introduced the idea of a semi-analytical solution for the poisson equation arising from the incompressible Navier-Stokes equations.

2. PROBLEM FORMULATION

The incompressible Navier-Stokes equations[1] are given as follows:

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \mu \nabla^2 \boldsymbol{v} + \rho \boldsymbol{f} \quad \text{for} \quad \boldsymbol{x} \in \mathcal{V} \quad \text{and} \quad t \geq 0 \tag{1a}$$

$$\nabla \cdot \boldsymbol{v} = 0 \quad \text{for} \quad \boldsymbol{x} \in \mathcal{V} \tag{1b}$$

where μ is the fluid dynamic viscosity and ρ is the fluid density and both properties are constants.

Equation (1a) is the momentum conservation equation and equation (1b) is the mass conservation equation, also called the incompressibility constraint.

In addition to the theoretical, analytic difficulties of Navier-Stokes equations, the equations also present practical problems. The equations form a nonlinear, coupled set of equations which admit analytical solutions in a limited number of cases. So being, the last resource for solving this system of equation (1) is by numerical techniques. The main issues of implementing numerically the solution of the Navier-Stokes equations is how exactly to implement the incompressibility constraint (1b) (Shirokoff, 2011) and the non-existence of an equation for the pressure.

2.1 Classical Projection Method

A more efficient way to solve incompressible flows was introduced by Chorin (1968) as Projection Methods. The key advantage of the projection method is that the computations of the velocity and pressure fields are decoupled. The algorithm of projection method is based on the Helmholtz-Hodge decomposition[2] (Wu *et al.*, 2007) of any vector field into a solenoidal part and an irrotational part.

In Chorin's (Chorin, 1968) original Projection Method, one should first compute an intermediate velocity $\hat{\boldsymbol{v}}$, explicitly using the momentum equation by ignoring the pressure gradient term:

$$\hat{\boldsymbol{v}} = \Delta t \left(\nu \nabla^2 \boldsymbol{v}^l + \rho \boldsymbol{f}^l - \boldsymbol{v}^l \cdot \nabla \boldsymbol{v}^l \right) - \boldsymbol{v}^l \tag{2}$$

In the second half of the algorithm, the projection step, the intermediate velocity is corrected to obtain the final solution of the time step $\boldsymbol{v}^{l+1}$:

$$\boldsymbol{v}^{l+1} = \hat{\boldsymbol{v}} - \frac{\Delta t}{\rho} \nabla p^{l+1} \tag{3}$$

Applying the divergence operator on both sides of the equations (3) and knowing that $\nabla \cdot \boldsymbol{v}^{l+1} = 0$ due to the mass conservation[3]:

$$\nabla^2 p^{l+1} = \frac{\rho}{\Delta t} \nabla \cdot \hat{\boldsymbol{v}} \tag{4}$$

[1] Some authors refer to the Navier-Stokes equations as just the momentum equation.
[2] Also known as Helmholtz decomposition and theorem of Ladyzhenskaya
[3] Note that $\nabla \cdot \hat{\boldsymbol{v}}$ is not necessarily equal to zero.

For the pressure poisson equation, the following boundary condition and Neumann-Poisson compatibility condition are required:

$$(\nabla p)\cdot\boldsymbol{n} = \left(\rho\frac{\partial \boldsymbol{v}}{\partial t} - \rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \mu\nabla^2\boldsymbol{v} + \rho\boldsymbol{f}\right)\cdot\boldsymbol{n} \qquad \boldsymbol{x}\in\partial\mathcal{V} \tag{5}$$

$$\int_{\mathcal{V}} p(\boldsymbol{x})\,\mathrm{d}\mathcal{V} = 0 \tag{6}$$

Equation (2) can be rewritten in the following form for a 2D cartesian domain:

$$\hat{u} = \Delta t\, F^l - u^l \qquad \text{and} \qquad \hat{v} = \Delta t\, G^l - v^l \tag{7}$$

where

$$F = \nu\left(\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial y^2}\right) + f_x - \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \qquad \text{and} \qquad G = \nu\left(\frac{\partial^2 v}{\partial^2 x} + \frac{\partial^2 v}{\partial y^2}\right) + f_y - \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) \tag{8}$$

In the present work, the staggered grid allows us to implement the discretization of F and G by approximating the derivatives using a second order finite difference approach for the advection and diffusion terms. However, it could be troublesome for very high Reynolds numbers (Versteeg and Malalasekera, 2007). Furthermore, for future works, a high order upwind scheme can be suggested.

The problem geometry is considered rectangular 2D in cartesian coordinates and it is discretized using staggered grids. The discretization parameters are given bellow for the pressure grid (volume center):

$$\Delta x = L/i_{\max} \qquad \Delta y = H/j_{\max} \qquad x_i = \Delta x\,(i-1/2) \qquad y_j = \Delta y\,(j-1/2) \tag{9}$$

Where L is de dimension in x-direction and H is the dimension in y-direction. The indices used for each dimension discretized are for x-direction: i, q, p; for y-direction: j, r, s and for the Classical Integral Transform Technique (CITT) are n, m, k.

3. SOLUTION OF THE PRESSURE POISSON EQUATION

The general Poisson equation for pressure in two-dimensions, using cartesian coordinates and compatible boundary conditions can be written in the following form:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = Q(x,y) \tag{10a}$$

$$\left(\frac{\partial p}{\partial x}\right)_{x=0} = -\rho\left[\left(\frac{\partial u}{\partial t}\right)_{x=0} + u(0,y)\left(\frac{\partial u}{\partial x}\right)_{x=0} + v(0,y)\left(\frac{\partial u}{\partial y}\right)_{x=0}\right] + \mu\left[\left(\frac{\partial^2 u}{\partial x^2}\right)_{x=0} + \left(\frac{\partial^2 u}{\partial y^2}\right)_{x=0}\right] + f_x(0,y) \tag{10b}$$

$$\left(\frac{\partial p}{\partial x}\right)_{x=L} = -\rho\left[\left(\frac{\partial u}{\partial t}\right)_{x=L} + u(L,y)\left(\frac{\partial u}{\partial x}\right)_{x=L} + v(L,y)\left(\frac{\partial u}{\partial y}\right)_{x=L}\right] + \mu\left[\left(\frac{\partial^2 u}{\partial x^2}\right)_{x=0} + \left(\frac{\partial^2 u}{\partial y^2}\right)_{x=0}\right] + f_x(L,y) \tag{10c}$$

$$\left(\frac{\partial p}{\partial y}\right)_{y=0} = -\rho\left[\left(\frac{\partial v}{\partial t}\right)_{y=0} + u(x,0)\left(\frac{\partial v}{\partial x}\right)_{y=0} + v(x,0)\left(\frac{\partial v}{\partial y}\right)_{y=0}\right] + \mu\left[\left(\frac{\partial^2 v}{\partial x^2}\right)_{y=0} + \left(\frac{\partial^2 v}{\partial y^2}\right)_{y=0}\right] + f_y(x,0) \tag{10d}$$

$$\left(\frac{\partial p}{\partial y}\right)_{y=H} = -\rho\left[\left(\frac{\partial v}{\partial t}\right)_{y=H} + u(x,H)\left(\frac{\partial v}{\partial x}\right)_{y=H} + v(x,H)\left(\frac{\partial v}{\partial y}\right)_{y=H}\right] + \mu\left[\left(\frac{\partial^2 v}{\partial x^2}\right)_{y=H} + \left(\frac{\partial^2 v}{\partial y^2}\right)_{y=H}\right] + f_y(x,H) \tag{10e}$$

where $Q(x,y)$ is the source term of the equation, which must be calculated in the same nodes as the pressure, and for the Projection Method it is given by:

$$Q(x,y,t) = \frac{\rho}{\Delta t}\left(\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y}\right) \tag{11}$$

The problem can be rewritten in a compact general form:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = Q(x, y, t) \tag{12a}$$

$$\left(\frac{\partial p}{\partial x}\right)_{x=0} = \mathcal{B}_{\mathrm{W}}(y); \qquad \left(\frac{\partial p}{\partial x}\right)_{x=L} = \mathcal{B}_{\mathrm{E}}(y); \qquad \left(\frac{\partial p}{\partial y}\right)_{y=0} = \mathcal{B}_{\mathrm{S}}(x); \qquad \left(\frac{\partial p}{\partial y}\right)_{y=H} = \mathcal{B}_{\mathrm{N}}(x) \tag{12b}$$

where $\mathcal{B}_{\mathrm{W}}(y)$, $\mathcal{B}_{\mathrm{E}}(y)$, $\mathcal{B}_{\mathrm{S}}(x)$ and $\mathcal{B}_{\mathrm{N}}(x)$ represent the function for each boundary condition showed previously.

3.1 Filtering Scheme for Integral Transformation

Before continuing to the solution via integral transformation, it is proposed a separation of the pressure on the following form:

$$p(x, y, t) = p^*(x, y, t) + p_f(x, y) \tag{13}$$

where p^* is the filtered pressure and p_f is a known filter function.

Replacing equation (13) in the Poisson equation and since the pressure boundary conditions are the same in the previous and current time-step, the resulting boundary conditions for the filtered problem are homogeneous:

$$\frac{\partial^2 p^*}{\partial x^2} + \frac{\partial^2 p^*}{\partial y^2} = Q^*(x, y) \tag{14a}$$

$$\left(\frac{\partial p^*}{\partial x}\right)_{x=0} = 0; \qquad \left(\frac{\partial p^*}{\partial x}\right)_{x=L} = 0; \qquad \left(\frac{\partial p^*}{\partial y}\right)_{y=0} = 0; \qquad \left(\frac{\partial p^*}{\partial y}\right)_{y=H} = 0 \tag{14b}$$

where: $Q^*(x, y) = Q^l(x, y) - Q^{l-1}(x, y)$

Instead of solving the general problem (12), the filtered homogeneous problem (14) will be solved, which for the integral transformation has a much better convergence, due to it being boundary homogeneous and the fact that the filtered pressure p^* is a small quantity. The filtered pressure can be interpreted as just a correction of the pressure in the previous time-step.

The discretization of Q is defined in the pressure mesh and is chosen to be of the following form for the Projection Method:

$$Q_{i,j} = \frac{\rho}{\Delta t}\left(\frac{\hat{u}_{i+1,j} - \hat{u}_{i,j}}{\Delta x} + \frac{\hat{v}_{i,j+1} - \hat{v}_{i,j}}{\Delta y}\right) \tag{15}$$

4. CLASSICAL INTEGRAL TRANSFORM TECHNIQUE: SINGLE TRANSFORMATION

In this work the Classical Integral Transform Technique (Mikhailov and Özişik, 1984) is used for the purpose of solving the filtered Poisson equation (14). This is an analytical technique that uses expansions of the sought solution in terms of an infinite orthogonal basis of eigenfunctions, keeping the solution process always within a continuous domain. In order to establish the transformation pair, the pressure field is written as function of an orthogonal eigenfunctions obtained from the following auxiliary eigenvalue problem known as the Helmholtz classic problem in cartesian coordinates (Mikhailov and Özişik, 1984), where $Y_m(y)$ are the eigenfunctions and β_m are the eigenvalues for the y-direction.

$$\frac{\mathrm{d}^2 Y_m(y)}{\mathrm{d}y^2} + \beta_m^2 Y_m(y) = 0; \qquad Y'(0) = 0; \qquad Y'(H) = 0 \tag{16a}$$

which has the following solution:

$$Y_m(y) = \cos(y\beta_m); \qquad \beta_m = \frac{\pi m}{H}; \qquad \text{for} \qquad m = 1, 2, 3, \ldots \tag{17a}$$

It should be noted that for these boundary conditions, one needs also to account for non-trivial solutions corresponding to $\beta_0 = 0$: $Y_0 = 1$ and $\beta_0 = 0$.

The norm Ny_m is defined by:

$$\mathrm{Ny}_m = \int_0^H Y_m^2\,\mathrm{d}y = \frac{H}{2} \qquad \text{for} \qquad m \neq 0 \qquad \text{and} \qquad \mathrm{Ny}_m = \int_0^H Y_0^2\,\mathrm{d}y = H \qquad \text{for} \qquad m = 0 \tag{18}$$

The transformation pair is defined by:

$$\text{Transformation} \quad \Rightarrow \quad \bar{p}_n^*(x) = \int_0^L p^*(x, y)\, Y_m(y)\,\mathrm{d}x \tag{19a}$$

$$\text{Inversion} \quad \Rightarrow \quad p^*(x, y) = \frac{\bar{p}_0^*(x)}{\mathrm{Ny}_m} + \sum_{m=1}^{\infty} \frac{Y_m(y)\bar{p}_m^*(x)}{\mathrm{Ny}_m} \tag{19b}$$

The final solution is given by two portions of the pressure: the average pressure in the x-direction, p^*_{avg} and the modified pressure p^*_{mod}:

$$p^*(x,y) = p^*_{\text{avg}}(x) + p^*_{\text{mod}}(x,y) \tag{20}$$

where p^*_{avg} comes from the solution of the eigenproblem when $\beta = 0$ and p^*_{mod} comes from the solution when $\beta > 0$, in other words:

$$p^*_{\text{avg}}(x) = \frac{\bar{p}^*_0(x)}{\text{Ny}_m} \qquad \text{and} \qquad p^*_{\text{mod}}(x,y) = \sum_{m=1}^{\infty} \frac{Y_m(y)\bar{p}^*_m(x)}{\text{Ny}_m} \tag{21}$$

The integral transformation of the governing differential equation is derived by applying the operator $\int_0^H (\bullet)\, Y_m \,\mathrm{d}y$ on the filtered Poisson equation (14), leading to the following transformed Poisson equation:

$$\frac{\partial^2 \bar{p}^*_m(x)}{\partial x^2} - \beta_m^2 \bar{p}^*_m(x) = \bar{Q}^*_m(x) \tag{22a}$$

$$\left(\frac{\partial \bar{p}^*_m(x)}{\partial x}\right)_{x=0} = 0; \qquad \left(\frac{\partial \bar{p}^*_m(m)}{\partial x}\right)_{x=L} = 0 \tag{22b}$$

where the transformation of the parameter Q^* is given by $\bar{Q}^*_m(x) = \int_0^H Q^*(x,y)\, Y_m \,\mathrm{d}y$.

Equation (22) admits a closed-form analytical solution:

- For $m = 0$:

$$\bar{p}^*_0(x) = \int_0^x \int_0^{K_2} \bar{Q}^*_0(K_1)\,\mathrm{d}K_1\,\mathrm{d}K_2 \tag{23}$$

- For $m > 0$:

$$\begin{aligned}\bar{p}^*_m(x) = \frac{1}{4} e^{-x\beta_m} \Bigg(&\left(-e^{2x\beta_m} - 1\right)\left(\coth(L\beta_m) - 1\right) \int_0^L \frac{e^{K_1\beta_m}\bar{Q}^*_m(K_1)}{\beta_m}\, dK_1 - \\ 2\int_0^x \frac{e^{K_1\beta_m}\bar{Q}^*_m(K_1)}{\beta_m}\, dK_1 - &\left(e^{2x\beta_m} + 1\right)\left(\coth(L\,\beta_m) + 1\right) \int_0^L \frac{e^{-K_2\beta_m}\bar{Q}^*_m(K_2)}{\beta_m}\, dK_2 + \\ &2e^{2x\beta_m} \int_0^x \frac{e^{-K_2\beta_m}\bar{Q}^*_m(K_2)}{\beta_m}\, dK_2 \Bigg)\end{aligned} \tag{24}$$

Where K_1 and K_2 are dummy integration variables.

In order to recover the original filtered pressure p^* we only need to used the inversion formula (19b):

$$p^*(x,y) = \frac{\bar{p}^*_0(x)}{\text{Ny}_m} + \sum_{m=1}^{m_{\max}} \frac{Y_m(y)\bar{p}^*_m(x)}{\text{Ny}_m} \tag{25}$$

The summation must be truncated to a finite value ($m_{\max}$) to be obtained. The main advantage of this method is that, besides requiring more analytical manipulation, the solution has only one summation.

4.1 Integration of the discrete fields

We already know from the previous section that the source term Q is discrete, so the Taylor series is used to obtain an approximation of Q inside each cell. The required order is 2, so the following approximation is chosen, resulting in the scheme:

$$\int_0^H \int_0^L Q^*(x,y)\,\Lambda(x,y)\,\mathrm{d}x\;\mathrm{d}y \approx \sum_{i=1}^{i_{\max}} \sum_{j=1}^{j_{\max}} Q^*_{i,j} \int_{y_j - \Delta y/2}^{y_j + \Delta y/2} \int_{x_i - \Delta x/2}^{x_i + \Delta x/2} \Lambda(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{26}$$

$$\begin{aligned}\int_0^{x_i} \int_0^H Q^*(K_1,y)\Lambda(K_1,y)\,\mathrm{d}y\,\mathrm{d}K_1 \approx \sum_{q=1}^{i-1} \sum_{r=1}^{j_{\max}} Q^*_{q,r} \left(\int_{x_q - \frac{\Delta x}{2}}^{\frac{\Delta x}{2} + x_q} \int_{y_r - \frac{\Delta y}{2}}^{\frac{\Delta y}{2} + y_r} \Lambda(K_1,y)\,\mathrm{d}y\,\mathrm{d}K_1 \right) + \\ + \sum_{r=1}^{j_{\max}} Q^*_{i,r} \left(\int_{x_i - \frac{\Delta x}{2}}^{x_i} \int_{y_r - \frac{\Delta y}{2}}^{\frac{\Delta y}{2} + y_r} \Lambda(K_1,y)\,\mathrm{d}y\,\mathrm{d}K_1 \right)\end{aligned} \tag{27}$$

$$\int_0^{x_i}\int_0^{K_2}\int_0^{H} Q^*(K_1,y)\Lambda(K_1,y)\,\mathrm{d}y\,\mathrm{d}K_1\,\mathrm{d}K_2 \approx$$
$$\approx \sum_{r=1}^{j_{\max}} Q^*_{i,r}\int_{x_i-\frac{\Delta x}{2}}^{x_i}\left(\int_{x_i-\frac{\Delta x}{2}}^{K_2}\left(\int_{y_r-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}+y_r}\Lambda(K_1,y)\,\mathrm{d}y\right)\mathrm{d}K_1\right)\mathrm{d}K_2+$$
$$+\sum_{s=1}^{i-1}\sum_{r=1}^{j_{\max}} Q^*_{s,r}\int_{x_s-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}+x_s}\left(\int_{x_s-\frac{\Delta x}{2}}^{K_2}\left(\int_{y_r-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}+y_r}\Lambda(K_1,y)\,\mathrm{d}y\right)\mathrm{d}K_1\right)\mathrm{d}K_2+$$
$$+\sum_{s=1}^{i-1}\sum_{r=1}^{j_{\max}} Q^*_{s,r}\frac{1}{2}\Delta x\int_{x_s-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}+x_s}\left(\int_{y_r-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}+y_r}\Lambda(K_1,y)\,\mathrm{d}y\right)\mathrm{d}K_1+$$
$$+\sum_{s=1}^{i-1}\sum_{q=1}^{s-1}\sum_{r=1}^{j_{\max}} Q^*_{q,r}\Delta x\int_{x(q)-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}+x(q)}\left(\int_{y_r-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}+y_r}\Lambda(K_1,y)\,\mathrm{d}y\right)\mathrm{d}K_1 \quad (28)$$

where $\Lambda(x, y)$ is a general function, the integrals are computed analytically and K_1 and K_2 are dummy integration variables.

5. RESULTS

In this chapter the results obtained from the computational code developed using the new formulation will be presented. The solution of the incompressible flow in a test case simulated by the proposed method will be presented. The codes were implemented in FORTRAN 90 using GFortran, they were compiled in serial computation and using the -O3 optimization flag.

The solution of the classic Lid-Driven Cavity problem is shown to validate the proposed formulation and evaluate its performance for more demanding problems. Three methods are computed and evaluated: CITT using single transformation with filtering scheme (CITT-ST-F), CITT using single transformation without filtering scheme (CITT-ST) and a code developed using Finite Volumes Methods (FVM) (Patankar, 1980) together with a Gauss-Seidel linear system solver (Moin, 2001).

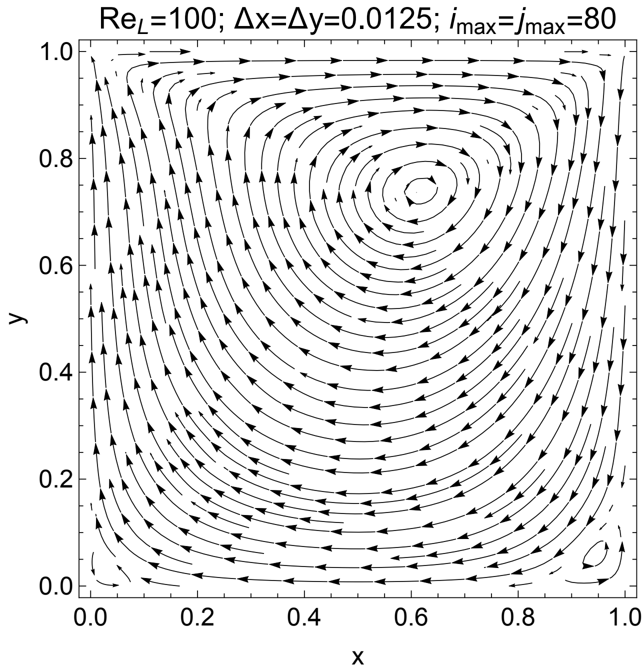

Figure 1. Streamlines of the flow inside the lid-driven cavity simulated by CITT using single transformation with filter scheme and with $i_{\max} = j_{\max} = 80$.

For the computation, all results were calculated with prescribed relative precision of 10^{-6} for the Gauss-Seidel iterative solver used in FVM and also for the CITT summation series convergence. In order to guarantee this precision in the truncated summations, an automatic truncation procedure was developed and implemented. The projection method was used and the time-steps Δt utilized were the same for all methods. The results computed in the current work were carried out using only uniform meshes and also considering $\Delta x = \Delta y$ and the Neumann boundary conditions for pressure on all

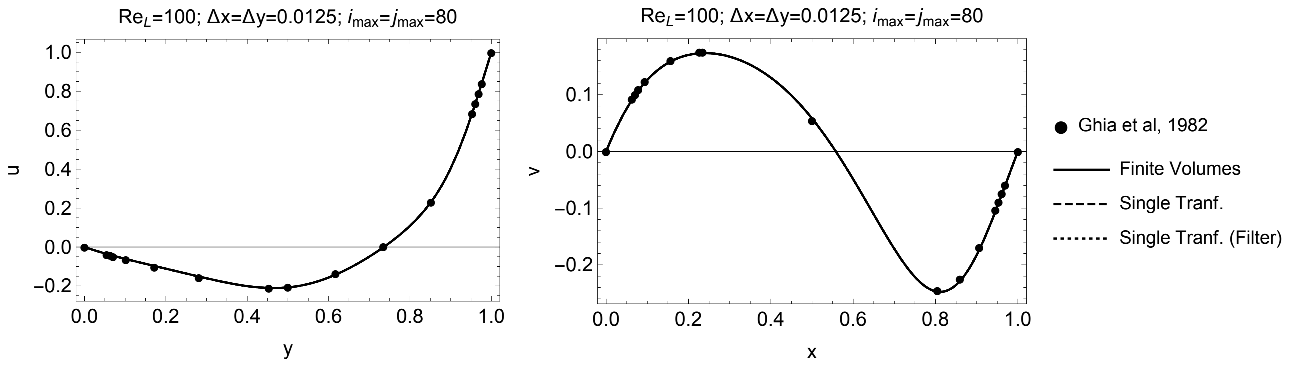

Figure 2. Lid-Driven Cavity: Profile of streamwise velocity at the midplane of the cavity ($x = 0.5L$ and $y = 0.5L$) for $\mathrm{Re} = 100$: Comparison of the present results with the work of Ghia *et al.* (1982).

impermeable/no slip walls were approximated to zero without violating the Poisson-Neumann compatibility condition.

In order to illustrate the solution, figure 1 shows the streamlines for the lid-driven cavity flow for the three solved Reynolds numbers with a mesh of 80×80. No visual difference was observed between the three methods, so only the case computed by CITT-ST-F is displayed. One can see that the corner eddies become more prominent when the Reynolds is increased as expected from the literature (Ghia *et al.*, 1982).

Figure 2 shows the streamwise velocity obtained from the novel methodologies at the midplane of the cavity ($x = 0.5L$ and $y = 0.5L$) for $\mathrm{Re} = 100$, where the results from Ghia *et al.* (1982) are also presented for comparison purposes. As can be seen, the results for the present work using a mesh of 80×80 are in good agreement with the literature data. From the graphics of the figure 2, it is easy to note that all the three methods shown are graphically identical.

In figure 3 presents the total computational time required to achieve steady state versus mesh size. It can be seen that CITT-ST without filter requires a lot more computational time compared to the other methods. For very coarse meshes, CITT-ST-F and FVM require approximately the same computational time, however, when the mesh is refined, CITT-F overcomes FVM performance.

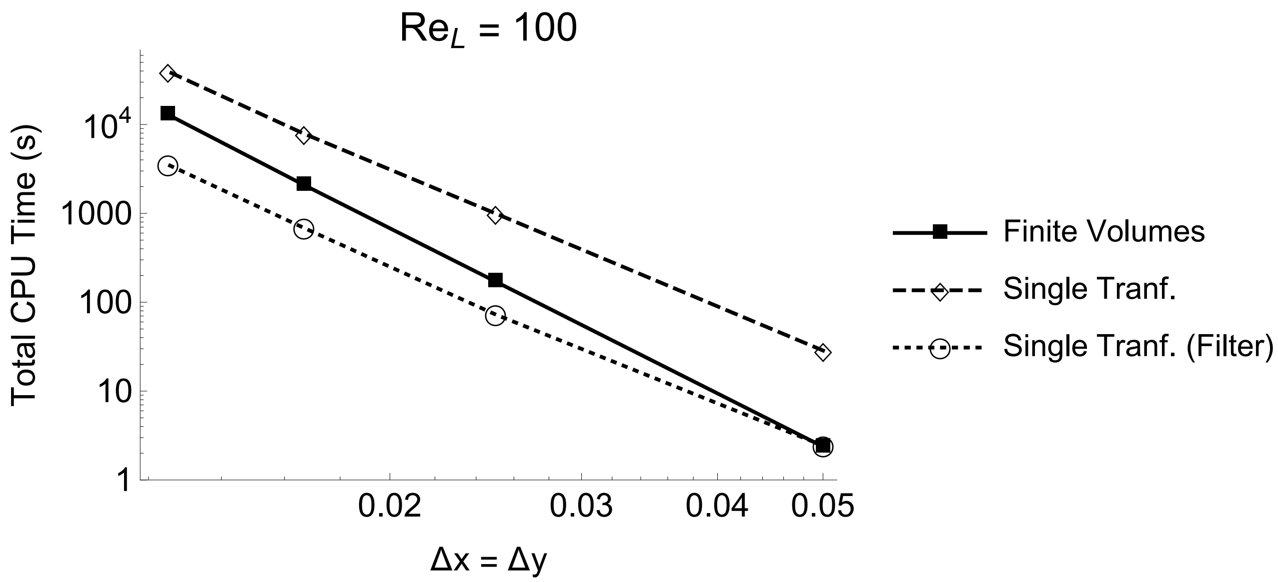

Figure 3. Lid-Driven Cavity: Total computational time consumed to achieve steady state.

6. CONCLUSIONS

The present work developed a numerical method for solving the unsteady incompressible Navier-Stokes equations with primitive variables in two dimensions, although it can be easily extended to tree dimensions. The novel methodology is based on projection methods schemes using a mixed approach through the Integral Transform Technique. Knowing that the main bottleneck of the classical pressure-correction approach is the solution of the Poisson equation for the pressure, this work proposed a new approach to overcome this constraint.

In this work the classic Projection Method (Chorin, 1968) was used to advnce in time the Navier-Stokes equations and the Classical Integral Transform Technique (Mikhailov and Özişik, 1984) was used for the purpose of solving the filtered Poisson equation (14) to find the pressure dependence on the discrete velocity field in a semi-analytical fashion, eliminating the need for an iterative solver. In order to establish the transformation pair, the pressure field is written as function of an orthogonal eigenfunctions obtained from an auxiliary eigenvalue problem known as the Helmholtz classic problem (Mikhailov and Özişik, 1984).

The Lid-Driven Cavity problem was analysed. In this section, three methods were computed and compared: CITT using single transformation with filtering scheme (CITT-ST-F), CITT using single transformation without filtering scheme (CITT-ST) and Finite Volumes Methods with Gauss-Seidel solver (FVM). The obtained results were compared with the literature showing a very good agreement for both test cases. Results showed a very similar qualitative behavior. One could see that for very poorly refined meshes CITT-ST-F and FVM had similar performances. However for more refined meshes, CITT-ST-F had a dramatically better performance and CITT-ST (without filter) had the worst performance overall.

The results showed that, although more investigations are needed, CITT-ST-F has a great potential of being a good substitute for the methodologies currently used to solve the pressure Poisson equation.

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