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COMPARATIVE ANALYSIS OF THREE DIFFERENT EIGENVALUE SOLVERS FOR LINEAR STABILITY ANALYSIS

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Abstract. *In local stability analysis, linear disturbance growth can be measured by solving either an eigenvalue problem or and initial value problem. Shooting method is one of the traditional ways to solve the eigenvalue problem. However, it can be quite hard to determine suitable initial guess and locate all the important modes. An alternative approach is known as the matrix forming method, which transforms the Linearized Navier-Stokes Equations (LNSE) into a generalized eigenvalue problem that provides all existing modes. Recently, a novel approach known as time stepping method marches the LNSE in time until the leading eigemode dominates the asymptotic temporal behaviour. The present paper provides an instability analysis for mixing layers using all three methods in order to compare them. All three methods provide consistent results.*

Keywords: *stability analysis, shooting method, matrix forming, time steppers, mixing layer.*

1. INTRODUCTION

There are several engineering problems, mainly in aerodynamic applications, where it is crucial to understand the flow transition from laminar to turbulent. For example, wing design in order to reduce drag on commercial and military aircraft, intensification of the fuel/oxidant in combustion chambers of rocket engines and so on. Laminar flows are always subject to small perturbations, due to different reasons such as structural vibration, surface roughness, noise among others. When small perturbations are introduced in the flow and one are not dispelled the state of the flow is changed and the perturbation is classified as unstable (Chandrasekhar, 1961). In this context, the question is for what perturbations the flow not return to initial state.

For the best part of the last century, instability analysis was performed using the classical theory of normal modes and it still has important relevance for several problems. For small perturbation, linear stability theory (LST) can be considered for spatial or temporal development decomposing the flow into a steady part called base flow, and an unsteady part. Introducing this decomposition intro the Navier-Stokes equations, subtracting the equations for the steady flow and disregarding nonlinear terms yields the linearized perturbation equations referred as linearized Navier-Stokes equations (LNSE). There are diferrent form to simplity the unsteady part of decomposition. At the strongest level of aproximation, disturbances are assumed homogeneous in two out of the three spatial directions and the perturbations are expanded in terms of Forier transform in one or two spacial dimensions. As the stability is determined separately for each mode, rising the term modal LST (Theofilis, 2011).

Applying these considerations about disturbances, the LNSE become the eigevalue problem for incompressible flows. Through algebraic manipulation the pressure perturbation can be eliminated, and then the system reduces to the Orr-Sommerfeld equation or Rayleigh equation for large Reynolds number (Drazin and Ried, 1981). Shooting method is one of the traditional ways to solve these equations and different problem have been studied with this method, for example the classic problems of mixing layer (Michalke, 1964, 1965), planar channel (Orszag, 1969) and boundary layer (Mack, 1984). The analogous expansion and LNSE can be found for compressible flow. Mendonca,2014; Mendonca *et al.*, 2015 has been used extensively this compressible formulation for study of compressible binary mixing layer. However, it can be quite hard to determine suitable initial guess to start the method.

The matrix forming is another form to solve the LNSE. In this method, the perturbations equations are spatially discretized getting a gerenarized eigenvalue problem EVP. There are in the literature a lot of suggestion to make the spacial discretization, Paredes *et al.,* 2013 realized a comparison among different types of spacial discretization which showed that high order finite difference method is competitive over others discretization of the multidimensional EVP. The main advantage of the matrix forming is its simplicity and flexibility to programme, it is easy to work with incompressible or compressible, viscous or invicid into the same code. The disadvantage of the matrix forming approach is the cost associated with matrix storage (Theofilis, 2003).

When use of matrix forming expensive is due computational limitations, the time stepping approach may be used alternatively. In this method the amplitude of the unsteady part in the decomposition vary with the time. This system may be integrad along time and the result of the time integration at $t \to \infty$ is the leading eigenmode of the steady basic flow (Theofilis, 2011). The main advanteage of time stepping approach is that a matrix is never formed or stored.

The present article presents the solutions of modal linear instability for viscous mixing layers using all three methods in order to evaluate their potential advantages and disdvantages when simulating instability problems for incompressible flows. Section 2 introduces the formulation of each approach. Section 3 to show the results with the differents methodologys. Section 4 presents some conclusions.

2. Formulation and methodology

In this section it is going to be presented the mathematical formulation of three methods for modal linear stability analysis. The case test choosen was a mixing layers for viscous fluid, where temporal linear stability was considered. In this analysis α is a real wave number and ω is a complex frequency.

2.1 Shooting method

The nondimensional $2D$ incompressible Navier Stokes equations are given by:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial v} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

Decomposing the flow into the steady and unsteady parts

$$u(x,y,t) = U(y) + \tilde{u}(x,y,t) \tag{4}$$

$$v(x,y,t) = \tilde{v}(x,y,t) \tag{5}$$

$$p(x,y,t) = P + \tilde{p}(x,y,t) \tag{6}$$

If Eqs. (4), (5) e (6) are introduced into the Navier Stokes equation and linearized, assuming the base flow is local,

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial v} = 0 \tag{7}$$

$$\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\frac{\partial U}{\partial y} = -\frac{\partial \tilde{p}}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2}\right) \tag{8}$$

$$\frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x} + = -\frac{\partial \tilde{p}}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial y^2}\right) \tag{9}$$

and assuming normal modes

$$\tilde{u}(x,y,t) = \frac{1}{2}[\bar{u}(y)exp(i\alpha x - i\omega t) + cc], \tag{10}$$

$$\tilde{v}(x,y,t) = \frac{1}{2}[\bar{v}(y)exp(i\alpha x - i\omega t) + cc], \tag{11}$$

$$\tilde{p}(x,y,t) = \frac{1}{2}[\bar{p}(y)exp(i\alpha x - i\omega t) + cc], \tag{12}$$

where i is the imaginary number and cc is the complex conjugate. This equations indicate that perturbation travel as waves with frequency ω and wavelength $\lambda = 2\pi/\alpha$ and amplitudes $\bar{u}$, $\bar{v}$ and $\bar{p}$.

Substituting the expression (10), (11) e (12) into the system (7), (8) e (9) yields the normal mode equations.

$$i\alpha\bar{u} + \frac{\partial\bar{v}}{\partial y} = 0 \tag{13}$$

$$i\alpha(U - \omega/\alpha)\bar{u} + \frac{\partial U}{\partial y}\bar{v} + i\alpha\bar{p} = \frac{1}{Re}\left(\frac{\partial^2\bar{u}}{\partial y^2} - \alpha^2\bar{u}\right) \tag{14}$$

$$i\alpha(U - \omega/\alpha)\bar{v} + \frac{\partial\bar{p}}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2\bar{v}}{\partial y^2} - \alpha^2\bar{v}\right) \tag{15}$$

Defining the variable vector $\mathbf{V}$ as $\mathbf{V} = \begin{bmatrix} \bar{u} \\ \frac{\partial\bar{u}}{\partial y} \\ \bar{v} \\ \bar{p} \end{bmatrix}$, the system (13), (14) e (15) may be written in matricial form with

$$\frac{\partial\mathbf{V}}{\partial y} = \mathbf{MV} \tag{16}$$

where $\mathbf{M}$ is 4×4 matrix as followed

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ \alpha^2 + i\alpha Re(U - \omega/\alpha) & 0 & Re\frac{\partial U}{\partial y} & i\alpha Re \\ -i\alpha & 0 & 0 & 0 \\ 0 & -\frac{i\alpha}{Re} & -\frac{1}{Re}\alpha^2 + i\alpha Re(U - \omega/\alpha) & 0 \end{bmatrix} \tag{17}$$

For mixing layer problem the boundary conditions are given by

$$\begin{cases} \bar{u} = \bar{v} = \bar{p} = 0 \quad para \quad y \to \infty \\ \bar{u} = \bar{v} = \bar{p} = 0 \quad para \quad y \to -\infty \end{cases} \tag{18}$$

whose asymptotic solutions are

$$\begin{bmatrix} \bar{u} \\ \bar{v} \\ \bar{p} \end{bmatrix} = \mathbf{A_1}\begin{bmatrix} -i \\ 1 \\ i(U_{+\infty} - \omega/\alpha) \end{bmatrix} e^{-\alpha y} + \mathbf{A_2}\begin{bmatrix} -i\gamma/\alpha \\ 1 \\ 0 \end{bmatrix} e^{-\gamma y} \tag{19}$$

or in vector form $V_+ = A_1V_1 + A_2V_2$

$$\begin{bmatrix} \bar{u} \\ \bar{v} \\ \bar{p} \end{bmatrix} = \mathbf{B_1}\begin{bmatrix} i \\ 1 \\ -i(U_{-\infty} - \omega/\alpha) \end{bmatrix} e^{\alpha y} + \mathbf{B_2}\begin{bmatrix} i\gamma/\alpha \\ 1 \\ 0 \end{bmatrix} e^{\gamma y} \tag{20}$$

or in vector form $\mathbf{V}_- = \mathbf{B_1V_{-1}} + \mathbf{B_2V_{-2}}$, where $\gamma = \sqrt{\alpha^2 + i\alpha Re(U_{\pm\infty} - \omega/\alpha)}$. The terms $\mathbf{V_{\pm 1}}$ is the inviscid solution and $\mathbf{V_{\pm 2}}$ is the viscuos solution.

Using the asymptotic solution as initial conditions, a numerical solution for system may be found by shooting method. Thereunto, α must be fixed and then a marching method is used to determinate the solution in $y = 0$. The correct eigenvalue is computed when the condition in Eq.(21) is true.

$$\mathbf{A_1V_1} + \mathbf{A_2V_2} = \mathbf{B_1V_{-1}} + \mathbf{B_2V_{-2}} \tag{21}$$

For viscous problems the invidcid and viscous solutions have discrepant magnitudes, which creates stifness. Hence, orthonarmalization method must be applied for a accurate results. In this article, a fourth order Runge Kutta method, and a Gram-schimid method for orthonarmalization were employed.

2.2 Matrix forming

With matrix forming the spatial derivates in equations (13), (14) e (15) were discretized, creating the generalized matrix eigenvalue problem $\mathbf{A}\mathbf{v} = \omega\mathbf{B}\mathbf{v}$. Where $\mathbf{v}$ is the vector $\mathbf{v} = [\bar{u}, \bar{v}, \bar{p}]^T$, and

$$\mathbf{A} = \begin{bmatrix} L_{1D} & U_y & i\alpha \\ 0 & L_{1D} & D_y \\ i\alpha & D_y & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{22}$$

where $L_{1D} = i\alpha U - \frac{1}{Re}(D_{yy} - \alpha^2)$, D_y being the first and D_{yy} the second derivative respect to y direction, Theofilis (2011).

For the pressure boundary condition equation (15), where the boundary condition $\bar{u} = \bar{v} = 0$ for $y \to \pm\infty$ was applied to generate:

$$\frac{\partial \bar{p}}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 \bar{v}}{\partial y^2}\right) \tag{23}$$

In this paper, second order central finite difference is used for discretizatiom, linear algebra subroutine ZGGEV of LAPACK, based on the QZ algorithm is used for the solution of the eigenvalue problem.

2.3 Time stepping

The normal mode assumption now is restricted to

$$\tilde{u}(x, y, t) = \bar{u}(y, t)exp(i\alpha x) \tag{24}$$

$$\tilde{v}(x, y, t) = \bar{v}(y, t)exp(i\alpha x) \tag{25}$$

$$\tilde{p}(x, y, t) = \bar{p}(y, t)exp(i\alpha x) \tag{26}$$

leading to

$$\mathrm{i}\alpha\bar{u} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{27}$$

$$\frac{\partial \bar{u}}{\partial t} + i\alpha\bar{p} + i\alpha U\bar{u} + \bar{v}\frac{\partial U}{\partial y} = -\frac{1}{\mathrm{Re}}\left(\alpha^2\bar{u} - \frac{\partial^2 \bar{u}}{\partial y^2}\right) \tag{28}$$

$$\frac{\partial \bar{v}}{\partial t} + \frac{\partial \bar{p}}{\partial y} + i\alpha U\bar{v} = -\frac{1}{\mathrm{Re}}\left(\alpha^2\bar{v} - \frac{\partial^2 \bar{v}}{\partial y^2}\right) \tag{29}$$

which, in matrix form, becomes

$$\mathbf{R}\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{E_i}}{\partial y} - \frac{\partial \mathbf{E_v}}{\partial y} + \mathbf{F} = 0 \tag{30}$$

or

$$\mathbf{R}\frac{\partial \mathbf{Q}}{\partial t} + \mathbf{G}(\mathbf{Q}) = 0 \quad \text{putting} \quad \mathbf{G}(\mathbf{Q}) = \frac{\partial \mathbf{E_i}}{\partial y} - \frac{\partial \mathbf{E_v}}{\partial y} + \mathbf{F} \tag{31}$$

where $\mathbf{R}$ is the diagonal matrix given by $\mathbf{R} = \text{diag}\ (0, 1, 1)$, and

$$\mathbf{Q} = \begin{bmatrix} \bar{p} \\ \bar{u} \\ \bar{v} \end{bmatrix} \quad \mathbf{E_i} = \begin{bmatrix} \bar{v} \\ 0 \\ \bar{p} \end{bmatrix} \quad \mathbf{E_v} = \begin{bmatrix} 0 \\ \frac{1}{Re}\frac{\partial \bar{u}}{\partial y} \\ \frac{1}{Re}\frac{\partial \bar{v}}{\partial y} \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} i\alpha\bar{u} \\ \alpha(iU + \frac{\alpha}{Re})\bar{u} + \bar{v}\frac{\partial U}{\partial y} + i\alpha\bar{p} \\ \alpha(iU + \frac{\alpha}{Re})\bar{v} \end{bmatrix}$$

The system in Eq.(31) it can not be solved directly because the matrix $\mathbf{R}$ is singular. Therefore, the variable $\bar{p}$ cannot be updated in time. In this article, dual time stepping has been used to solve this singularity problem Merkle and Athavale (1987). A pseudo time τ is introduced in the Eq.(31) generating the partial differential system

$$\frac{\partial \mathbf{Q}}{\partial \tau} + \mathbf{R}\frac{\partial \mathbf{Q}}{\partial t} + \mathbf{G}(\mathbf{Q}) = 0 \tag{32}$$

The steady state solution of Eq.(32) in pseudo time reduces precisely to the solution of Eq.(31). For temporal discretization in physical time the crank-Nicholson was employed. In pseudo time an Explicit Euler scheme was used whereas for spatial discretization a second order central finite difference scheme was used. The discretized equation is residual form is given by

$$\Delta \mathbf{Q^{p+1}} = \frac{\Delta \tau}{\Delta t}\mathbf{R}(\mathbf{Q^p} - \mathbf{Q^n}) - (\theta \mathbf{G}(\mathbf{Q^p}) + (1-\theta)\mathbf{G}(\mathbf{Q^n}))\,\Delta\tau \tag{33}$$

where p and n are respectively the superscript for pseudo and physical time. The same pressure boundary condition of Matrix forming method (equation 23) was used in this approach.

3. Results

This section presents some results about temporal stability analisys of all three metodology to evaluate advantages and disadvantages of each method. However, it is first presented the case test.

3.1 Case test

The mixing layer problem was chosen to carry out the study of the temporal stability analysis, and consequently, the comparison among the three approaches. The database for this study were taken from Seo (1995). The hyperbolic tangent profile in Eq. (34) was used as base flow.

$$U(y) = 1 - R\tanh(y) \quad \text{where} \quad R = \frac{U_\infty - U_\infty}{U_\infty + U_\infty} \tag{34}$$

3.2 Shooting method

To validate the shooting method a spatial stability analisys of the planar mixing layer was performed and compared with the results of Seo (1995). The computed two dimensional spatial amplification it was obtained for the hyperbolic tangent profile with velocity ratio $R = 0.31$ into Eq. (34). The result of this simulation is plotted in the Figure 1. A verification of order was also performed to ensure the second order accuracy of the code. The result is plotted in Figure 2.

3.3 Matrix forming

After validating the spatial shooting method code, small adaptations were performed to create the temporal stability code. Then, the results of temporal stability analysis by shooting method have been used for validate the matrix forming code for the same planar mixing layer. This comparison is plotted in the Figure 3 and the numerical verification order is plotted in Figure 4.

3.4 Time stepping

Equations 33 were marched for a long time, 100 seconds, with $dt = 10^{-4}$. Data was extracted at the point $y = 3$ in the domain, where it was possible to observe the oscillatory behavior of the variables $\mathbf{Q}$. Figure 5, shows the behavior for velocity disturbance $\bar{u}$. The same behavior has been observed for others disturbances.

Figure 6 shows a zoom of Figure 5 for the physical time of up to 40 segunds. It was observed in this figure, the presence of other oscillating frequencies which are excited due to numerical. For large simulation times these frequencies become negligible, since the dominant mode grows faster than everything else..

The results of amplification rate and frequency are presented in the next section. These values were obtained through the interpolation of the dataset seen in figure 5. The FIT tool of software Mathematica has been used to perform the task.

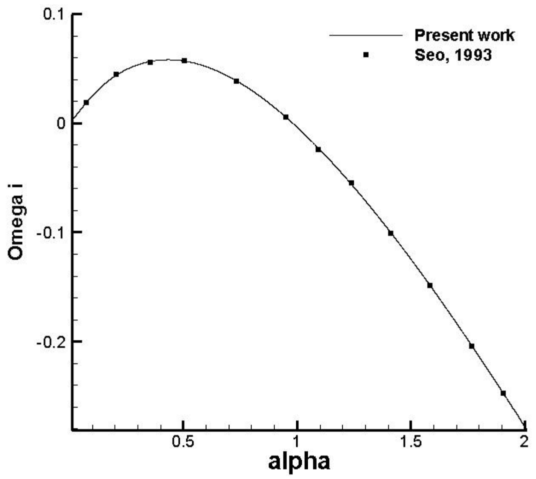

Figure 1. Amplification rate (α) *vesus* frequency (ω) for spatial analysis.

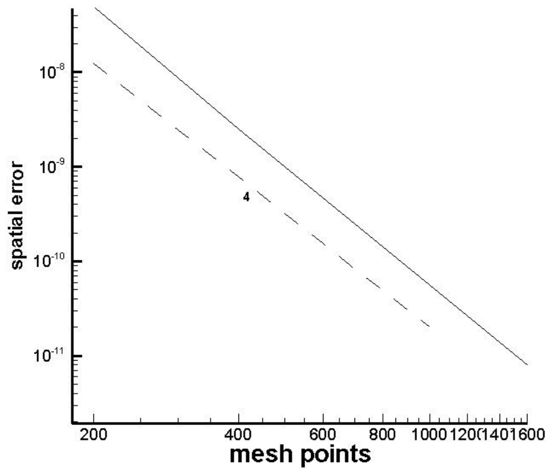

Figure 2. Spatial numerical order of shooting method

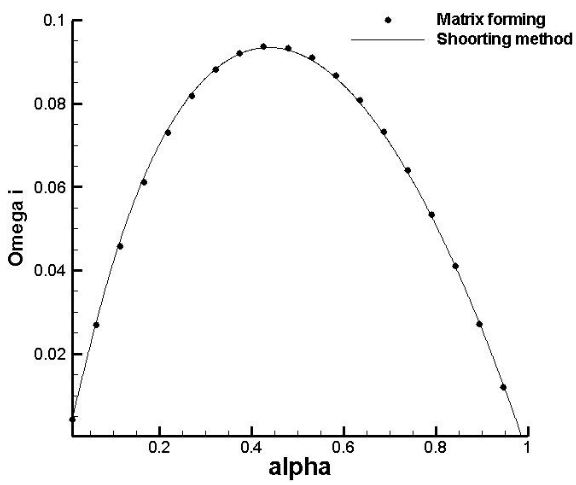

Figure 3. Amplification rate (α) *vesus* frequency (ω) for temporal analysis.

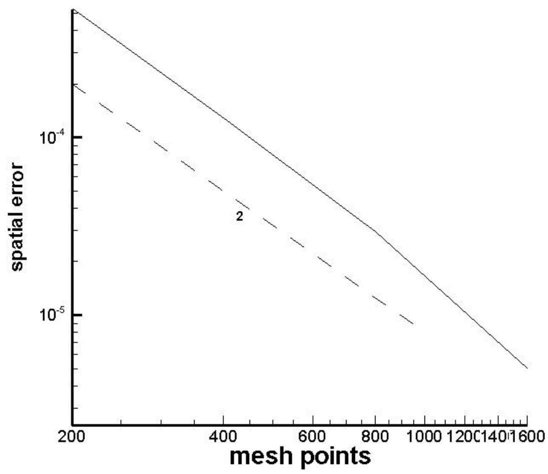

Figure 4. Spatial numerical order of matrix forming.

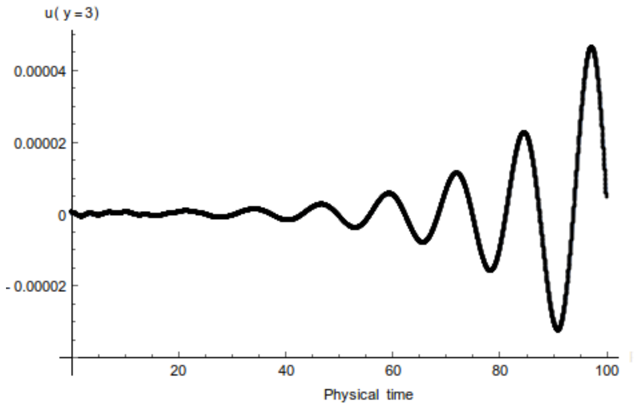

Figure 5. Velocity disturbance $\bar{u}$ over time.

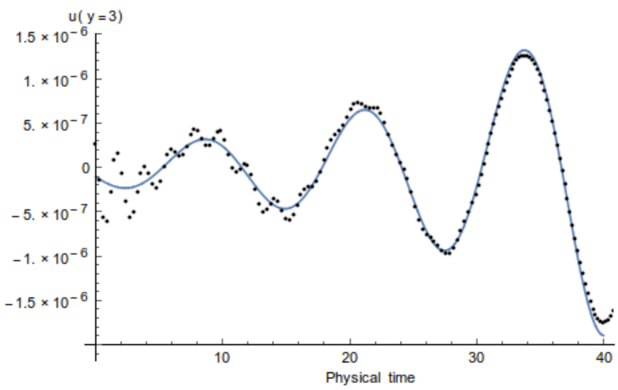

Figure 6. Zoom of velocity disturbance $\bar{u}$ up to 40 seconds.

3.5 Comparative analysis of all three methods

Table 1 shows results for amplificate rate and frequency for different approach. Has been observed an agreement on results for three methodology. It is known from the literature that shooting method is the most accurate approach. However, It is not always easy to find a good initial guess for locate all the important modes. The matrix forming approach find all important modes, but it is required a large allocation of memory. The time stepping approach does not require large memory allocation but, on the other hand, it needs long simulation times.

Table 1. Amplificate rate and frequency for different approach.

$\alpha = 0.5$	Shooting method	Matrix forming	Time stepping
ω_r	0.500000	0.500000	0.499298
ω_i	0.056430	0.056379	0.056131
$\alpha = 0.35$	Shooting method	Matrix forming	Time stepping
ω_r	0.350010	0.349999	0.348087
ω_i	0.055663	0.055461	0.055758

4. Conclusions

In this paper three different eigenvalue solvers for linear stability analysis were presented. The stability analisys of mixing layer was simulated and results for three methodologies are shown to be in agreement. It was evident that the matrix forming and time stepping may be used alternatively to the traditional shooting method. However, the choice of approach depends on the complexity of the problem and a more detailed study of time stepping it is necessary to verify computer costs.

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