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ANALYTICAL SOLUTION FOR HEAT CONDUCTION IN MOVING SOLIDS USING GREEN FUNCTIONS

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Abstract. *Thermal problems involving moving sources occur in various engineering applications, such as welding processes, heat treatment furnaces and other treatments. Typically, in these cases, numerical software are used due to the complexity to modelling of the heat diffusion equation and boundary conditions. This paper propose a mathematical analysis and analytical solution of a 1D transient thermal model based on Green functions, verification and validation to the analytical solution, comparison with numerical solutions and an example of welding process considering a moving solid at a constant speed.*

Keywords: *moving solid, heat conduction, analytical solution, Green function.*

1. INTRODUCTION

Analytical solutions in heat conduction is a strong and robust tool for the temperature calculation in a thermal problems. Analytical solutions has this force due to the fact not to rely on approximations, estimates or fails of truncations, because they are accurate. And easy computational implementation and the calculation process when implemented have low computational cost.

Obtaining analytical solutions to this problem requires more elaborate procedures to those used in fixed problems because of the additional terms that arise in the governing equation. The proposed method for solving partial differential equations with the boundary conditions modeling the problem will be Green's function method.

Thus, this work propose an study and develop an analytical solution and mathematical analysis of a 1D heat conduction problem in a body heated by a moving heat source. In this case exemplified a welding problem.

It also objective of this work is to develop verification from the solutions obtained analytically with problems of finite or semi-infinite geometries in order to ensure reliability and safety of the calculated solutions.

2. MATHEMATICAL MODEL

First we propose an approach for the modelling, change of variables and method for analytical solution using Green function. The $1D$ mathematical model that describes heat conduction in a heated solid by a moving source is:

$$k\frac{\partial^2 T}{\partial x^2} + \frac{1}{k}g(x,t) = \rho c_p \left(\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x}\right) \tag{1}$$

where v is moving speed, T temperature, k coefficient of thermal conductivity, ρ specific mass, C_p specific heat, t time, $g(x,t)$ heat generation coefficient.

The boundary conditions of the first kind are given by:

$$\begin{aligned} T(0,t) = &\quad T_{x1}, \\ T(L,t) = &\quad T_{x2} \end{aligned} \tag{2}$$

and boundary conditions of second and third kinds are given by:

$$\begin{aligned} -k\frac{\partial T}{\partial x}\Big|_{x=0} &= h_{x1}[T_\infty - T(0,t)] + q_{x1}, \\ k\frac{\partial T}{\partial x}\Big|_{x=L} &= h_{x2}[T(L,t) - T_\infty] + q_{x2}, \end{aligned} \tag{3}$$

and initial condition:

$$T(x,0) = F(x). \tag{4}$$

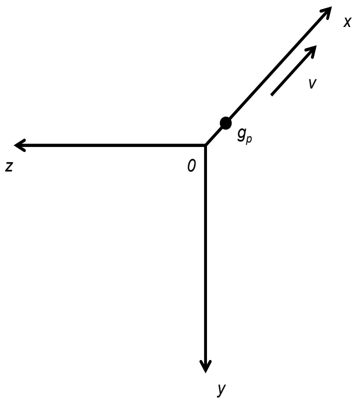
According (Özişik, 1993), we use an variable $\xi = x - vt$, and consider the internal generation term $g(x,t) = constant$ in any position between $0 < x < L$, and spatial variation to be a Dirac delta function, we have

$$g(x,t) = g(x,t)\delta(x - vt), \tag{5}$$

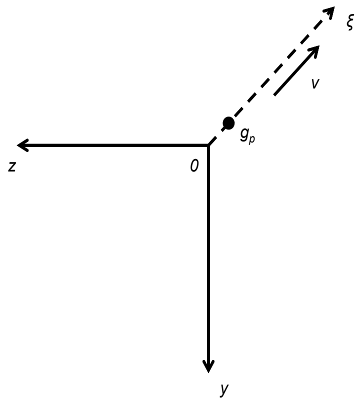
after change of variable, Eq.(5) become

$$g(\xi,t) = g(\xi,t)\delta(\xi - P_\xi), \tag{6}$$

the consequence of variable change is the moving heat generation term become fixed in P_ξ point. The next figures illustrated this operation.



(a)



(b)

Figure 1. (a) Moving heat source; (b) Fixed heat source .

Thus Eq.(1) take the form

$$k\frac{\partial^2 T}{\partial \xi^2} + \frac{1}{k}g(\xi,t) = \rho c_p\left(\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial \xi}\right), \tag{7}$$

and boundary conditions of first kind:

$$T(0,t) = T_{\xi 1}, \tag{8}$$
$$T(L,t) = T_{\xi 2}, \tag{9}$$

and boundary conditions of second and third kids are:

$$-k\frac{\partial T}{\partial \xi}\Big|_{\xi=0} = h_{\xi 1}[T_\infty - T(0,t)] + q_{\xi 1}, \tag{10}$$
$$k\frac{\partial T}{\partial \xi}\Big|_{\xi=L} = h_{\xi 2}[T(L,t) - T_\infty] + q_{\xi 2}, \tag{11}$$

and initial condition:

$$T(\xi,0) = F(\xi). \tag{12}$$

To apply the Green function method for obtaining the analytical solution, we performed a change of variable from T to W using

$$exp\left(\frac{v\xi}{2\alpha}-\frac{v^2t}{4\alpha}\right), \tag{13}$$

then T will be calculated by

$$T(\xi,t)=W(\xi,t)exp\left(\frac{v\xi}{2\alpha}-\frac{v^2t}{4\alpha}\right). \tag{14}$$

To change of variable, according (Beck and McMasters, 2004), we first do

$$\begin{aligned}
\frac{\partial T}{\partial x} &= \frac{\partial W}{\partial x}exp\left(\frac{vx}{2\alpha}-\frac{v^2t}{4\alpha}\right)+W\frac{v}{2\alpha}exp\left(\frac{vx}{2\alpha}-\frac{v^2t}{4\alpha}\right)\\
\frac{\partial^2 T}{\partial x^2} &= \frac{\partial^2 W}{\partial x^2}exp\left(\frac{vx}{2\alpha}-\frac{v^2t}{4\alpha}\right)+\frac{\partial W}{\partial x}\frac{v}{2\alpha}exp\left(\frac{vx}{2\alpha}-\frac{v^2t}{4\alpha}\right)\\
&\quad+\frac{\partial W}{\partial x}\frac{v}{2\alpha}exp\left(\frac{vx}{2\alpha}-\frac{v^2t}{4\alpha}\right)+W\frac{v^2}{4\alpha}exp\left(\frac{vx}{2\alpha}-\frac{v^2t}{4\alpha}\right),\\
\frac{\partial T}{\partial t} &= \frac{\partial W}{\partial t}exp\left(\frac{vx}{2\alpha}-\frac{v^2t}{4\alpha}\right)-W\frac{v^2}{4\alpha}exp\left(\frac{vx}{2\alpha}-\frac{v^2t}{4\alpha}\right)
\end{aligned} \tag{15}$$

Replace Eq.(15) in Eq.(1)

$$\begin{aligned}
&exp\left(\frac{vx}{2\alpha}-\frac{v^2t}{4\alpha}\right)\left(\frac{\partial^2 W}{\partial x^2}+2\frac{\partial W}{\partial x}\frac{v}{2\alpha}+W\frac{v}{4\alpha}\right) \quad = \\
&\frac{1}{\alpha}\left\{exp\left(\frac{vx}{2\alpha}-\frac{v^2t}{4\alpha}\right)\left(\frac{\partial W}{\partial t}-W\frac{v^2}{4\alpha}\right)+\right.\\
&\left.v\left[exp\left(\frac{vx}{2\alpha}-\frac{v^2t}{4\alpha}\right)\left(\frac{\partial W}{\partial x}+W\frac{v}{2\alpha}\right)\right]\right\}
\end{aligned} \tag{16}$$

After simplify Eq.(16) we have:

$$k\frac{\partial^2 W}{\partial x^2}=\rho c_p\frac{\partial W}{\partial t}, \tag{17}$$

similarly applying Eq.(15)-(15) we get the boundary conditions

$$W(0,t) \quad = \quad T(0,t)exp\left(\frac{v^2t}{4\alpha}\right), \tag{18}$$

$$W(L,t) \quad = \quad T(L,t)exp\left(-\frac{vL}{2\alpha}+\frac{v^2t}{4\alpha}\right), \tag{19}$$

boundary conditions of first kind,

$$-k\frac{\partial W}{\partial x}\bigg|_{x=0}+W|_{x=0}\left(\frac{-kv}{2\alpha}+h_{x1}\right)=exp\left(\frac{v^2t}{4\alpha}\right)(h_{x1}W_\infty+q_{x1}), \tag{20}$$

$$k\frac{\partial W}{\partial x}\bigg|_{x=L}+W|_{x=L}\left(\frac{kv}{2\alpha}+h_{x1}\right)=exp\left(\frac{-v^2t}{4\alpha}+\frac{v^2t}{4\alpha}\right)(h_{x2}W_\infty+q_{x2}), \tag{21}$$

boundary conditions of second and third kinds.

And initial condition are given by

$$W(x,0)=F(x)exp\left(-\frac{vx}{2\alpha}\right). \tag{22}$$

It is observed that boundary conditions of second kind become third kind, while boundary conditions of first and third kinds remain the same

The terms

$$h_{eff} = W|_{x=0}\left(\frac{-kv}{2\alpha} + h_{x1}\right), \tag{23a}$$

$$h_{eff} = W|_{x=L}\left(\frac{kv}{2\alpha} + h_{x1}\right), \tag{23b}$$

According (Beck and McMasters, 2004) are effective heat transfer coefficient.

In next section we propose the unidimensional solution for one boundary condition case, verification of analytical solution and comparison with numerical results.

3. ANALYTICAL SOLUTION FOR X11 CASE

Figure (2) show the problem that will be analysed here

Figure 2. $X11$ *problem.*

The equation in the variable W and ξ that modelling this case are given by:

$$\frac{\partial^2 W}{\partial \xi^2} + \frac{1}{k} g_p \delta(\xi - P_\xi) exp\left(\frac{v\xi}{2\alpha} - \frac{v^2 t}{4\alpha}\right) = \frac{1}{\alpha}\frac{\partial W}{\partial t}, \tag{24}$$

under the boundary conditions of first kind:

$$W(0,t) = 0, \tag{25}$$

$$W(L,t) = 0, \tag{26}$$

and the initial condition

$$W(\xi, 0) = 0. \tag{27}$$

The Green's functions solution is proposed by:

$$W(\xi,t) = \frac{\alpha}{k}\int_{\tau=0}^{t}\int_{\xi'=0}^{L} G(\xi,t|\xi',\tau) g_p \delta(\xi' - P) exp\left(\frac{v\xi'}{2\alpha} - \frac{v^2\tau}{4\alpha}\right) d\xi' d\tau. \tag{28}$$

The Green Function can be easily found in (Beck *et al.*, 1992, p.584) and take the form:

$$G(\xi,t|\xi',\tau) = \frac{2}{L}\sum_{m=1}^{\infty} exp\left(-\frac{m^2\pi^2\alpha(t-\tau)}{L^2}\right) sin\left(\frac{m\pi\xi}{L}\right) sin\left(\frac{m\pi\xi'}{L}\right), \tag{29}$$

Substituting Eq.(29) in Eq.(28) and using the property:

$$\int_{-\infty}^{+\infty} F(x)\delta(x-a)dx = F(a), \tag{30}$$

$$\begin{aligned} W(\xi,t) = \; & \frac{\alpha}{k}\int_{\tau=0}^{t}\frac{2}{L}\sum_{m=1}^{\infty} exp\left(-\frac{m^2\pi^2\alpha(t-\tau)}{L^2}\right) sin\left(\frac{m\pi\xi}{L}\right) \times \\ & sin\left(\frac{m\pi P}{L}\right) g exp\left(\frac{vP_\xi}{2\alpha} - \frac{v^2\tau}{4\alpha}\right) d\tau. \end{aligned} \tag{31}$$

after time integration, the solution is given by

$$W(\xi,t) = \frac{2\alpha g_p}{Lk}\sum_{m=1}^{\infty} sin\Big(\frac{m\pi\xi}{L}\Big) sin\Big(\frac{m\pi P_\xi}{L}\Big) exp\Big(\frac{vP_\xi}{2\alpha}\Big)\left(\frac{exp\Big(\frac{-vt^2}{4\alpha}\Big) - exp\Big(\frac{-m^2\pi^2\alpha t}{L^2}\Big)}{\frac{m^2\pi^2\alpha}{L^2}+\frac{v^2}{4\alpha}}\right), \tag{32}$$

The solution in T variable are

$$T(\xi,t) = W(\xi,t)exp\Big(\frac{v\xi}{2\alpha} - \frac{v^2 t}{4\alpha}\Big), \tag{33}$$

This problem will be verified using other analytical solution, in this case, it is using a solution of fixed source, and compare with this solution here calculate. The analytical solution for a fixed source can be easily found in literature and is given by

$$W_f(\xi,t) = \frac{2\alpha g_p}{Lk}\sum_{m=1}^{\infty} sin\Big(\frac{m\pi\xi}{L}\Big) sin\Big(\frac{m\pi P_\xi}{L}\Big)\left(\frac{1 - exp\Big(\frac{-m^2\pi^2\alpha t}{L^2}\Big)}{\frac{m^2\pi^2\alpha}{L^2}+\frac{v^2}{4\alpha}}\right), \tag{34}$$

the subscript f in Eq.(34) indicate that is solution for a fixed heat source.

We will easily note that the solution of moving source is algebraically equal to solution of fixed heat source when $v \to 0$, this means, that the two solutions they behave in the same way, which is an equivalence which enables verification of the computed solution.

4. RESULTS

For example we use solution given by Eq.(32) in MATLAB software for simulate a welding process, in 1D geometry. We use the physical properties: length $L = 0.1m$, solid speed $v = 0.001m/s$, thermal conductivity $k = 80.2W/mK$, thermal diffusivity $\alpha = 23.1 \times 10^{-6}m^2/$ and an heat generation by the welding process $g(x,t) = 1 \times 10^5 W/m^3$ in fixed point $\xi = P = 0.02m$. The times of simulation are 0, 10, 30, 60 and 120 seconds.

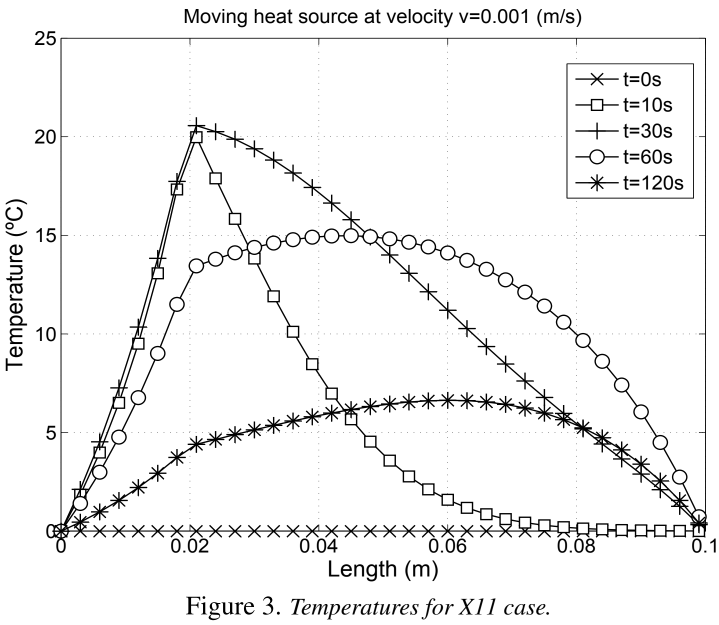


Figure 3. *Temperatures for X11 case.*

Figure (4). show the thermal behavior, note the speed effect for each time step.

5. NUMERICAL COMPARISON

The same welding problem was implemented in commercial software COMSOL Multphysics using the same conditions and thermal proprieties, which will be compared to results obtained analytically (previous section). For this comparison we return in x variable, because the COMSOL do not make this variable change. The results are.

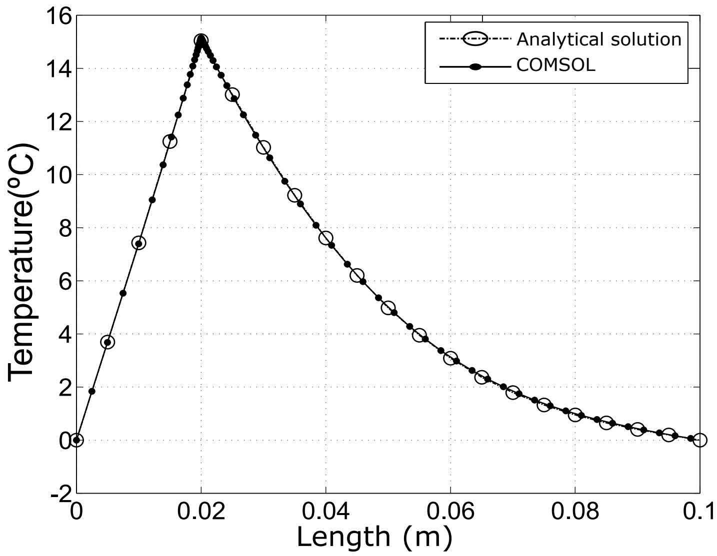


Figure 4. *COMSOL × analytical solution.*

We can observe in Fig.(4) that two solutions, analytical and numerical show the same behaviour in time step $60s$. The absolute difference between analytical and numerical solution are less than $0.1C$, or 0.62% which shows a good approximation between the two solutions.

6. CONCLUSION

Was proposed in this paper the approach in mathematical model, analysis, analytical solution and numerical comparison of an thermal problem involving an welding process. We concluded that analytical solution showed is a strong tool and reliable for thermal analysis when involve moving or fixed heat sources, surfaces exposed to a heat flux, thermal insulation or any boundary condition. And also can be used for comparison and validation of numerical algorithms, due to accuracy, easy implementation and reliability, present in analytical solutions.

7. ACKNOWLEDGEMENTS

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