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SIMULATION OF THE STEADY-STATE CONDUCTION-RADIATION HEAT TRANSFER PROCESS USING FINITE ELEMENT METHOD

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Abstract. *This work deals the steady-state conduction-radiation heat transfer process in rigid, opaque and convex body at rest. The phenomenon is mathematically modeled by a linear second order partial differential equation subjected to nonlinear boundary conditions arising from the thermal radiation heat transfer between the body and the environment. The simulation will be carried out by means of a sequence of linear problems, each of them simulated with the aid of a finite element approximation. The approach employs triangular finite elements. Some typical results are show to illustrate the methodology.*

Keywords: *Conduction/Radiation, Numerical Simulation, Finite Element Method*

1. INTRODUCTION

In materials science many constitutive equations are derived assuming constant thermal conductivity, however it is noteworthy that this could lead to considerable errors (Mintsa *et al.*, 2009).

Conduction heat transfer problems are usually simulated assuming temperature independent thermal conductivity. Such approximation strongly simplifies the simulation of the considered problems. However, temperature-dependent thermal conductivity is present in many problems with engineering relevance.

Nanofluids (fluids with suspended ultrafine particles with very small size) have opened a new area of study in heat transfer processes. Recent investigations confirm the potential of nanofluids in enhancing heat transfer required for present age technology. Das *et al.* (2003) details the increase of thermal conductivity with temperature for nanofluids with water as base fluid and particles of Al_2O_3 or CuO as suspension material. Mintsa *et al.* (2009) shows the considerable effect of temperature variation in the value of thermal conductivity for nanofluids and presents the most relevant tracks.

Song *et al.* (2003) presents the standard curves $k(T)$ for monolithic tungsten and its alloys, as well as demonstrate its direct relationship with the addition of alloying elements. Tungsten and tungsten alloys are widely used in high temperature environments where arc ablation or mechanical deformation and damage are the main sources of materials failure. As a refractory metal with the highest melting point, tungsten is technologically attractive and widely used in various fields. Especially, a small coefficient of thermal expansion and good high temperature mechanical properties are attractive characteristics for important structural applications at high temperatures.

In addition to the temperature factor, Song *et al.* (2003) details other factors affecting the non-constant thermal conductivity in tungsten composites base, especially the phase dispersions, impurities, porosity, microcracks, limits of solid grains and grain size.

The subject of this work is the energy transfer process in a rigid, opaque and nonconvex tungsten body surrounded by an atmosphere-free space. This coupled conduction/radiation heat transfer process is an inherently nonlinear phenomenon, where the coupling between conduction and radiation on the body boundary is mathematically represented by a nonlinear relation between the absolute temperature and its exterior normal derivative, in which the unknown is the temperature distribution in the body.

A numerical simulation for this problem employing a finite element approximation will be presented. The solution to the problem is given by the limit of a sequence whose elements are obtained from the minimization of functionals like the usual ones employed in mechanics. Each of these elements represents the solution of a heat transfer problem with a prescribed external thermal radiant source (instead of a temperature-dependent thermal radiant source). The limit of the sequence is the solution of the considered problem with a temperature dependent thermal radiant source.

2. MATHEMATICAL MODEL

One of the simplest classes of problems involving energy transmission is conduction in a stationary solid. Since the velocity vector is identically zero, for steady state in the region Ω the differential energy balance reduces to (Slattery, 1999)

$$div(k\nabla T) + Q = 0 \tag{1}$$

whith Q the internal energy generation (per unit time and volume) and the thermal conductivity k of the material, positive and scalar quantity. This problem uses non-constant tungsten conductivity depending on the temperature.

The coupling between the conduction heat transfer, convection and thermal radiant heat transfer is done on the body boundary. For a black convex body (emittance equal to 1) surrounded by an atmosphere-free space, since there is no jump in the normal energy flux across the boundaries $\partial\Omega$, have nonlinearity on the boundary condition, such as (Siegel and Howell, 1992):

$$-k\nabla T.\mathbf{n} = \sigma T^4 + hT - hT_\infty - \theta \quad on \quad \partial\Omega \tag{2}$$

in wich $\mathbf{n}$ is the unit outward normal vector, σ is the Stefan-Boltzmann constant, T is the absolute temperature, T_∞ is a temperature of reference (in general, the enviroment temperature), h is the convection heat transfer coefficient and θ is the effect of an external thermal radiant source.

In the Eq.(2) will be employed the term $|T|^3.T$ in place of the term T^4 in order to ensure the coerciveness of the operator in infinite dimension, preserving the physical structure of the phenomenon.

2.1 Kirchhoff transformation

If k and Q are functions of space only, Eq.(1) becomes a linear differential equation with variable coefficients. The solution of such an equation requires no additional mathematics. But if k is dependent on temperature and independent of space, however, Eq.(1) becomes non-linear and difficult to solve. Usually, numerical methods have to be employed. A number of analytical methods are also available (Arpaci, 1966).

Kirchhoff's method says that Eq.(1) may be reduced to a linear differential equation by introducing a new temperature ω related to the temperature T of the problem by the Kirchhoff transformation:

$$\omega = f(T) = \int_{T_0}^{T} k(\epsilon)d\epsilon \tag{3}$$

For the tungsten empirical curve Fig.(1), there was obtained with the help of least squares method the adjusted curve $k(\epsilon)$ to be integrated in kirchhoff transformation.

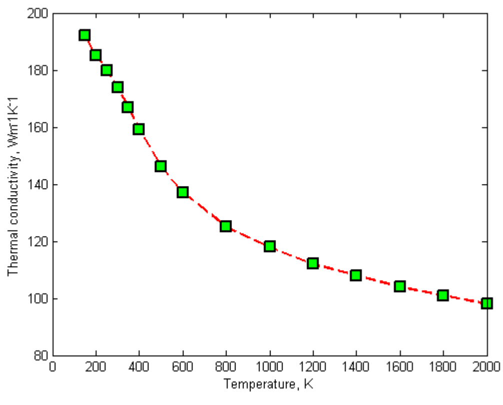

Figure 1. Tungsten thermal conductivity

The kirchhoff transformation is showed below

$$\omega = f(T) = \int_{T_0}^{T} k(\epsilon)d\epsilon = \frac{c}{(d+1)}.T^{(d+1)} \tag{4}$$

being c and d were obtained with least square method in MatLab curve fitting.

According to

$$grad\omega = kgradT \tag{5}$$

the following partial differential equation can be written.

$$div[grad\omega] + Q = 0 \quad in \quad \Omega \tag{6}$$

The inverse of the above Kirchhoff transformation can be easily obtained as

$$T = f^{-1}(\omega) = e^{\frac{1}{d+1}\log\frac{\omega(d+1)}{c}} \tag{7}$$

with $\lambda = \frac{1}{d+1}\log\frac{\omega(d+1)}{c}$, the boundary condition becomes

$$-(grad\omega).\mathbf{n} = h.e^{\lambda} + \sigma|e^{\lambda}|^3.e^{\lambda} \quad on \quad \partial\Omega. \tag{8}$$

The positiveness of the thermal conductivity ensures that ω is an increasing function of T, while T is an increasing function of ω.

2.2 Solution formulation

In order to ensure the existence of a minimum, solution for (6), is necessary and sufficient to show that I, is continuous, convex and coercive functional (Gama, 1991) and (Gama, 2004).

Assuming that Q is constant, the solution of problem in Eq.(3) may be reached as the limit of a sequence whose elements ate obtained form the minimization of a quadratic functional. In other words, the solution ω of problem given by Eq.(3) is given by

$$\omega = \lim_{i\to\infty} \Phi_i \tag{9}$$

in wich the elements of sequence $[\Phi_0, \Phi_1, \Phi_2, ..., \Phi_i]$ are obtained from

$$div\big(grad\Phi_{i+1}\big) + Q = 0 \quad in \quad \Omega \tag{10}$$

$$-\big(grad\Phi_{i+1}\big)\mathbf{n} = \alpha\Phi_{i+1} - \beta_i \quad on \quad \partial\Omega \tag{11}$$

$$\beta_i = \alpha\Phi_{i-1} - \left(\sigma\left|\Phi_{i-1}\right|^3\Phi_{i-1} - h\left(\Phi_{i-1} - T_\infty\right)\right) \tag{12}$$

$i = 0, 1, 2, ...$

where α is sufficiently large positive constant and $\Phi_0 \equiv 0$. This constant is evaluated from an priori estimate for the upper bound of the solution in Eq.(3) and ensures a bounded and nondecreasing sequence $[\Phi_0, \Phi_1, \Phi_2, ..., \Phi_i]$ (Gama *et al.*, 2013).

The above problem is equivalent to the minimization of the following quadratic functional in Eq.(7)

$$I_{i+1} = \int_\Omega (\nabla v).(\nabla v)dV - 2\int_\Omega QvdV + \int_{\partial\Omega} \alpha v^2 dA - 2\int_{\partial\Omega} \beta_i v dA \tag{13}$$

Gama (2004) describes the variational formulation of the problem addressed that may be mathematically represented by the minimization of a functional coercive. The existence of this minimum principle provides an easy and accurate tool for numerical simulations of such heat transfer phenomena.

In other words, the minimum of I_{i+1} is reached for the field $v = \Phi_{i+1}$ which satisfies in Eq.(6). It is to be noticed that Φ_i is a known function, when we look for the minimum of the functional. The tools employed for minimizing $I_{i+1}[v]$ are exactly the same ones employed for solving linear heat transfer problems.

The presented functinal is convex (ensuring the uniqueness of the solution, provided it exists) and coercive (ensuring the existence of at least on solution).

The constant α must satisfy the following relationship

$$\alpha \geq \frac{d}{d\eta}|\eta|\left(h.e^{\lambda} + \sigma|e^{\lambda}|^3.e^{\lambda}\right) \quad for \quad 0 < \eta < \sup_{\partial\Omega} \omega \tag{14}$$

that is equivalent

$$\alpha \geq \frac{h}{\delta}, \quad \delta = e^{\lambda} + \sigma|e^{\lambda}|^3.e^{\lambda} \tag{15}$$

It is remarkable that any α satisfying (15) ensures convergence, but this is a sufficient condition. The convergence may be reached even for values of α which do not satisfy (15).

The temperature is given by:

$$T = \lim_{i \to \infty} \Phi_i \quad in \quad \overline{\Omega} \tag{16}$$

The elements of the sequence $[\Phi_0, \Phi_1, \Phi_2, ..., \Phi_i]$ are obtained from the minimization of a continuous, convex and coercive functional. this assures the existence and the uniqueness of each element and provides an efficient way for obtaining them. In addition, it is assured the existence and the uniqueness of the sequence $[\beta_0, \beta_1, \beta_2, ..., \beta_i]$.

The sequences $[\Phi_0, \Phi_1, \Phi_2, ..., \Phi_i]$ and $[\beta_0, \beta_1, \beta_2, ..., \beta_i]$ are, for each i, non-decreasing, provides only the solution with physical sense. This fact allows us to conclude that the solution is the unique non-negative, since, from classical thermodynamics, the absolute temperature must be a non-negative value real field, we conclude that Eq.(1) and Eq.(10) are thermodynamically equivalent.

The algorithm associated to the minimization of the functional I provide an efficient procedure for simulating the considered energy transfer phenomena. In addition, this procedure provides only the (desired) solution with thermodynamical sense.

3. NUMERICAL MODEL

In this section a very simple problem presenting an exact solution which can be obtained in a very easy way is considered, in order to lend support and illustrate the previously presented estimates. The problem is the steady-state (bi-dimensional) conduction heat transfer process in a tungsten slab with temperature-dependent thermal conductivity, represented as

$$\Omega \equiv \{(x, y) \in \Re^2 \quad such \quad that \quad 0 < x < 1 \quad and \quad 0 < y < 1\} \tag{17}$$

This problem presents two constants, unlike and localized internal heat sources, the external radiation heat source on body boundary will be assumed null.

Some results obtained with the aid of a finite element approximation are presented in Fig.2, Fig.3 and Fig.4 for $i = 1$ with differents α, showing that more closer to the great α faster convergence occurs. It is to be noticed that the convergence was reached for i=5. In other words, for i=6, i=10, i=100 no significant changes were observed. Gama (1997) demonstrates such analytical convergence process.

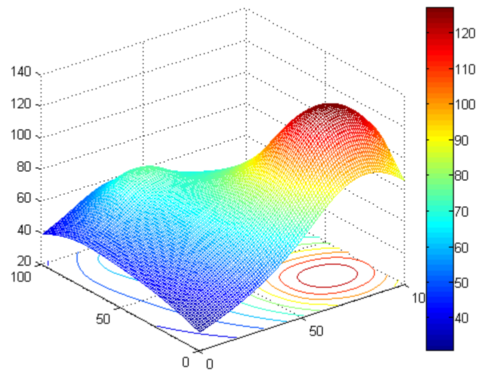

Figure 2. $\alpha = 1, T_{max} = 126,9102$

4. CONCLUSIONS

Besides the solution existence and uniqueness (even outside the physically admissible range) this new model provides additional mathematical advantages. This systematic mechanism for the numerical simulation of a nonlinear problem uses an infinite dimensional results in a context of finite elements (finite dimension) for the solution of a nonlinear problem from a sequence of linear problems. In other words, with a simple and basic tool we obtained the solution of more complex problems. Other problems may be solved using the same methodology in conjunction with other discretizations that can be used in the Finite Element methods and the block storage array using linear system solution methods using preconditioners.

This powerful methodology for solving n-dimensional nonlinear problems displays features such as low storage costs and low computational effort, when compared to other numerical procedures to approximate nonlinear problems.

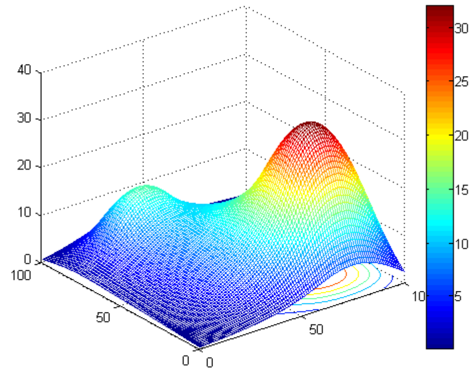

Figure 3. $\alpha = 10, T_{max} = 31,9698$

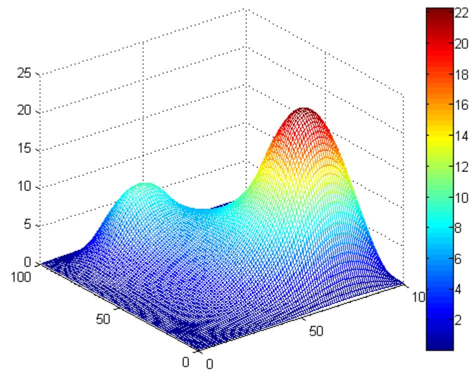

Figure 4. $\alpha = 100, T_{max} = 22,2327$

5. ACKNOWLEDGEMENTS

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