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SIMULATION OF THE NONLINEAR HEAT TRANSFER PROCESS THROUGH A LINEAR SCHEME IN FINITE DIFFERENCES

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Abstract. *In this work the nonlinear conduction-radiation heat transfer process is considered in a two dimensional plane context and simulated by means of a finite difference linear scheme. The original nonlinear problem is regarded as the limit (which always exists) of a sequence of linear problems like the classical conduction-convection ones. Such a limit is reached in an easy way by means of standard procedures, allowing the employment of more realistic hypotheses, like some nonlinear boundary conditions, since the mathematical complexities are not a constraint for simulating the elliptic partial differential equation. This work simulates the steady state heat transfer process in a rigid body subjected to convective and radiative boundary conditions*

Keywords: *Nonlinear Heat Transfer, Sequence of linear problems, Finite Difference*

1. INTRODUCTION

Heat transfer is a global phenomenon and the precise control of its magnitude in different materials and geometries has become increasingly imperative. The influence of the temperature over many fundamental quantities justify the importance of this knowledge.

The advent of thermoelectric devices and semiconductor materials has been useful for large practical applications. An important phenomenon is the thermoelectric refrigeration that occurs when a temperature differential is established between the hot and cold ends of the semiconductor material, in this case a voltage is generated. This voltage is called the Seebeck voltage and it is directly proportional to the temperature differential. The constant of proportionality is referred as the Seebeck coefficient (Riffat and Ma, 2003).

The thermoelectric devices convert thermal energy into electric energy, this phenomenon, discovered in 1821, is called Seebeck effect. Actually, the Seebeck effect is an inverse effect of Peltier effect. Based on this, the thermoelectric devices can also act as power generators. If heat supplied to one junction gives rise to an electric current flow in the circuit, the electrical power will be available. In practice, a large number of such thermocouples are connected electrically in series to form a module (Riffat and Ma, 2003).

Uchida *et al.* (2008) has shown the importance of the Seebeck effect. Recent studies prove the effectiveness of thermal utilization in obtaining energy, taking as principle the precise control of the local temperature. In his study, the generation of electric voltage in a conductor is provoked by a temperature gradient. Its efficiency is represented by the Seebeck coefficient, S, which is defined as the ratio between the generated electric voltage and the temperature difference and is determined by the scattering rate and the density of the conduction electrons. The effect can be exploited, for example, in thermal electric-power generators and for temperature sensing, connecting two conductors with different Seebeck coefficients, a device called thermocouple.

Another important illustration for the temperature effects is the Curie-Weiss law. Arrott (1985) has pointed out that experimentalists has used the Curie-Weiss law to extract magnetic moments from the susceptibility at high temperatures. In the limit of high temperature, the Curie-Weiss law would be applicable to a material with an infinity range of interaction. It is not valid to the limit of high temperature of real materials that have a finite range of interaction, but the generalization of the Curie-Weiss law is valid over a range of temperatures for real materials that are described by the Heisenberg model of ferromagnetism above the Curie temperature T_c.

Emelyanov *et al.* (2002) explicites size effects on the microstructure and physical properties of ferroelectric crystals, thin films, and ceramics, these are presently attracting great interest in view of numerous current and potential applications of ferroelectric materials in the modern microelectronics. Various possible causes for the existence of the crystal-size effects in ferroelectrics were discussed in his literature. For ferroelectric ceramics, the dependence of dielectric response on the grain size was first observed in 1954. Kinoshita and Yamaji (1976) studied the dielectric properties of $BaTiO_3$ ceramics with the grain size down.

Koutroulis *et al.* (2001) showed the relationship between the efficiency in the generation of electricity and the distribution of temperature in the photovoltaic board. Jaboori *et al.* (1991) confirms the importance of local temperature in photovoltaic energy production.

Several heat transfer problems have closed form analytical solutions obtained by Duhamel's Theorem, Green Functions and Laplace Transform. However, many other real problems do not have their respective analytical solutions, making it necessary numerical methodologies with good approximations. The main objective of this work is to present an alternative mathematical description for the heat transfer phenomena. This new description, thermodynamically equivalent to the classical one, is mathematically more convenient, since it allows the construction of the solution from a sequence of linear problems. The new mathematical approach does not require physical restrictions while solving the problem.

2. GOVERNING EQUATIONS

In this section the mathematical description for the steady-state heat transfer process in rigid and opaque body in the rest will be present employing the classical form of the physical laws.

The elliptic partial differential equation bellow will be presented in generic coordinate system.

2.1 Mathematical formulation

The heat equation for steady-state heat transfer process is (Ozisik, 1993):

$$-\nabla.\mathbf{q}(\mathbf{r}) + g(\mathbf{r}) = 0 \;\; in \;\; \Omega \tag{1}$$

in which $\mathbf{q}(\mathbf{r}, t)$ is the conduction heat flux per unit time and unit area and $g(\mathbf{r}, t)$ is the internal energy supply per unit time and unit volume (assumed constant and positive).

The basic law that gives the relationship between the heat flow and the temperature gradient, based on experimental observations, is generally named after the French mathematical physicist Joseph Fourier, who used it in his analytic theory of heat. For a homogeneous, isotropic solid (i.e., material in which thermal conductivity is independent of direction) the Fourier law is given in the form

$$\mathbf{q}(\mathbf{r}) = -k\nabla T(\mathbf{r}) \tag{2}$$

where the temperature gradient is a vector normal to the isothermal surface, the heat flux vector $\mathbf{q}(\mathbf{r}, t)$ represents heat flow per unit time, per unit area of the isothermal surface in the direction of the decreasing temperature, and k is called the thermal conductivity of the material which is a positive, scalar quantity (assumed constant).

The coupling between the conduction heat transfer, convection and thermal radiant heat transfer is done on the body boundary. For a black convex body ($\epsilon = 1$) surrounded by an atmosphere-free space, since there is no jump in the normal energy flux across the boundaries $\partial\Omega$, we must have nonlinearity on the boundary condition, such as:

$$-k\nabla T.\mathbf{n} = \sigma|T|^3 T + h(T - T_\infty) - c \;\; on \;\; \partial\Omega \tag{3}$$

in wich $\mathbf{n}$ is the unit outward normal vector, σ is the Stefan-Boltzmann constant, T is the absolute temperature, T_∞ is a temperature of reference (in general, the enviroment temperature), h is the convection heat transfer coefficient and c is the effect of an external thermal radiant source.

2.2 Methodology of solution

The nonlinear problem presented in Eq.(3) will be solved starting from the limit of a sequence whose elements are linear problem solutions.

Gama (1997) explains and prove the constructing the sequences $[\Psi_0, \Psi_1, \Psi_2, ..., \Psi_i]$ with the limit of this sequence satisfing the Eq. (1), such as:

$$div(\nabla\Psi_i) + \frac{g(\mathbf{r})}{k} = 0 \quad in \quad \Omega \tag{4}$$

$$-k\frac{\partial\Psi_i}{\partial x} = \gamma\Psi_i - \left(\gamma\Psi_{i-1} - \sigma|\Psi_{i-1}|^3\Psi_{i-1} - h(\Psi_{i-1} - T_\infty) + c\right) \quad on \quad x = L_x \tag{5}$$

$$k\frac{\partial\Psi_i}{\partial x} = \gamma\Psi_i - \left(\gamma\Psi_{i-1} - \sigma|\Psi_{i-1}|^3\Psi_{i-1} - h(\Psi_{i-1} - T_\infty) + c\right) \quad on \quad x = 0 \tag{6}$$

$$-k\frac{\partial \Psi_i}{\partial y} = \gamma\Psi_i - \bigg(\gamma\Psi_{i-1} - \sigma|\Psi_{i-1}|^3\Psi_{i-1} - h(\Psi_{i-1} - T_\infty) + c\bigg) \quad on \quad x = L_y \tag{7}$$

$$k\frac{\partial \Psi_i}{\partial y} = \gamma\Psi_i - \bigg(\gamma\Psi_{i-1} - \sigma|\Psi_{i-1}|^3\Psi_{i-1} - h(\Psi_{i-1} - T_\infty) + c\bigg) \quad on \quad y = 0 \tag{8}$$

with:

$$T = \lim_{i\to\infty} \Psi_i \tag{9}$$

The term $|\Psi_{i-1}|^3\Psi_{i-1}$ is the coercitivity term that ensures the existence of at least one solution. The problem proposed in its originality has boundary condition type Robin, but the proposed methodology simplifies way on the convergence of the solution, imposing a boundary condition of Neumann type without physical meaning, but rather as a effective mathematical tool.

The auxiliar term β:

$$\beta_i = \gamma\Psi_{i-1} - \bigg(\sigma|\Psi|^3\Psi_{i-1} - c + h(\Psi_{i-1} - T_\infty)\bigg) \tag{10}$$

Rewriting the Eq.(1) and Eq.(3) with respect β

$$div(kgrad\Psi_{i+1}) + \dot{q} = 0 \quad in \quad \Omega \tag{11}$$

$$-kgrad\Psi_{i+1}.\mathbf{n} = \gamma\Psi_{i+1} - \beta_i \quad on \quad \partial\Omega \tag{12}$$

$$\Psi_0 \equiv 0 \tag{13}$$

The sequence $[\Psi_0, \Psi_1, \Psi_2, ..., \Psi_i]$ converges in $\partial\Omega$, Gama (1991) and Gama (1997) makes the analytical demonstration of this series.

2.3 Discretization

To handle of the computational model is necessary to express adequately equations on continuous region, or domain in which they are valid. For obtain numerical solutions on a continuous region with infinite points is necessary discretize the domain, that is, divide it into points. Only in these points is that the solutions will be obtained, and to all the discrete points is given the mesh name. The proper distribution of points in the field is critical to obtain a representative numerical solution of the modeled phenomenon. Then, terms that appear written on the amounts of the unknowns in adjacent discrete points. The result is a set of algebraic equations, generally linear, which may or may not be coupled. This step introduces the problem of boundary conditions modifying the equations for points close to the boundaries, the boundary conditions along the initial conditions and the physical properties of the fluid and the flow parameters specify the problem to be treated (Fortuna, 2000).

Sperandio *et al.* (2003) describes the method of finite differences as a technique for obtaining a numeric solution of a partial differential equation in replacing continuous derivatives (and boundary conditions and the initial) of the differences by formulas involving only discrete values associated with mesh positions. In all numerical solution of partial differential equation is replaced by a discrete approximation, i.e. the numerical solution is only known for a finite number of points in the physical domain, while the analytical solution must satisfy the partial differential equation at each point of the region. The discrete approach results in a set of algebraic equations are calculated for unknown discrete values.

The finite difference method (FDM) is nothing more than rewriting the differential equations so that the derivatives are taken into finite intervals and not infinitesimal. Assuming isotropic and convex body, the finite difference method is obtained by truncation of the Taylor (Ozisik, 1993).

Representing analytically the term of the internal local temperature with Δx and Δy spacings of the cartesian mesh:

$$\Psi(i,j) = \frac{[\Psi(i+\Delta x, j) + \Psi(i-\Delta x, j)] + \Delta x^2[\Psi(i, j+\Delta y) + \Psi(i, j-\Delta y)] + \Delta x^2\Delta y^2[\frac{\dot{q}}{k}]}{2\Delta x^2 + 2\Delta y^2} \tag{14}$$

The following equations denote explicitly the terms for local temperature on the boundary for a board contour, where γ is the convergence factor sufficiently large positive.

$$x = 0: \quad \Psi(1,j) = \frac{k\Psi(2,j) + \beta\Delta x}{\gamma\Delta x + k} \tag{15}$$

$$x = L_x: \quad \Psi(i_{max},j) = \frac{k\Psi(i_{max-1},j) + \beta\Delta x}{\gamma\Delta x + k} \tag{16}$$

$$y = 0: \quad \Psi(i,1) = \frac{k\Psi(i,2) + \beta\Delta y}{\gamma\Delta y + k} \tag{17}$$

$$y = L_y: \quad \Psi(i,j_{max}) = \frac{k\Psi(i,j_{max-1}) + \beta\Delta y}{\gamma\Delta y + k} \tag{18}$$

Is noteworthy that the vertex nodes are obtained through tween approximations.

2.4 Verification and validation

Programming errors or inadequate boundary conditions can make a simulation provide visually plausible results, but physically incompatible with the problem presented (Fortuna, 2000).

The need often arises for evaluating the definite integral of a function that has no explicit antiderivative or whose antiderivative is not easy to obtain. The basic method involved this approximating is called numerical quadrature. The methods of quadrature are based on the interpolation polynomials, the basic idea is to select a set of distinct nodes from the interval. Then integrate the Lagrange interpolating polynomial (Burden and Faires, 2001):

$$P_n(x) = \sum_{i=0}^{n} f(x_i)L_i(x) \tag{19}$$

and its truncation error term over $[a,b]$ to obtain

$$\int_a^b f(x)dx = \int_a^b \sum_{i=0}^{n} f(x_i)L_i(x)dx + \int_a^b \prod_{i=0}^{n}(x - x_i)\frac{f^{n+1}(\epsilon(x))}{(n+1)!}x \tag{20}$$

with error given by:

$$E(f) = \frac{1}{(n+1)!}\int_a^b \prod_{i=0}^{n}(x - x_i)f^{(n+1)}(\epsilon(x))dx \tag{21}$$

Using the first lagrange polynomials with equally-spaced nodes gives trapezoidal's rule, this feature used when can not determine an exact function and that function is approximated at one stretch by a straight. Equation(22) is the trapezoidal rule when f is a function with positive values.

$$\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12}f^{'''}(\epsilon) \tag{22}$$

This study will make energy conservation tests and trapezoid rule approximations in order that notes accuracy and objectivity of this methodology.

Integrating the Eq.(1) in all body

$$\int_\Omega div(kgradT) + g)dV = 0 \quad in \quad \Omega \tag{23}$$

And substituing Eq.(12) in Eq.(23)

$$\int gdS = \int (\gamma\Psi_{i+1} - \beta_i)dl \tag{24}$$

Comparing consecutive iterations for the inside of body (Ω)

$$\int_i^{i+1} gdS = \left(\gamma\Psi_i + \gamma\Psi_{i+1} - \beta_i - \beta_{i+1}\right)\left(\frac{\delta x}{2}\right) \tag{25}$$

Analysing the outline, for the Eq.(12)

$$-kgrad\Psi_{i+1}.\mathbf{n} = \gamma\left(\Psi_{i+1} - \frac{\beta_i}{\gamma}\right) \quad on \quad \partial\Omega \tag{26}$$

Thence for ($\partial\Omega$)

$$\Psi_{i+1} \rightarrow \frac{\beta_i}{\gamma} \tag{27}$$

Numerical and analytical comparisons confirm the accuracy of the methodology

3. NUMERICAL EXAMPLE

In this section steady-state a nonlinear conduction/radiation heat transfer problem is simulated by means of very simple solution linear problem.

The problem proposed in this work is represented by uniform, structured and cartesian grid, a mesh with evenly spaced points. Intuitively, the greater the number of discrete points, that is, the finer mesh is most faithful model in the numerical result, this work uses mesh 30x30.

Results obtained with aid of finite difference method approximation are presented in Fig. 1, Fig. 2, Fig. 3, Fig. 4 and Fig. 5. In this case the value of the thermal conductivity is equal to $1W/mK$, Stefan Boltzman constant $\sigma = 1.10^{-4}W/m^{-2}K^{-4}$ and convection heat transfer coefficient $h = 1W/m^2K$. It is to be noticed that the convergence was reached for $\gamma = 10$. In other words, for values greater than $\gamma = 10$ no significant changes were observed.

Table 1 illustrates the numerical convergence process, presents Φ_1, Φ_2, Φ_3, Φ_4, Φ_5, Φ_6 and Φ_∞ obtained with three different values of γ ($\gamma = \gamma_1 = 1.10^1 W/m^2K$, $\gamma = \gamma_2 = \gamma_1^2$ and $\gamma = \gamma_3 = \gamma_1^3$) with $L_x = L_y = 10$.

Gama (1997) demonstrates such analytically convergence process.

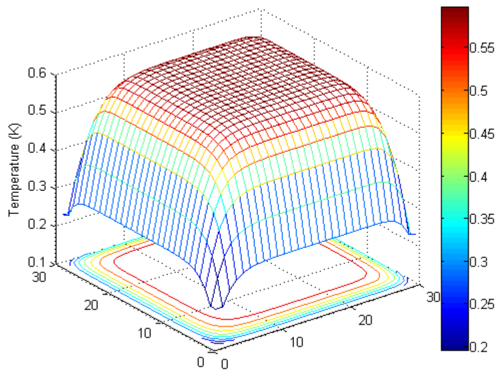

Figure 1. Iteration 10, $T_{max} = .5945$

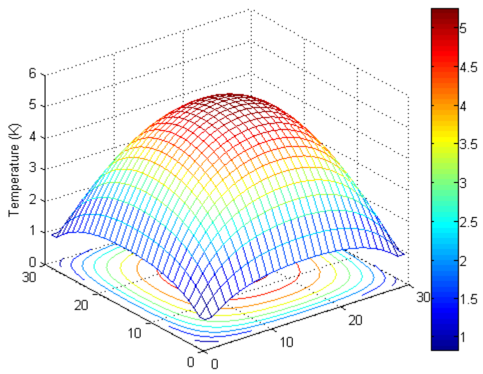

Figure 2. Iteration 100, $T_{max} = 5.2373$

Table 1. Numerical convergence verification.

Φ_i	$\gamma = \gamma_1$	$\gamma = \gamma_2$	$\gamma = \gamma_3$
i=10	.5945	0.5941	0.5940
i=100	5.2373	5.2245	5.0802
i=500	9.8306	9.8270	9.7566
i=1000	10.0053	10.0052	10.0021
i=1200	10.0081	10.0076	10.0059
i=1387	10.0081	10.0081	10.0079
i$\rightarrow \infty$	10.0081	10.0081	10.0081

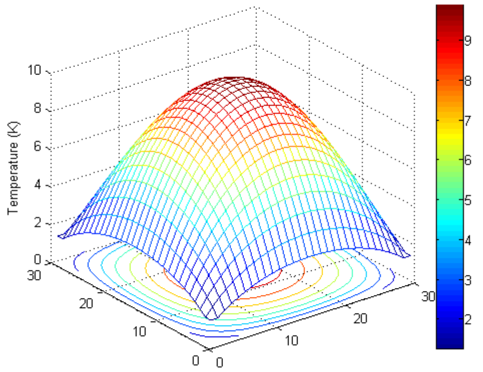

Figure 3. Iteration 500, $T_{max} = 9.8306$

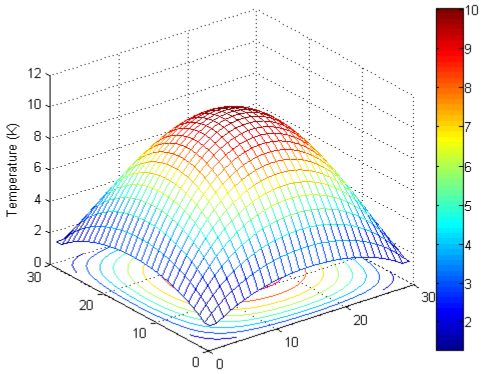

Figure 4. Iteration 1000, $T_{max} = 10.0053$

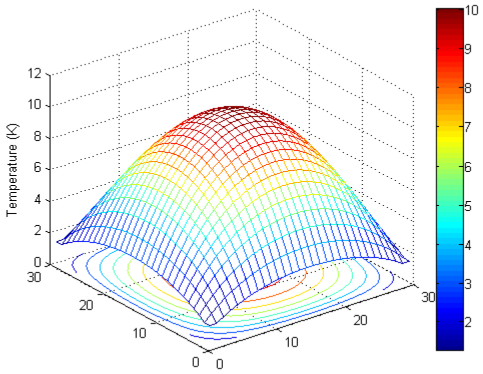

Figure 5. Iteration 1387, $T_{max} = 10.0081$

4. CONCLUSIONS

The mathematical model proposed here represents a new viewpoint for describing nonlinear heat transfer problems in rigid bodies, presenting several advantages, being physically equivalent and mathematically more convenient. Verified the effectiveness, this methodology is applicable for any dimensions and geometry types, the solution methodology simplifies the problem considerably, and thus can be tested and used to problems of larger spans.

5. ACKNOWLEDGEMENTS

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6. REFERENCES

Arrott, A.S., 1985. “Generalized curie-weiss law”. *Physical Review B*, Vol. 31, No. 5, p. 2851.

Burden, R.L. and Faires, J.D., 2001. “Numerical analysis”. *Brooks/Cole, USA*.

Emelyanov, A.Y., Pertsev, N., Hoffmann-Eifert, S., Böttger, U. and Waser, R., 2002. “Grain-boundary effect on the curie-weiss law of ferroelectric ceramics and polycrystalline thin films: Calculation by the method of effective medium”. *Journal of electroceramics*, Vol. 9, No. 1, pp. 5–16.

Fortuna, A., 2000. “Técnicas computacionais para dinâmicas dos fluidos”. *Conceitos Básicos e Aplicações, EDUSP*.

Gama, R.M.S., 1991. “Existence uniqueness and construction of the solution of the energy transfer problem in a rigid and nonconvex black body”. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, Vol. 42, No. 3, pp. 334–347.

Gama, R.M.S., 1997. “The nonlinear conduction/radiation heat transfer phenomenon represented as the limit of a sequence of linear problems”. *International communications in heat and mass transfer*, Vol. 24, No. 1, pp. 119–128.

Jaboori, M.G., Saied, M.M. and Hanafy, A.A., 1991. “A contribution to the simulation and design optimization of photovoltaic systems”. *Energy Conversion, IEEE Transactions on*, Vol. 6, No. 3, pp. 401–406.

Kinoshita, K. and Yamaji, A., 1976. “Grain-size effects on dielectric properties in barium titanate ceramics”. *Journal of Applied Physics*, Vol. 47, No. 1, pp. 371–373.

Koutroulis, E., Kalaitzakis, K. and Voulgaris, N.C., 2001. “Development of a microcontroller-based, photovoltaic maximum power point tracking control system”. *Power Electronics, IEEE Transactions on*, Vol. 16, No. 1, pp. 46–54.

Ozisik, M.N., 1993. *Heat conduction*. John Wiley & Sons.

Riffat, S.B. and Ma, X., 2003. “Thermoelectrics: a review of present and potential applications”. *Applied Thermal Engineering*, Vol. 23, No. 8, pp. 913–935.

Sperandio, D., Mendes, J.T. and e Silva, L.H.M., 2003. *Cálculo numérico: características matemáticas e computacionais dos métodos numéricos*. Prentice Hall.

Uchida, K., Takahashi, S., Harii, K., Ieda, J., Koshibae, W., Ando, K., Maekawa, S. and Saitoh, E., 2008. “Observation of the spin seebeck effect”. *Nature*, Vol. 455, No. 7214, pp. 778–781.

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