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COMPUTATIONAL MODEL FOR VIBRATION AND BUCKLING OF THIN AND THICK ANNULAR PLATES

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Abstract. *This paper uses a version of the Rayleigh-Ritz method with specialized functions for the analysis of thin and thick circular and annular plates subjected to out-of-plane and in-plane loads. The approximation functions for displacements are polynomials in the radial direction combined with trigonometric functions in the circumferential direction. A convenient feature is the use of linear nodal functions, which allow for easy consideration of nodal loads and boundary conditions (including follower forces), enriched by higher order polynomials without inclusion of additional nodes. The model allows for thickness variation and was implemented in MAPLE18, enabling the calculation of displacements and stresses under constant and linearly varying load, frequencies of vibration and buckling loads with a few commands. The examples show the effectiveness of this approach in the analysis of such structures and bring new light to this classical problem, allowing for interesting and novel comparisons illustrating the effect of shear deformation and thickness variation in circular and annular plates, including displacements, moments and shear forces, vibration frequencies and buckling loads.*

Keywords: *Circular annular plate; variable thickness circular plates; natural frequencies; buckling of annular plates; Rayleigh-Ritz method.*

1. INTRODUCTION

Circular and annular plates are used as structural components in various engineering applications. They can be analyzed based on two-dimensional theories, such as the classical thin plate theory (CPT). The CPT neglects the effects of transverse shear deformation and rotary inertial. This simplification not only hinders the application to thick plates, but also may lead to overestimation of the higher vibration frequencies in thin plates. A similar situation is found in buckling analysis (Timoshenko e Gere, 1963). In preliminary design procedures the use of tabulated results (when available) may be misleading.

A summary of research studies on free vibration of circular plates made by (Leissa, 1969), (Ventsel and Krauthammer, 2001) and (Chakraverty, 2009) indicates that (CPT) has certain limitations due to the Kirchhoff hypothesis. (Liew and Yang, 1998) they have studied the vibrational characteristic of annular plates by using the three-dimensional and based on polynomial-ritz model, including the thick deformation. For the analyses of critical load have (Koohkan, *et al.*, 2010) presented an analitycal approach on the buckling analysis of circular plates whith functionally graded and for verified by conparing the results resolve the equation differential for evaluation of the bucklin loads of annular plates with varius boundary conditions.the estability equations based on the (CPT).

The energy-based Rayleigh-Ritz method is used herein to provide an expedite calculation of displacements, stresses, natural frequencies and buckling loads of circular and annular plates with different boundary conditions and thickness ratios. In addition, though not discussed in the paper for space limitations, this method is naturally suited to include the variation of properties of the plate in the transverse, radial and tangential directions, in order to analyze plates made of functionally graded materials.

2. MATHEMATICAL FORMULATION

2.1 Description of the geometric parameters

Consider, for simplicity, a homogeneous isotropic annular plate of uniform thickness h, with inner and outer radius are denoted by b and a respectively. The origin of the coordinate system is taken at the center of the plate in the middle plane, as show in Fig.1. Thus, the plate geometry and dimensions are defined in by a cylindrical coordinate system (r, θ, z). The corresponding displacement components at a generic point are u_r, u_θ and u_z in the radial, circumferential and transversal directions, respectively.

2.2 Displacement field

Based on the assumptions of Mindlin- Reissner thick plate theory (Reddy, 2004), the displacement field in the plate may be described as follows:

$$
\begin{aligned}
u &= u_0 + z\psi_r \\
v &= v_0 + z\psi_\theta \\
w &= w_0
\end{aligned}
\tag{1}
$$

In the above expressions, u_0, v_0 and w_0 are the radial, circumferential and transverse displacement components of the mid-surface of the circular plate, respectively; ψ_r and ψ_θ are rotations in the radial and circumferential directions, respectively. In simple bending analysis, u_0 and v_0 may be set equal to zero. The approximation for the displacement field is polynomial in the radial direction and trigonometric in the circumferential direction. The parameters are displacements and rotations in $r = a$ and $r = b$ ("nodal" displacements), plus the magnitudes of an arbitrary number of polynomial functions of increasing order which are zero at the ends. This approach satisfies exactly the essential boundary conditions (displacements and rotations) at $r = a$ and $r = b$, and is assured to converge for other boundary conditions.

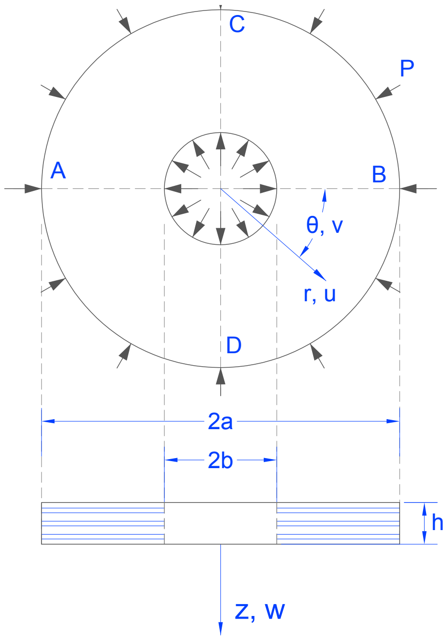

Fig 1. Geometry and dimensions of an annular plate in cylindrical coordinate system

2.3 Strain displacement relations

For small deflections, the strain-displacement relations may be written as:

$$
\begin{aligned}
&\varepsilon_{rr} = \frac{\partial u}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u}{r} + \frac{\partial v}{r\,\partial\theta}, \quad \varepsilon_{zz} = 0, \\
&\varepsilon_{r\theta} = \frac{\partial v}{\partial r} + \frac{\partial u}{r\,\partial\theta} - \frac{v}{r}, \quad \varepsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \quad \varepsilon_{\theta z} = \frac{\partial v}{\partial z} + \frac{\partial w}{r\,\partial\theta}
\end{aligned}
\tag{2}
$$

Where ε_{rr} and $\varepsilon_{\theta\theta}$ are the normal strains and ε_{rz}, ε_{rz} and $\varepsilon_{\theta z}$ are the shear strains.

2.4 Hooke´s law

The stress-strain relations for the elastic plate can be written as

$$\sigma_{rr} = \frac{E}{(1-\nu^2)}\left(\varepsilon_{rr} + \nu\,\varepsilon_{\theta\theta}\right),\quad \sigma_{\theta\theta} = \frac{E}{(1-\nu^2)}\left(\varepsilon_{\theta\theta} + \nu\,\varepsilon_{rr}\right),\quad \sigma_{r\theta} = \frac{E}{2\,(1+\nu)}\varepsilon_{r\theta}$$
$$\sigma_{rz} = k^2\,\frac{E}{2\,(1+\nu)}\varepsilon_{rz},\quad \sigma_{\theta z} = k^2\,\frac{E}{2\,(1+\nu)}\varepsilon_{\theta z} \tag{3}$$

Where E and ν are the Young´s modulus and Poisson`s ratio, respectively; k^2 denotes the transverse shear correction factor. In the present analysis, this coefficient is adopted as $k^2 = \pi^2/12$ (Reddy, 2007).

2.5 Displacement functions for Rayleigh-Ritz approach

The plate model contains three degrees of freedom for both end nodes. The accuracy of the Rayleigh-Ritz model is controlled by the number of functions (N) used to provide the description of the displacement field, in addition to the basic linear functions (for the thick plate model) or cubic functions (for the thin plate model). There is considerable freedom in the choice of the additional functions, except that they should be zero at the nodes (the thin plate functions must also have zero derivative at the nodes).

$$w_n^{(1)} = \sum_{i=1}^{N} c_i\, w_i(x) \qquad \psi r^{(1)} = \sum_{i=1}^{N} d_i\, \psi r_i \qquad \psi\theta^{(1)} = \sum_{i=1}^{N} e_i\, \psi\theta_i$$
$$w_n^{(2)} = \sum_{i=1}^{N} f_i\, w_i(x)\, sen\theta \quad \psi r^{(2)} = \sum_{i=1}^{N} k_i\, \psi r_i \cos\theta \quad \psi\theta^{(2)} = \sum_{i=1}^{N} m_i\, \psi\theta_i\, sen\theta \tag{4}$$

Where the coefficients (*c, d, e, f, k, m*) in Eq. (4) are weighting terms associated with their shape functions $w_n, \psi r$ and $\psi\theta$ which correspond to the transverse displacement and the two rotations at each point along the radius in the mid surface. The boundary conditions are imposed by adding adequate spring stiffness values at the diagonals of the stiffness submatrix associated with the nodal degrees of freedom, zero representing a free displacement and a very large value (penalty approach) corresponding to a restraint.

2.6 Equation of motion

The kinetic energy T and the strain energy U of an elastic annular plate are expressed as

$$T = \frac{1}{2}\int_b^a \int_0^{2\pi} \int_{-h/2}^{h/2} \rho\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) r dr\, d\theta\, dz \tag{5}$$

And

$$U = \frac{1}{2}\int_b^a \int_0^{2\pi} \int_{-h/2}^{h/2} \rho\left(\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + \sigma_{rz}\varepsilon_{rz} + \sigma_{\theta z}\varepsilon_{\theta z} + \sigma_{r\theta}\varepsilon_{r\theta}\right) r dr\, d\theta\, dz \tag{6}$$

Where ρ is the specific mass of the plate material and the upper dot represents the differentiation with respect to the time variable t.

The potential energy corresponding to the conservative component V is given by

$$V = \frac{1}{2}\int_b^a \int_0^{2\pi} \int_{-h/2}^{h/2} \left(N_r \frac{\partial w}{\partial r} + N_\theta \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right) r dr\, d\theta\, dz \tag{7}$$

Where N_r and N_θ are the stresses depend to the uniform compressive load P is considered along its edge.

2.7 Formulation of the eigenvalue problem

Let Π be the energy functional given by

$$\Pi = U + V - T \tag{8}$$

For linear small-strain assuming harmonic motion, the global displacement vectors can be written as,

$$x = \bar{x}\, e^{i\omega t} \tag{9}$$

$$\bar{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{10}$$

Here $\bar{x}$ is a nonzero vector of constants to be determined, ω is the natural circular frequency. Substituting Eq. (5) and (6) into the expressions (8) and not considering V . Leading to the governing eigenvalue equation of the form

$$\left(\overline{K} - \lambda^2 \overline{M}\right)\bar{x} = 0 \tag{11}$$

Where $\overline{K}$ is the stiffness matrix $\overline{M}$ is the mass matrix and $\lambda = \omega a^2 \sqrt{\rho h / D}$, here D is the flexural rigidity of the plate.

Substituting Eq. (6) and (7) into the expressions (8) and not considering T

$$\left(\overline{K} + \lambda_{cr}\overline{G} + \lambda_{cr}\overline{G}_L{}^{s}\right)\bar{x} = 0 \tag{12}$$

Where $\overline{G}$ is the geometric matrix, λ_{cr} is the parameter load, and $\overline{G}_L{}^{s}$ is the symmetric load matrix. In the case of nonconservative loads, a nonsymmetrical matrix is added to the equation. Clearly, this matrix cannot be obtained from the energy functional, but results from consideration of equilibrium in the deformed configuration.

3. RESULTS AND DISCUSSION

Due to space limitations, a few comparison results are presented herein, to illustrate the adequacy of the model. A plate with outer radius over thickness ratio of 45 with $E = 20\,GPa, \nu = 0.3$ is considered. In this case, Kirchhoff and Reissner –Mindlin theories give results which differ just slightly.

Table 1. Frequencies of annular plate for values $b/a = 0.3$

Boundary conditions	Source of results	Modes of Vibration			
		1	2	3	4
C - F	Leissa^{CPT}	6,33	6,66	7,96	13,27
	Authors ^{K}	6,55	6,66	7,96	13,28
	Authors ^{RM}	6,55	6,65	7,96	13,25
C - S	Leissa^{CPT}	29,90	31,40	36,20	45,40
	Authors ^{K}	29,97	31,39	36,24	45,47
	Authors ^{RM}	29,83	31,32	36,15	45,01

Table 2. Critical buckling loads of circular plate

Boundary conditions	Source of results	Pcr / 10^5 (N)
C	Timoshenko^{CPT}	26,5622
	Authors ^{K}	26,5585
	Authors ^{RM}	26,5043
S	Timoshenko^{CPT}	7,6019
	Authors ^{K}	7,5933
	Authors ^{RM}	7,5888

^{K} Solution obtained without shear deformation; ^{RM} Solution obtained considering shear deformation; ^{CPT} Analytical solution based on the classical thin plate theory.

Table 1 presents a comparison of results for an annular plate, with clamped inner edge and outer edge free (C-F), and with clamped inner edge and simply supported outer edge (C-S) of uniform thickness. The results obtained using the present method, with four additional functions in the radial direction, practically coincide with those given by (Leissa, 1969). It should be mentioned that here transverse shear deformation and rotary inertia are taken into consideration.

Table 2 presents the results for buckling load for circular plates with outer edge clamped and simply-supported. The Rayleigh-Ritz model with just four additional functions also provides a good approximation to analytical results available for a thin plate (Timoshenko and Woinowski-Krieger, 1959). Here, the model requires consideration of an annular plate with a very small inner radius, which explains why the calculated thin plate loads are slightly lower than those obtained from the analytical model.

The vibration mode shapes of the annular plate for clamped inner edge and outer edges free are presented in Fig. 2. the all modes is not axisymmetric, this values has no units.

The plots in Fig 3 show the variation of the buckling load and the first three frequencies in the clamped circular plate, relative to the thin plate result (values of the thick plate in relation to the values of the thin plate), with the ratio of thickness to radius of the plate. It should be noted the substantial reduction in the higher modes, which are more influenced by shear deformation and rotary inertia.

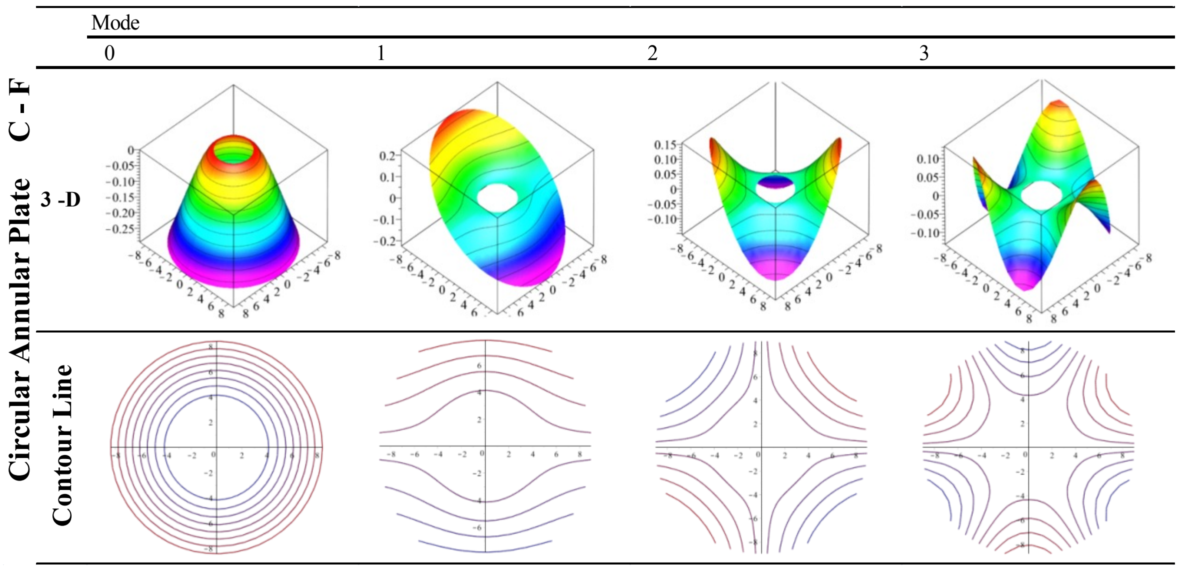

Fig 2. Deformed mode shape of an annular plate with clamped the inner and outer edges free

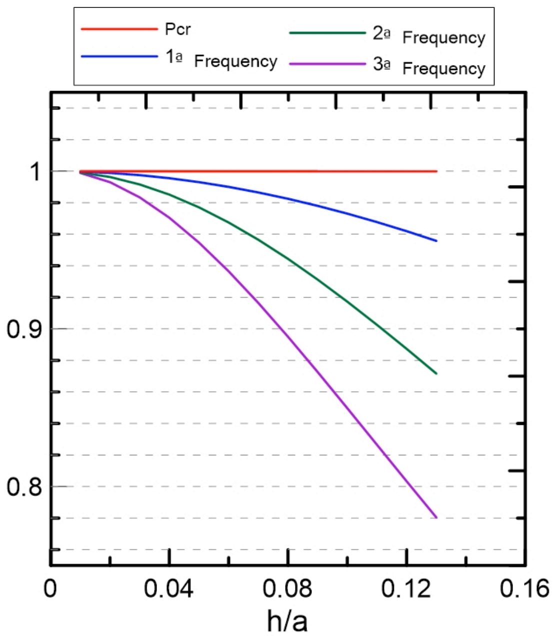

Fig 3. Variation of the critical load values and frequency, relative to thin plate results

Table 3. Displacements and moments of the circular plate subject to an axisymmetric load, with geometry $a = 9\,m$, and $h = 0.2\ m$.

Boundary conditions	Source of results	Displacement (m)	Moment $(N \cdot m)$	
			Mr / 10^3	Mθ / 10^3
C	Timoshenko^{CPT}	0,1968	120,2344	208,8281
	Authors ^{K}	0,1968	120,2342	208,8277
	Authors ^{RM}	0,1974	120,2339	208,8278

Table 4. Displacements and moments of the circular annular plate subject to an axisymmetric load, with geometry $a = 9\,m$, $b = 3m$ and $h = 0.2\ m$.

Boundary conditions	Source of results	Displacement (m)	Moment $(N \cdot m)$	
			Mr / 10^3	Mθ / 10^3
C - F	Timoshenko^{CPT}	0,2402	186,7538	316,3615
	Authors ^{K}	0,2402	186,7517	316,3839
	Authors ^{RM}	0,2410	186,7604	316,3643

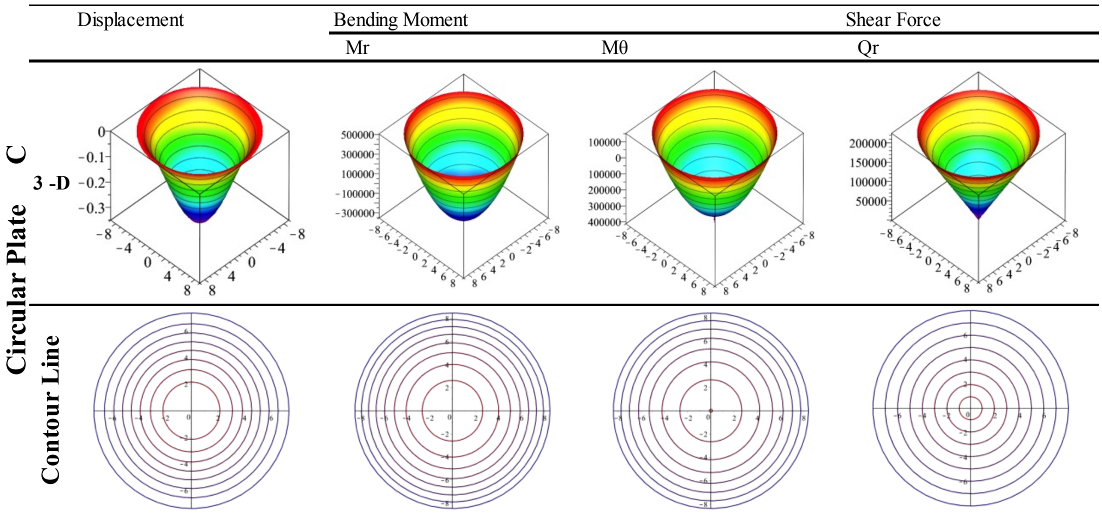

Fig 4. Deformed mode shape of an annular plate with clamped edges

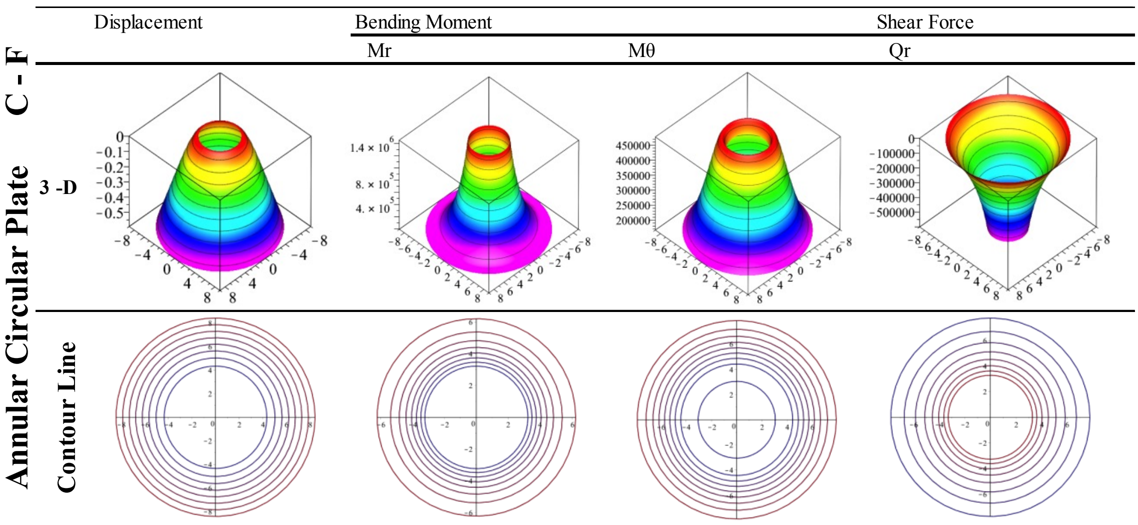

Fig 5. Deformed mode shape of an annular plate with clamped the inner and outer edges free

For the clamped circular, Tables 3 and 4 present the displacements and moments measured at a distance of 4.5m for circular plate and 6m for circular annular plate of the center. Figs. 4 and 5 show the deformed shape and the distribution of moments and shear forces. It should be mentioned that for good accuracy in the stress resultants use of additional radial modes is advised.

4. CONCLUSIONS

The results demonstrate the efficiency and accuracy of the Rayleigh-Ritz procedure in the formulation developed with nodal and additional modes. The practical usefulness of the method lies in the application to expedite analysis of annular plates with variable properties. Another convenient feature is the straightforward treatment of nonconservative (follower) loads. Hence, the application of this method in the preliminary design of thick annular plates, such as the supporting structure of towers for wind turbines, is preconized.

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