Progress in the Solution of the Richards Equation in Infiltration Problems

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Abstract. *In this paper, we consider a transient one-dimensional flow problem of water in unsaturated porous media, modeled by the nonlinear Richards equation. Constitutive relations of Van Genuchten will be employed. A parameterized solution for Richards equation is presented with the objective to define an initialization of the recursion for an alternative Adomian scheme. This solution is optimized via least squares and Newton's nonlinear method and evaluated by the nonlinear Richards equation i.e. a self consistency test. The results are analyzed for typical cases found in the literature.*

Keywords: *Porous media, Padé, Adomian decomposition, parametrised solution*

1. INTRODUCTION

In infiltration problems the soil-water parametrisations are strong non-linear functions of the dependent variables. Usually the Richards od Focker-Planck equations are used to interpret mathematically this process, where both equations are highly non-linear. Thus, analytical solutions to the equations are extremely difficult to find, so that linearisation or numerical schemes are frequently used to get approximate descriptions of the problem. In order to cause prognostics more efficient, it is important to consider field observations, because these are necessary for identification of constitutive relations governing the phenomenon and are used in theoretical formulations for computer simulations. Basically, the best-known models that relate soil parameters of conductivity K, matrix potential ψ and soil moisture θ, are the exponential model by Gardner (1958), the power model by Brook and Corel (1964) and the Van Genuchten model (1980) which can be found in refs. (Brooks and Corey, 1964; Genuchten, 1980; Gardner, 1958), respectively. The Van Genuchten model provides more satisfactory results than others when compared with experimental data, but due to its functional form, proposed solutions have limited applicability for this model (Nasseri *et al.*, 2010). On the other hand, the other two cited models can result in simplified equations, leading to a case of linearised governing equation and deriving a series of analytical solutions (Broadbridge and White, 1988; Srivastava and Yeh, 1991; Basha, 1999; Chen *et al.*, 2001; Basha, 2002). However, most of these solutions are limited to cases with uniform initial conditions and infinite domain. Furthermore, they are bound to considerations restricted to information on the soil-water parameter.

Thus, for a more general approach quite often it was necessary to resort to numerical models. A model discretisation equation, which describes the flow in the unsaturated zone, can be performed numerically by finite difference techniques or finite elements. In these techniques, a scheme of discretisation is applied to a system of nodal points that map the region-time soil depth considered. Implementing the initial and boundary conditions appropriately, leads to a set of algebraic equations that can be solved by various techniques (Feddes *et al.*, 1988), however in general without mathematically sound error estimates.

One way to obtain solutions in analytical representation and without resorting to linearisation or perturbation approximation is by the Adomian decomposition method. Solutions obtained by the decomposition method for problems considering non-saturated flux may be found in the literature, in reference (Serrano, 1998) the authors used exponential relations for the soil-water parameter to construct a solution by the decomposition method for a one dimensional vertical case, emphasizing that these relations are not adequate for the cases with dry soil. The diffusivity with a power function was used in (Pamuk, 2005, 2006) that allowed to obtain the horizontal flux by the decomposition method, where the solution was satisfactory already with only a few terms. The Brooks-Corey model, with a power approximation was implemented by the authors of (Nasseri *et al.*, 2008) and the equation for the flux was solved by the Adomian method considering a travelling wave solution. Note, that so far no solution in analytical representation is found in the literature for a general case of soil-water relations or for the model based on the van Genuchten relation.

In this contribution, we analyse a problem of transient flow of water in unsaturated media, modelled by the Richards equation. To this end, the constitutive relations of van Genuchten are employed and a hybrid method of Padé approximants and Adomian decomposition is applied. Although Adomian states in his work that this method should be used to any non-linear problem, in this application the method in its original form fails. In order to circumvent this shortcoming, we propose a construction of a functional solution method for the Richards equation, that shall replace the recursion initialisation of the decomposition method. The found solution is then optimized and its accuracy evaluated by the non-linear Richards equation and the profile of the potential matrix is also compared to numerical solutions for two types of soil textures. It is remarkable that for the present parameter set the recursion initialisation is already considerably close to the true solution, so that in the present case no further recursion steps are necessary.

2. The model

2.1 The Richards equation

The governing equation describing infiltration in porous media established in (Richards, 1928) is a result of the combination by the Darcy-Buckingham and the continuity equation.

$$\vec{q} = -K(\theta)\vec{\nabla}\Phi \quad \text{and} \quad \frac{\partial\theta}{\partial t} = -\vec{\nabla}\vec{q} \tag{1}$$

Here $\vec{q}$ in (m/s) is the specific flow, $K(\theta)$ in (m/s) is the hydraulic conductivity depending on soil moisture θ and Φ signifies the hydraulic potential in units of (m). Equation (1) has a considerable mathematical complexity due to the non-linearity present in the hydraulic conductivity. Moreover, this equation is established for steady state condition or dynamical equilibrium. Most situations though are transients, and to describe such scenarios time dependence is introduced by the continuity equation. For convenience one may split $\Phi = \psi + z$ into the matrix potential that contains the essential effects attributed to porosity, and the gravitational potential represented by the soil depth. Combining the Darcy-Buckingham and continuity equation, allows to cast the problem in

$$C(\psi)\frac{\partial\psi}{\partial t} = \vec{\nabla}[K(\psi)\nabla\psi] + \frac{dK(\psi)}{d\psi}\vec{\nabla}\psi \quad \psi \in \Gamma \times [0,T]; \quad Q\psi = J \tag{2}$$

where $C(\psi) = \frac{d\theta}{d\psi}$ is called hydraulic capacity (m^{-1}); $\Gamma \subset R^3$ is the spatial domain; t is time; $0 < T < \infty$; Q are the boundaries and J is a known function. Equation (2) is known as the Richards equation, that governs the movement of water in unsaturated soil and can be applied in the whole domain even for distinct saturated and unsaturated areas (Wendland and Pizarro, 2010).

2.2 Van Genuchten model

The equation system (1) needs an additional relation so that the system can be solved with one unique solution for ψ. Parametric relations of $\theta(\psi)$ and $K(\psi\, or\, \theta)$ are used to mimic hydraulic soil-water properties. Mualen (Mualen, 1976) derived a model for hydraulic conductivity from retention curve of soils. The utility of this model attributes the fact that condutivity is difficult to be measured, so that this model is used in parametrisations of van Genuchten and Brooks-Corey, respectively.

The validity of the classical van Genuchten model wa discussed in (Fuertes *et al.*, 1992). There it was shown, that the curves are highly sensitive especially for $n < 2$ with consequence on the numerical solution's quality. In (Vogel *et al.*, 2001) modified model was presented which incorporates ar inlet and is discussed in (Vogel *et al.*, 2001; Marcel and Van Genuchten, 2005). The modifications resulted in better predictions when compared to those of the original model. To this end a minimum capilarity ψ_s together with an arbitrary parameter $\theta_m > \theta_s$ was introduced.

$$\theta(\psi) = \theta_r + \frac{\theta_m - \theta_r}{(1 + |\alpha\psi|^n)^m} \quad \psi < \psi_s; \quad \theta_s \quad \psi \geq \psi_s \tag{3}$$

where $\theta_m = \theta_r + (\theta_s - \theta_r)(1 + |\alpha\psi_s|^n)^m$, n, m and α are parameters that have to be adjusted according to the case in consideration. Using Eq.(3) yields the hydraulic conduction from the modified model

$$K(S_e) = K_s S_e^\tau \left[\frac{1 - F(S_e)}{1 - F(1)}\right]^2, \quad \psi < \psi_s; \quad K_s, \quad \psi \geq \psi_s \tag{4}$$

where $F(S_e) = [1 - (S_e)^{1/m}]^m$ and K_s (m/s) is a measure for saturation, $S_e(\theta) = (\theta - \theta_r)/(\theta_s - \theta_r)$ with θ_s and θ_r the saturated and residual soil water content. Vogel (2001) suggested the use of ψ_s approximately -1 or $-2cm$ and Marcel et al.(2005) optimised the value to $-4cm$ and suggested that ψ_s should also be used as fit degree of freedom (for details see Ippisch *et al.* (2006)).

2.3 Padé approxiamts

Note, that the model of Van Genuchten is widely used in numerical simulations, but due to its complexity in functional form algebraic manipulations are rather complicated that seem to make analytical solutions unattractive. To circumvent some of these difficulties functional Padé approximants were used to substitute the hydraulic parameters. Attempts like this one were already presented in (Cheng *et al.*, 2015) but the authors used a simple Taylor expansion to estimate the soil-water parameter (see also (Mohammad and Mohammad, 2010; Mohammad *et al.*, 2010)).

Note, that these parameters are by its nature phenomenological relations, so that the representation by approximants can be used without loss of generality. The simplest Padé representation for K and C in the region of interest are $[1/4]$ and $[2/4]$, respectively. A comparison of the original expressions and the Padé approximations for K and C are shown in Fig. 1 and 2 for two types of texture of soils.

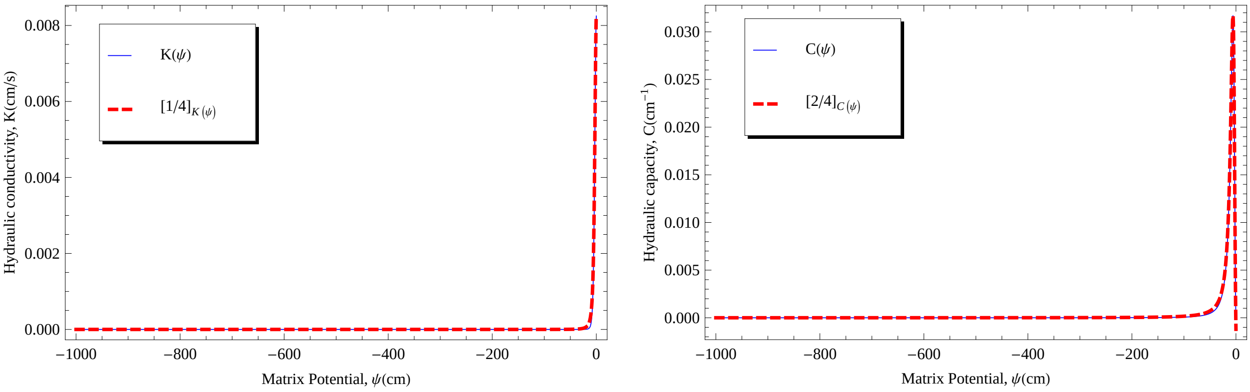

Figure 1. Padé approximants for $K(\psi)$ and $C(\psi)$ with the original expressions for the sand soil.

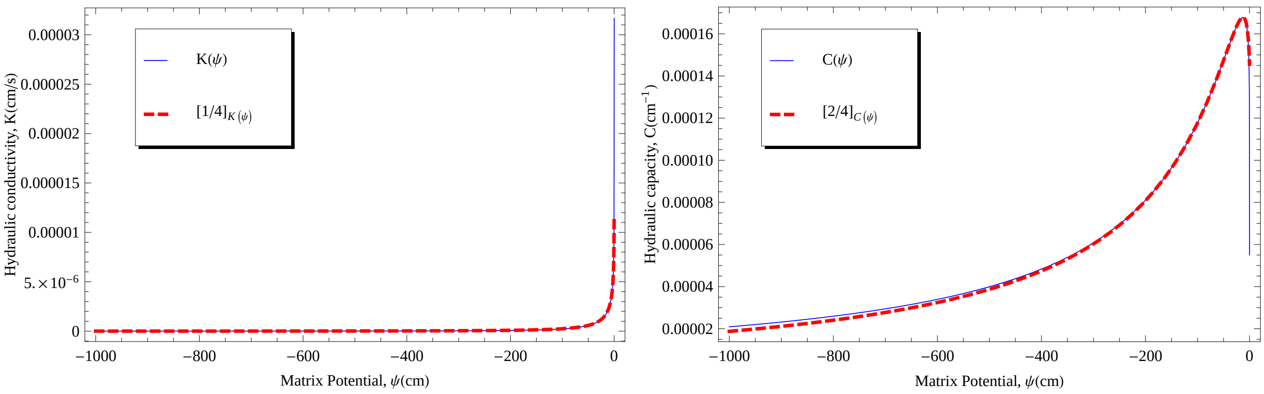

Figure 2. Padé approximants for $K(\psi)$ and $C(\psi)$ with the original expressions for the clay soil.

2.4 Adomian decomposition

In this work an approach is proposed to determine the flow of water in a non-saturated medium without linearisation. In general, the technique is based on the decomposition of a solution in a number of functions. The non-linear term is approximated by a recursive scheme defined according to the form of the non-linearity. It is noteworthy that, as an approximate solution for an infinite series, there is need for a truncation of the series for an operational solution. Because the solution is a truncated series, the validity of the method is restricted to the radius of convergence of the series.

The equation to be solved for the matrix potential is the Richards equation

$$C(\psi)\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial z}\left[K(\psi)\left(\frac{\partial\psi}{\partial z}+1\right)\right] \tag{5}$$

Let $C(\psi) = \overline{C} + \mathcal{C}(\psi)$ and $K(\psi) = \overline{K} + \mathcal{K}(\psi)$, where $\overline{C}$ and $\overline{K}$ are constants and medium $\mathcal{C}(\psi)$ and $\mathcal{K}(\psi)$ functions that carry information of the matrix potential (ψ). The linear and non-linear terms are reshuffled in order to elaborate the recursive scheme.

$$\frac{\partial^2\psi}{\partial z^2} - \frac{\overline{C}}{\overline{K}}\frac{\partial\psi}{\partial t} = \frac{1}{\overline{K}}\left[\mathcal{C}(\psi)\frac{\partial\psi}{\partial t} - \frac{\partial}{\partial z}\left(\mathcal{K}(\psi)\frac{\partial\psi}{\partial z}\right) - \frac{\partial\mathcal{K}(\psi)}{\partial z}\right]. \tag{6}$$

Thus, the left side of Eq. (6) represents the linear terms, which are used to find the initialization of the recursion from the decomposition. The right side is the non-linearity and is used as source term in the recursion scheme. Note that, in front of the correction term, there is an amplification factor $1/\bar{K}$, i.e., if $\bar{K}$ is a small term, then $1/\bar{K}$ becomes an amplification factor because it produces high values and makes difficult the determination of Adomian polynomials. As shown in (Furtado, 2013), for the presend consideration, i.e. the Richards-Genuchten equation, the Adomian procedure, does not provide satisfactory results because of strong divergence. Even trying to implement the method of Adomian in its alternative ways, do not result in satisfactory solutions. From these findings, the authors suggest a new formulation for obtaining a solution. It is built on the following steps: First, one determines a parametrised approximate initial solution, this solution is optimized by the Richards equation and evaluated by a self-consistency test. If the new solution shows a large remainder from the self-consistency test then recursion steps shall provide correction terms, which is initialized by the previously determined solution.

3. Construction of a parametrised solution

In the further we consider the $1 \oplus 1$ dimensional space-time version of the Richards equation with the afore mentioned initial and boundary conditions. We first construct a solution that shall be a reasonable initial solution for the recursive scheme so that all remaining corrections are sufficiently small and the scheme converges.

3.1 Transient and steady state regimes

From comparison to experimental findings one expects the matrix potential to assume negative values in the range of $[-10, 0]$. From inspection one observes that for a restriction $\psi \in [-10, -2]$ the hydraulic conductivity may be approximated by a constant $K(\psi) \approx K$ and the hydraulic capacity may be approximated by a polynomial $C(\psi) \approx a(\psi + 10)^6$. For convenience we introduce the substitution $\psi + 10 \to \phi$ and solve the resulting equation

$$a\phi^6 \frac{\partial \phi}{\partial t} = K \frac{\partial^2 \phi}{\partial z^2}. \tag{7}$$

This equation has an implicit travelling wave solution,

$$\lambda^2 \int \frac{d\phi}{F(\phi) + C_1} = t + \lambda z + C_2 \ , \qquad F(\phi) = \int \frac{a\phi^2}{K} \, d\phi \ , \tag{8}$$

where λ, C_1 and C_2 are constants which are determined from the initial and boundary conditions of the problem and the functional from of the integrals are given in term of the shifted matrix potential ψ. Due to the implicit character of Eq.(8) it is necessary to use an iterative procedure to determine the constants, in the present case by the use of Newton's method.

In (Zlotnik *et al.*, 2007) the authors assumed that th solution of the Richards equation has a traveling wave form, i.e. there exists an unknown parameter u such that $\theta(x, t) = u(\xi)$ for $\xi = x + ct$, which defines a linear relation between the variales of the problem. This finding allows to reduce the original PDE into a ODE with $u(\xi)$. Zlotnik (2007) solved the ODE by numerical integration. Nasseri (Nasseri *et al.*, 2010, 2012) supposed that the solution of the resulting ODE has the shape of a hyperbolic tangent function. In this contribution this method is used as a tool to solve the auxiliary problem, Eq. (7).

The found solution already allows to analyse some properties of the dynamics of the system namely the transient and stationary regime of the matrix potential. Figure 3 shows the plot of contours with constant matrix potential ϕ as a function of time and depth. One observes that contours with $\phi \geq 3$ (or equivalently $\psi \geq -7$) effectively reduce the dimension of the problem by one so that the time variable may be eliminated in the problem and hence can be attributed to stationarity.

Upon inserting the second into the first Eq. (8), we can say that ϵ is the displacement of the surface of the water head, i.e. the region border to the zone of saturation. In other words, it is the displacement of the matrix potential solution until the limit of the transient process.

3.2 The stationary solution

From the finding that a stationary regime exists for the expected values of ψ and thus ϕ, we solve the time independent problem but this time using a polynomial expression for the logarithmic hydraulic conductivity $\ln[K(\psi)] = a(\psi - b)^2 - c$, which is valid for $\psi \in]-10m, -2m]$. So $a\frac{\partial}{\partial z}(\psi - b)^2 \left(\frac{\partial \psi}{\partial z} + 1 \right) + \frac{\partial^2 \psi}{\partial z^2} = 0$.

Here, a, b and c are parameters determined such as to minimize the difference between the polynomial function to the original expression. Upon substitution of $\psi \to \phi = \psi - b$ we solve the resulting ordinary differential equation using the decomposition method, where $\phi = \sum_{i=0}^{\infty} \phi_i$.

$$0 = \underbrace{\frac{\partial^2 \phi}{\partial z^2} + a\frac{\partial \phi^2}{\partial z}}_{A\ (Initialisation)} + \underbrace{a\frac{\partial \phi^2}{\partial z}\frac{\partial \phi}{\partial z}}_{B\ (Correction)} \tag{9}$$

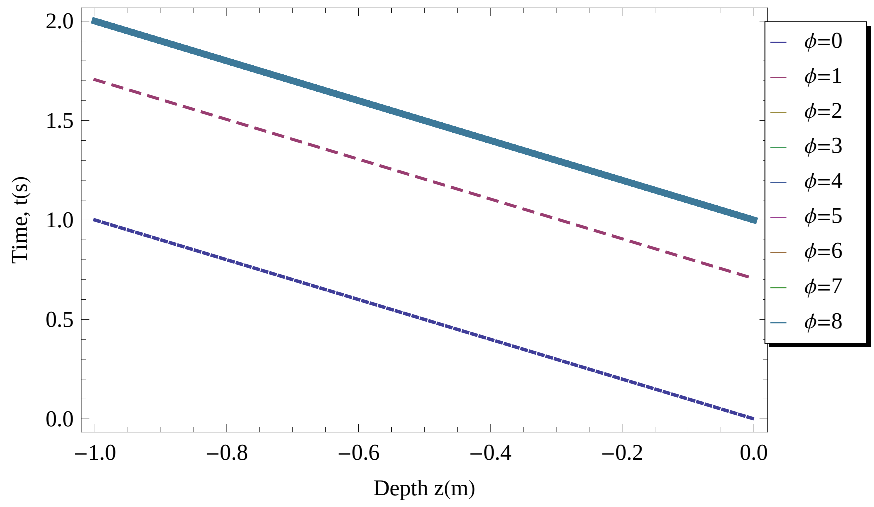

Figure 3. Transient to steady state evolution of constant matrix potential contours ϕ.

Equation (9) can be solved using a recursive method, where the terms A are used to determine the initialisation and terms of B are considered as a correction for the subsequent recursions. Therefore, if we assume that ϕ_0 is the first term of the recursion then ϕ_0 is solution of the following equation $\frac{\partial^2 \phi_0}{\partial z^2} + a\frac{\partial \phi_0^2}{\partial z} = 0$ with

$$\phi_0(z,t) = -\sqrt{\frac{c_1}{a}} \tanh\left(\sqrt{ac_1}(-z+c_2)\right) \ . \tag{10}$$

To determine ϕ_1 in the second recursion step, the term B is now considered as a source using the previously determined solution ϕ_0.

$$\frac{\partial^2 \phi_1}{\partial z^2} + a\frac{\partial \phi_1^2}{\partial z} = -a\frac{\partial \phi_0^2}{\partial z}\frac{\partial \phi_0}{\partial z} \tag{11}$$

Since ϕ_0 is the homogeneous solution of Eq. (11) the method of parameter variation (O'Neil, 2011) leads to the particular solution. Thus, the particular solution $\phi_p = v(z)\phi_0$ with $v(z)$ is determined from $v'(z)\phi_0(z) = -a\frac{\partial \phi_0^2}{\partial z}\frac{\partial \phi_0}{\partial z}$ and $\phi_1(z) = \phi_0 c_3 + \phi_0(z)\int_0^z \frac{1}{\phi_0(z')}\left(-a\frac{\partial \phi_0^2}{\partial z'}\frac{\partial \phi_0}{\partial z'}\right)\, dz'$ in this case $c_3 = 0$ because of the zero boundary conditions. Note, that the boundary conditions of the problem were already absorbed in the determination of the solution ϕ_0 so that all remaining extra boundary conditions are zero, which is a peculiarity of the implemented decomposition method. For all the terms ϕ_i $(i > 1)$ the procedure is repeated in an analogue fashion. By inspection one finds, that the first term of the form $\psi_0(z,t) = a_1 \tanh(a_3 z + a_4) + a_2$ is the dominant one and the result of the first recursion only a correction. The constants may be found using the Richards equation and minimizing the error.

3.3 The time dependent solution

Phenomenological arguments allow now to extend the stationary solution including a time dependence as follows. With increasing infiltration the surface region approaches local saturation so that the scenario characterized by the initial condition is shifted towards increasing depth. Saturation is already present in the asymptotic behaviour of the hyperbolic tangent function and because of the initial condition ($\psi(z,0) = -10, -L \leq z < 0$) the argument of $\tanh$ shall be singular for $t = 0$. The simplest way to introduce a shift is adding a term a_4/t to z in the argument of the hyperbolic tangent function. Last, we apply some "adjustment" to our solution by observing that there exists an asymmetry between the convex and concave parts of the profile, i.e. the edge towards the saturated region is sharper than the one at the edge where the matrix potential assumes a numerical value of approximately $-10\ m$. This may be achieved by multiplying the hyperbolic tangent's argument by a factor $1 + \exp\left(a_3 z + a_4 + \frac{a_5}{t}\right)$. Thus we arrive at a solution in parametrised form, that we evaluate using the original Richards equation.

$$\psi(z,t) = -a_1 \tanh\left(\left(1 + e^{a_3 z + a_4 + \frac{a_5}{t}}\right)\left(a_3 z + a_4 + \frac{a_5}{t}\right)\right) + a_2 \tag{12}$$

Now, the matrix potential ψ is given as a parametrised function $\psi = \psi(z, t; \{a_i\})$ with parameter a_i $(i = 1, 2, 3, 4, 5)$, where the unknown parameter have to be determined.

3.4 Optimization

To adjust the parameter set we insert the parametrised solution (ψ_P) given in Eq. (12) into the governing equation, which for convenience we write in a form where all terms are on the left hand side and consequently the right hand side shall be zero. Let Ω_R be the differential operator that represents the Richards equation with all terms to the left, then for the true solution $\Omega_R[\psi_T] = 0$ holds. Since our solution is an approximate solution the right hand side differs from zero by a residual term $\Omega_R[\psi_P] = R(z,t)$. Thus, the solution presented in Eq. (12) is optimized minimizing $R(z,t)$ using the method of non-linear least squares optimisation and refined by Newton's method.

Some constants can be determined á priori the optimization. We can fix the constants a_1 and a_2 directly using the boundary conditions where $a_1 = (\psi(0,t) - \psi(L,t))/2$ and $a_2 = \psi(0,t) - a_1$. The remaining parameter are determined using the afore mentioned minimisation of R. The objective function that is to be minimised is

$$\sum_{i=0}^{M}\sum_{j=0}^{N}\left[C(\psi)\frac{\partial\psi}{\partial t} - \left(\frac{\partial}{\partial z}[K(\psi)\frac{\partial}{\partial z}\psi] + \frac{\partial K(\psi)}{\partial\psi}\frac{\partial\psi}{\partial z}\right)\right]^2\Bigg|_{(z_i,t_j)} \qquad \rightarrow \qquad \text{min} \; .$$

Since the asymptotics of the solution was fixed using the boundary conditions we use a discrete set of points in the range that contains maximum curvature and the inflection point to optimise $\{a_3,\ldots,a_5\}$. The optimisation may then be simplified using an expansion of the hyperbolic tangent function around the inflection point (at z_0), i.e. where the argument of the function is zero $a_3 z_0 + a_4 + \frac{a_4}{t} = 0$, which allows to solve the minimisation problem in a straight forward fashion.

4. Results

The parameter set that was used for this application refers the two types of soils to a situation that considers the infiltration of water in a column of initially dry and homogeneous soil. We considered as depth range in soil $[0, L = 1m]$ and initial and boundary conditions $\psi(z,0) = -10m$, $-L \le z \le 0$; $\psi(0,t) = -0.75m$ and $\psi(-L,t) = -10m$ for $t > 0$. Further, the results are compared with numerical solutions implemented in the HYDRUS -1D software (Simunek *et al.*, 2005).

Average values of the van Genuchten model soil hydraulic parameters for two soil texture groups according to (Carsel and Parrish, 1988) are: (a) Sand soil $\theta_r = 0.045$, $\theta_s = 0.430$, $\alpha = 0.145cm^{-1}$, $n = 2.68$, $K_s = 712.80cm/day$ (b) Clay soil $\theta_r = 0.068$, $\theta_s = 0.380$, $\alpha = 0.008cm^{-1}$, $n = 1.09$, $K_s = 4.80cm/day$. Figures 4 and 5 show the computed matrix potential for a sequence of times and again one observes the already analysed transition from the transient to the steady state regime and the self-consistency test along the vertical coordinate for the parametrised to the true solution.

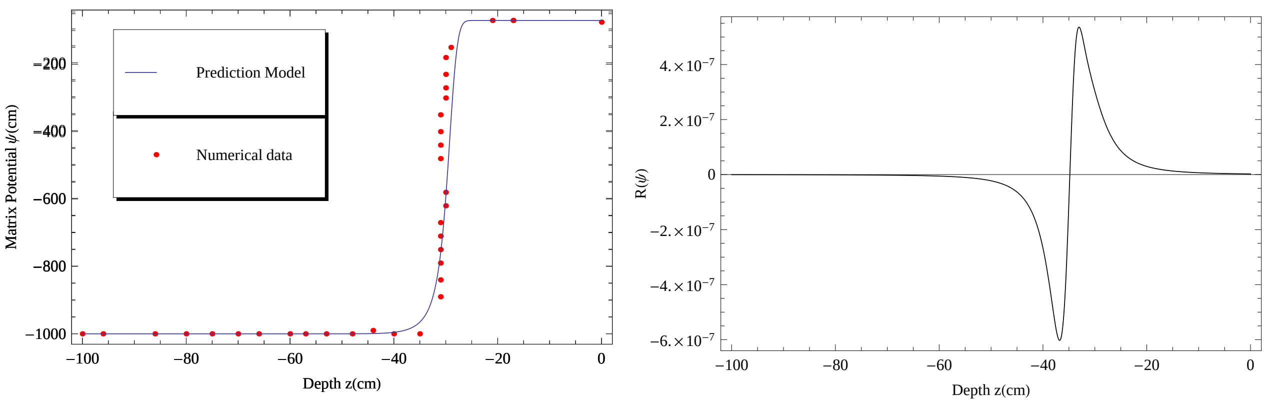

Figure 4. Matrix potential profile and self-consistency with depth for a selection of times for a sand soil (1).

5. Conclusions

In the present work we discussed a methodology to construct a parametrised solution for the Richards equation as a functional solution. It is remarkable that already the initial solution for a recursive scheme given in a relatively compact formula is close to the true solution, so that one may efficiently simulate one dimensional flow of water in unsaturated and saturated porous media. The main difficulties of the problem were the non-linearity and the initial condition. In the present case, the Adomian decomposition scheme strongly diverges and does not solve the non-linear Richards equation. As an ingredient of the present procedure, one may argue that using Padé approximations should simplify the decomposition, in an analytical sense, however the numerically increasing source term with increasing recursion depth clearly turns this procedure a divergent sequence. In this sense the Richards-Genuchten equation is a counterexample to the statement,

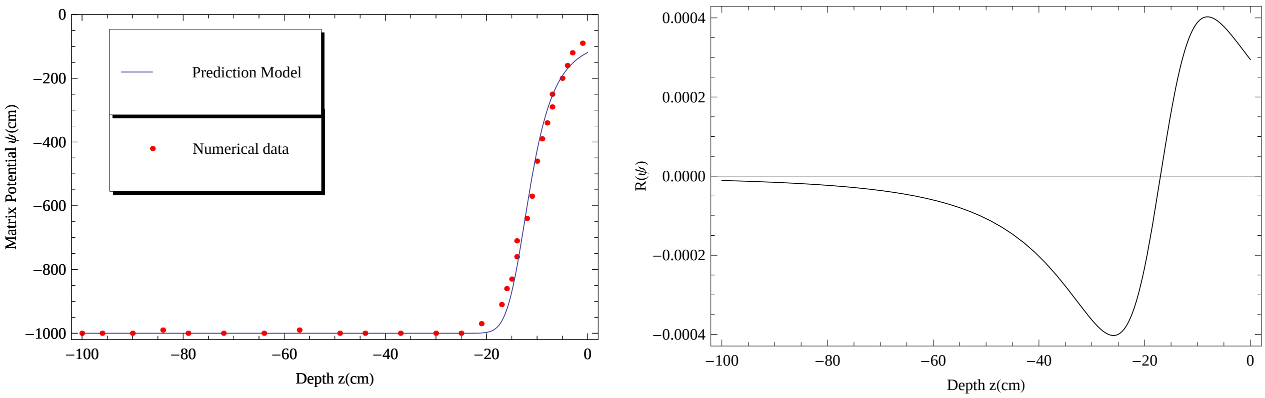

Figure 5. Matrix potential profile and Self-consistency with depth for a selection of times for a clay soil (2).

that Adomians prescription should in principle work for any non-linearity and lead to a convergent recursion procedure. Various implementations of Adomian's idea suffered from the same problem, the increasing source term contributions to the corrections of the solution per recursion step. Our modification of the decomposition method led to a parametrised solution, which was presented in Eq. (12), when optimized by the method of least squares and non-linear Newton's method, gave fairly good results for the matrix potential profile. A self-consistency test accused only small differences between the true and the parametrised solution. For the soils specification used to perform the numerical calculations we found that no further recursion was of need. Moreover, even for other soil compositions and their associated parameter sets the hyperbolic function formula was proven a good approximation. Nevertheless, it is well plausible that using the hyperbolic tangent expression as recursion initialisation will necessitate only a few recursion steps to obtain an acceptable solution within a pre-defined precision, for other soil textures that were not considered in this discussion. In future works (in preparation), we will provide further tests for a more complete collection of soil textures together with an analysis of possible relations between the physical and the optimisation parameters.

6. ACKNOWLEDGEMENTS

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