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GENERALIZED FRAMEWORK FOR ELASTOPLASTIC MODELS: A THEORETICAL AND COMPUTATIONAL APPROACH

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Abstract. *The present article is inserted on the research segment for numerical and computational methods for engineering and it is result from the implementation of elastoplastic constitutive models with isotropic and kinematic hardening in a computing environment named INSANE (INterative Structural ANalysis Environment), a free software for structural analysis, based on object-oriented programming paradigm, developed by the Structural Engineering Department of Federal University of Minas Gerais. The present paper refers to theory of plasticity, a necessary extension of the theory of elasticity that provide more close estimates to the real behavior of the structure's from the constitutive point of view. The theoretical framework necessary to the implementation of the mathematical theory of plasticity is approached, and the formulation based on stress of the von Mises plasticity model is described in detail, highlighting the expressions that shape the surface to isotropic, kinematic and mixed hardening. Finally, the paper shows numerical simulations based on Finite Element Method, considering the constitutive model presented, with linear hardening laws. The results of these analyzes were compared with tests results found in the literature.*

Keywords: *Computational Plasticity, Constitutive Models, Hardening Laws, Return Mapping*

1. INTRODUCTION

The structural engineer's work has always been related to the comprehension of the physical simplifications represented by mathematics models, being responsible for determine the force field that is acting over the material and to see the response of it to these forces. For a long time, structures were planned based on mathematical theory of elasticity, using linear static analysis, where the shifts are made infinitesimally and the Hooke's Law is valid. In the project's conception, the structure should be subjected to a admitted stress lower than its elastic limit, often neglecting an structural analysis better suited and a deep knowledge about the properties of the material.

To comprehend the real behavior of an structure is very complex, due to the variation of the acting tensions in the piece, that often surpass the elastic range limit. It's in this context that appears the plasticity theory, a necessary extension of the elasticity theory, that give us an realistic approach from the material's behavior and consequently, the structure.

The plasticity theory seeks to describe, mathematically, instant and non-reversive deformations, that happens in a solid piece, so, the deformations that don't disappears when removed the forces from which they originate (Chen and Han, 1988; Lubliner, 1990; Souza Neto *et al.*, 2008).

The development of models for materials and its uses in computational plasticity is of great importance ant interest (Bathe and Montáns, 2004). The analysis of this kind of material behavior becomes complex due to the fact that different materials requires different constitutive models for its correct characterization. The description of the elastoplastic behavior from the macroscopic viewpoint (designated as phenomenological behavior) for multiaxial stress states is set for the following asumptions:

(1) The existence of an elastic field, that means, a region of tensions in which the material behaves as It is purely elastic without the appearance of permanent deformations. The elastic domain is delimited by a flow function written in terms of yield stress.

(2) The occurrence of inelastic deformation, or the deformations associated with tensions above the yield stress, whose growth can be described by a flow rule.

(3) The occurrence of the phenomenon of strain hardening of the material, in other words, the possibility of increasing flow tensions, following the evolution of inelastic deformations.

It's important to note that the hypotheses mentioned can be observed not only in metals but also in several types of materials such as concrete, rocks and soils, among others. It should be noted that, according to the type of material, different experimental procedures may be required for the verification of such properties (Souza Neto *et al.*, 2008).

The steady growth of research in computational mechanics area contributes to the emergence of many softwares used by engineers and researchers, able to perform sophisticated analysis, taking as a backdrop rather complex constitutive models such as those using the theory of plasticity. In this context, the computational environment INSANE (INteractive Structural ANalysis Environment) is being developed in the Structuctural Engineering Department of Federal University of Minas Gerais.

In the literature there are various constitutive models that seek to represent a more realistic behavior of materials and structures. One of the most known criteria is the one proposed by von Mises to describe the plastic flow in metals and materials that don't depend of the hydrostatic pressure.

Thus, the aim of this work is to present a generic theoretical and computational framework for the implementation the elastoplastic constitutive model as well as its application to the von Mises model. Such implementation was performed in the INSANE system, also featuring the return mapping algorithm Closest-Point-Projection necessary for the integration of constitutive relations that rules the material's behavior in the context of computational plasticity.

The article is organized as follows. Section 2 presents the tension-based mathematical formulation, of a framework for constitutive models. In Section 3, we introduce the formulation of the von Mises model, as proposed by the framework. In Section 4, the return mapping algorithm is described in details. We conclude the article with numerical simulations of physically non-linear structural behavior models via Finite Element Method considering the constitutive model presented and taking into account linear hardening law.

2. GENERIC THEORETICAL FRAMEWORK FOR CONSTITUTIVE MODELS

The constitutive models generally have their own rating and, although in many cases keep similarities, the lack of unity of formulations prevents a generic and objective computational implementation (Penna, 2011).

From the work of Penna (2011), the implementation of constitutive models in INSANE system became described by potential functions and their gradients, which result in a series of tensor operations capable to handle the constitutive tensors, the gradients of the functions and conditions of loading and unloading. The operations of unified theoretical framework and the particularities of each model are then assigned to a set of classes, called Filters, created to account for parcels that make up common operations. The implementations are such that each Filter contains the information of the constitutive model he wants to represent and assumptions of the analysis model. The constitutive model is handled by the Filter in the tensor format, enabling analysis model to provide the components of each tensor (stiffness, flexibility, stress and strain) required for its assembly. The formulation of constitutive models and notations will be stress-based and follow the theoretical framework proposed by Carol et al. (1994).

The rate equations for the stress-based elastoplastic formulation can be expressed as

$$\dot{\sigma}_{ij} = E^0_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^p_{kl}), \tag{1}$$

with

$$\dot{\varepsilon}^p_{kl} = \dot{\lambda}_m m_{mkl}, \tag{2}$$

where E^0_{ijkl} are the components of elastic tensor stiffness, $\dot{\lambda}_m$ are the plastic multipliers, m_{mkl} are the gradients of plastic potentials functions Q (as $m_{kl} = \partial Q/\partial\sigma_{kl}$ and specifies the flow rule direction) and $\dot{\varepsilon}^p_{kl}$ is the variation of plastic strain.

Assuming of a yielding surface that can be expressed as $F_n(\boldsymbol{\sigma}, \mathbf{p})$, where $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{p}$ is a vector of internal variables, we get a consistency condition expressed by

$$n_{ij}\dot{\sigma}_{ij} - H_{mn}\dot{\lambda}_m = 0 \quad \text{with} \quad n_{ij} = \left.\frac{\partial F_n}{\partial \sigma_{ij}}\right|_p \quad \text{and} \quad H_{mn} = -\left.\frac{\partial F_n}{\partial \lambda_m}\right|_{\boldsymbol{\lambda}} = -\left.\frac{\partial F_n}{\partial p_q}\right|_{\boldsymbol{\sigma}} \frac{\partial p_q}{\partial \varepsilon^p_{kl}} m_{mkl} \tag{3}$$

where n_{ij} represents F_n derivatives for constant values of the plastic multiplier λ (so $\dot{\lambda} = \varepsilon^p_{kl} = 0$, does not occur any change of plastic strains and internal variables) and geometrically, it represents a normal vector to the current loading $F_n = 0$ in the stress space. In associative model the loading function F and the flow rule are defined in such a way that n_{ij} and m_{ij} are fully proportionals (often stated alternatively as $Q = F$). H is the linear isotropic hardening-softening modulus obtained from F derived for constant values of σ, being positive in case of hardening (an expanded flow surface); zero for perfect plasticity case (without changing the flow surface) and negative for softening (contraction of the flow surface).

By introduction equations 1 and 2 into 3, it has to the classical strain-driven format of the plastic multiplier, which is given by

$$\dot{\lambda}_m = \frac{n_{ncd}E^0_{cdkl}\dot{\varepsilon}_{kl}}{H_{mn} + n_{npq}E^0_{pqrs}m_{mrs}} \tag{4}$$

The expressions derived here provide the basic formulae for the tangent operators required in the assembly of the tangent stiffness tensor. Finally, the rate form of the constitutive equation can be expressed as

$$\dot{\sigma}_{ij} = E^t_{ijkl}\dot{\varepsilon}_{kl} \tag{5}$$

where E^t_{ijkl} is the so-called elastoplastic, or continuum, tangent modulus tensor and is given by

$$E^t_{ijkl} = E^0_{ijkl} - \frac{E^0_{ijab}m_{mab}n_{ncd}E^0_{cdkl}}{H_{mn} + n_{npq}E^0_{pqrs}m_{mrs}} \tag{6}$$

3. APPLICATION TO THE VON MISES'S CONSTITUTIVE MODEL

The criterion proposed by von Mises in 1913, requires that the yielding starts when the critical value of distortion energy per unit volume is reached. This is equivalent to consider that the second principal invariant of the stress deviator tensor, J_2, reaches a critical value. The mathematical representation of the von Mises constitutive model and the flow rule for mixed hardening is given by:

$$F(\sigma,\alpha,\beta) = \sqrt{3J_2(\eta(\sigma,\beta))} - (\sigma_y - H\alpha), \tag{7}$$

where H is the linear isotropic hardening-softening modulus, α is the function of flow accumulated plastic and the stress tensor η, given by:

$$\eta(\sigma,\beta) = s(\sigma) - \beta, \tag{8}$$

with

$$\mathbf{s} = \sigma - \frac{1}{3}(tr[\sigma])I \tag{9}$$

where $tr[\sigma]$, in principal stress space, is $tr[\sigma] = \sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$. η is defined as the difference between the stress deviator tensor s and the stress tensor β, known as "*back-stress*" and whose evolution law is given by

$$\beta = K\dot{\varepsilon}^p, \tag{10}$$

where K is the kinematic hardening-softening modulus.

β is the tensor relating to thermodynamic variables associated with kinematic hardening and represents the evolution of the plasticity surface center position in the stress space. When $\beta = 0$, we have $\eta = s$, and the flow surface defined by $F = 0$ is the von Mises with isotropic hardening and yield stress σ_y.

Also in equation 7, we have the second principal invariant of the stress deviator tensor given by:

$$J_2(\eta) = -I_2(\eta) = \frac{1}{2}tr[\eta^2] = \frac{1}{2}\eta : \eta = \|\eta\|^2 \tag{11}$$

So, we have

$$n_{ij} = \frac{\partial\Phi}{\partial\sigma} = \sqrt{\frac{2}{3}}\frac{\eta}{\|\eta\|} \tag{12}$$

where $\|\eta\| = \sqrt{\eta:\eta}$. In an associative model

$$n_{ij} = m_{ij} = \sqrt{\frac{2}{3}}\frac{1}{\sqrt{\eta:\eta}}\begin{bmatrix} \frac{2}{3}\sigma & 0 & 0 \\ 0 & -\frac{1}{3}\sigma & 0 \\ 0 & 0 & -\frac{1}{3}\sigma \end{bmatrix} \tag{13}$$

The law associated with hardening or softening phenomenon is given by

$$H = -\frac{\partial F}{\partial\sigma_y}\frac{\partial\sigma_y}{\partial\varepsilon^p_{kl}} \tag{14}$$

with

$$\frac{\partial F}{\partial\sigma_y} = -1 \quad \text{e} \quad \frac{\partial\sigma_y}{\partial\varepsilon^p_{kl}} = \mathscr{H} + \mathscr{K} \tag{15}$$

with $\mathscr{H}$ being the plastic modulus of isotropic hardening/softening and $\mathscr{K}$ is the kinematic hardening/softening modulus.

4. INTEGRATION ALGORITHM FOR PLASTICITY: CLOSEST-POINT PROJECTION

The main problem of elastoplastic analysis lies in the isolated determination of elastic and plastic portions of the total strain increment, because the classical plasticity models do not have a full relationship between stress and strain, resulting in the need to perform the integration of the constitutive law, due to the fact that the elastoplastic module is a function of the plastic deformation, so elapsing an iterative process.

Due to the incremental nature of the numerical models of plasticity, we should use a return mapping class algorithm able to get the update of the stresses needed to balance the internal forces in the non-linear analysis.

Ortiz and Popov (1985) point out three basic requirements that efficient algorithms must complete:

(1) to ensure consistency of the constitutive equations to be integrated;

(2) to have numerical stability;

(3) and to ensure consistency of the incremental model.

The choice of an inappropriate algorithm can not only lead to inaccurate solution to stress and may also delay the convergence of equilibrium iterations or even lead to divergence of the process.

There is a great variety of integration methods with different levels of complexity (Taqieddin, 2008), among them, the closest-point projection is one of the most important. From a physical point of view, the method is an implicit systematic process, considering a flow of plastic waste to ensure that the flow function in the current step "n + 1" is equal to zero. The geometric interpretation of the iteration scheme is shown in Fig. 1.

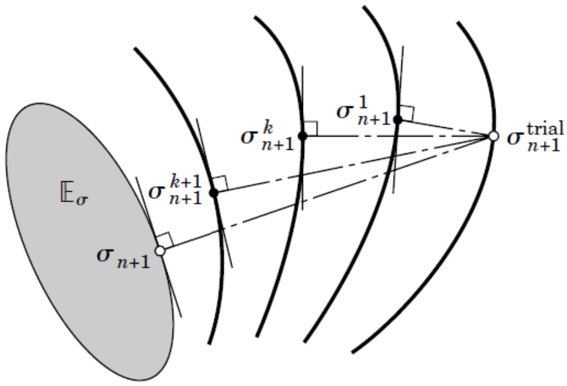

Figure 1. A geometric interpretation of the closest point projection algorithm in stress space. At each iterate $(\bullet)^k$, the constraint is linearized to find the intersection (cut) with $f = 0$. The next iterate $(\bullet)^{k+1}$, located on level set $f^{k+1}_{n+1} > 0$ is the closest point of that level set to the previous iterate $(\bullet)^k$ (Simo and Hughes, 1998).

Conceptually, the idea of the algorithm is quite simple and consists of a systematic application of Newton's method to the system of equations 16, which results in the calculation of the projection from the nearest point of a state of stress test of the flow surface. The algorithm is explained below on a general framework proposed by Simo and Hughes (1998).

(1) Assume the existence of a plastic loading, that is, $f^{trial}_{n+1} > 0$, such as the consistency parameter $\Delta\gamma > 0$. Set the Residual plastic flow R_{n+1} and the flow condition.

$$R^{(k)}_{n+1} := \left\{ \begin{array}{c} -\varepsilon^{p^{(k)}}_{n+1} + \varepsilon^p_n \\ -\alpha^{(k)}_{n+1} + \alpha_n \end{array} \right\} + \Delta\gamma^{(k)}_{n+1} \left\{ \begin{array}{c} \partial_\sigma f_{n+1} \\ \partial_q f_{n+1} \end{array} \right\} \quad \text{e} \quad f^{(k)}_{n+1} := f(\alpha^{(k)}_{n+1}, q^{(k)}_{n+1}) \tag{16}$$

where $\sigma^{(k)}_{n+1} := \nabla W(\varepsilon_{n+1} - \varepsilon^{p^{(k)}}_{n+1})$, $q^{(k)}_{n+1} := -\nabla H(\alpha^{(k)}_{n+1})$ and W the represents stored elastic energy.

(2) Compute elastic moduli C and consistent tangent moduli D :

$$C^{(k)}_{n+1} := \nabla^2 W(\varepsilon_{n+1} - \varepsilon^{p^{(k)}}_{n+1}) \quad \text{and} \quad D^{(k)}_{n+1} := -\nabla^2 H(\alpha^{(k)}_{n+1}) \tag{17}$$

$$[A^{(k)}_{n+1}]^{-1} := \begin{bmatrix} [C^{(-1)}_{n+1} + \Delta\gamma_{n+1}\partial^2_{\sigma\sigma} f_{n+1}] & \Delta\gamma_{n+1}\partial^2_{\sigma q} f_{n+1} \\ \Delta\gamma_{n+1}\partial^2_{q\sigma} f_{n+1} & [D^{(-1)}_{n+1} + \Delta\gamma_{n+1}\partial^2_{qq} f_{n+1}] \end{bmatrix}^k \tag{18}$$

where $[A^{(k)}_{n+1}]^{-1}$ is a matrix, in which the arguments refer to elastic and tangent modulus.

(3) Obtain increment of consistency parameter

$$\Delta^2\gamma_{n+1}^{(k)} := \frac{f_{n+1}^{(k)} - [\partial_\sigma f_{n+1} \partial_q f_{n+1}]^T A_{n+1}^{(k)} R_{n+1}^{(k)}}{[\partial_\sigma f_{n+1} \partial_q f_{n+1}]^T A_{n+1}^{(k)} \left\{ \begin{array}{c} \partial_\sigma f_{n+1} \\ \partial_q f_{n+1} \end{array} \right\}^k} \tag{19}$$

(4) Obtain increments of plastic strain and internal variables

$$\left\{ \begin{array}{c} \Delta\varepsilon_{n+1}^{p^{(k)}} \\ \Delta\alpha_{n+1}^{(k)} \end{array} \right\} = \begin{bmatrix} [C_{n+1}^{(-1)} + \Delta\gamma_{n+1}\partial_{\sigma\sigma}^2 f_{n+1}] & 0 \\ 0 & [D_{n+1}^{(-1)} + \Delta\gamma_{n+1}\partial_{qq}^2 f_{n+1}] \end{bmatrix}^k A_{n+1}^k \left[R_{n+1}^{(k)} + \Delta^2\gamma_{n+1}^{(k)} \left\{ \begin{array}{c} \partial_\sigma f_{n+1} \\ \partial_q f_{n+1} \end{array} \right\}^{(k)} \right] \tag{20}$$

(5) Update the state variables and consistency parameter

$$\varepsilon_{n+1}^{p^{(k+1)}} = \varepsilon_n^{p^{(k)}} + \Delta\varepsilon_{n+1}^{p^{(k)}} \quad , \quad \alpha_{n+1}^{p^{(k+1)}} = \alpha_n^{p^{(k)}} + \Delta\alpha_{n+1}^{p^{(k)}} \quad \text{and} \quad \Delta\gamma_{n+1}^{p^{(k+1)}} = \Delta\gamma_n^{p^{(k)}} + \Delta^2\gamma_{n+1}^{p^{(k)}} \tag{21}$$

5. NUMERICAL EXAMPLES

5.1 Uniaxial stress loading

In order to analyze different forms of hardening (isotropic, kinematic and mixed) that shape von Mises surface with linear hardening/softening laws, some simulation are present. A rectangular plate in plane stress state is adopted geometry configuration, load the boundary conditions and FEM mesh of the plate are shown in Fig. 2.

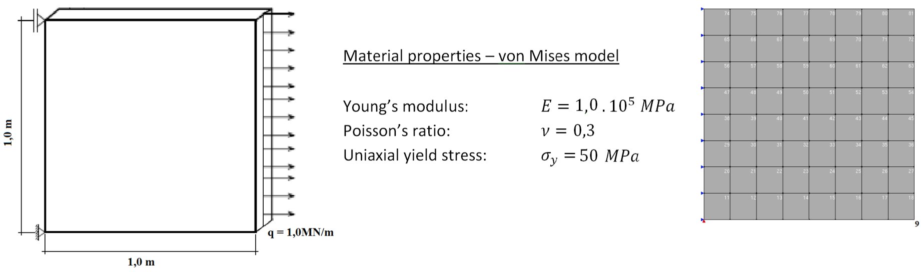

Figure 2. Geometry, loads and boundary conditions for the uniaxial stress loading simulations

Adopting the mesh with 64 quadrilateral elements of 4 knots, we proceeded with the parameter setting for the reproduction of perfect plasticity, hardening and softening behaviors. The nonlinear analysis was performed under Plane Stress conditions and direct displacement control method, where the horizontal displacement of the loaded face was controlled, with tolerance of 1×10^{-4} for unbalanced force.

We examine the behavior of the plate for linear hardening law case, and using module $1500MPa$ ($\mathscr{H} = 500MPa$ e $\mathscr{K} = 1000MPa$). The results can be seen in Fig. 3.

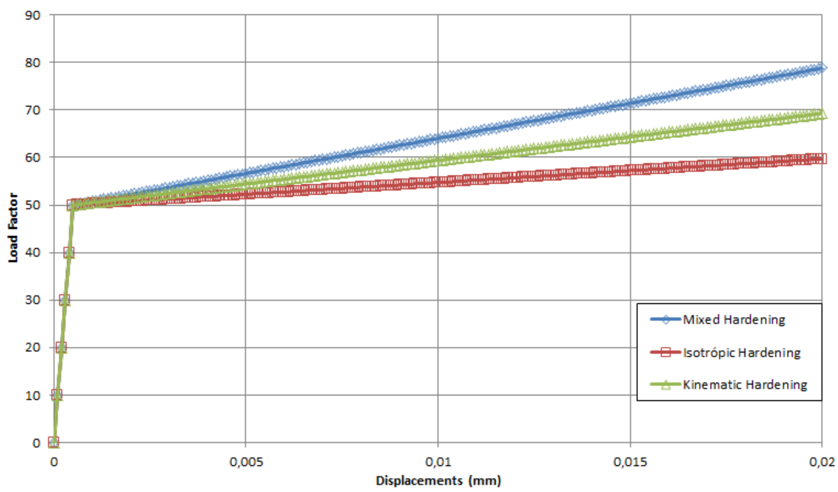

Figure 3. Curves Load Factor x Displacement for isotropic, kinematic and mixed Hardening

It's possible to see from the obtained curves that, when using the mixed hardening, stresses after hit the flow, achieves a higher level for a same level of deformation as compared with the isotropic and kinematic hardening curves alone. This result was expected, since the mixed strain hardening module is the combination of the other modules. Another thing to note is that in the mixed hardening, the tensions evolved quicker than the isotropic hardening, however, this point can not be taken as a general rule, since the hardening modules are not the same for both cases.

From the stress values, surfaces flow of the plate were generated, as can be seen in Fig 4. In this figure, we can clearly observe the elliptical initial surface, generated before any loading was applied. After the application of loading, other surfaces are obtained, one for each type of hardening (isotropic, kinematic and mixed).

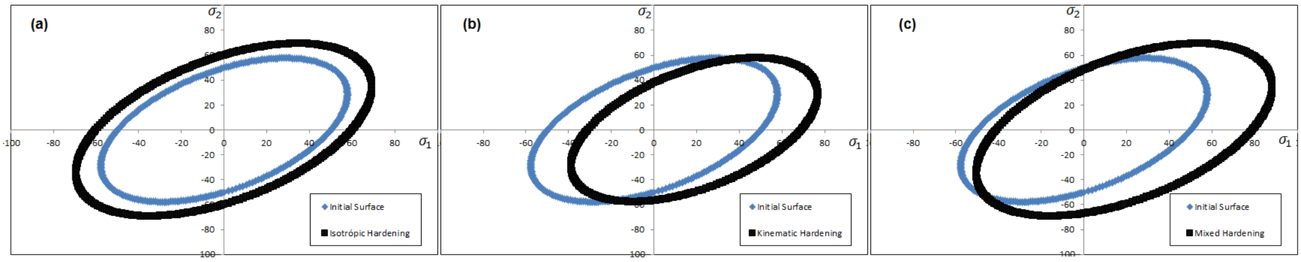

Figure 4. Von Mises's yield surface

The obtained results are consistent when compared with theory. In the case of the isotropic hardening (figure 4-a) the surface suffered uniform expansion, maintaining its center position unchanged, observing the associative characteristic of isotropic hardening, that is, since the stress evolves in one direction, it evolves equally in all others. In the case of the kinematic hardening moduli (figure 4-b), the surface maintained its shape and size, compared to the surface of isotropic hardening, ocurring only its translation. It is also observed isotropic hardening, that is, since the tension evolves in one direction, it evolves equally in all others, the kinematic hardening is not associative, meaning, the yield stress is only changed in the direction in which loading it is being applied. In figure 4-c, we can see the effects of mixed strain hardening where the flow surface expands and moves simultaneously.

5.2 Bending of a V-Notched Bar

The analysis of the failure of a rectangular metal bar is made. The bar contain a V notch and is subjected to pure bending. The results are compared with the analytical results of Green (1953) apud Souza Neto *et al.* (2008).

The geometrical and physical characteristics of the problem as well as the finite element mesh are illustrated in Fig. 5. Due to the symmetry, only half of the bar was discretized. The loading conditions were imposed by the application of a binary of opposing forces $F = 233625N$, prescribed in knots 3 and 4.

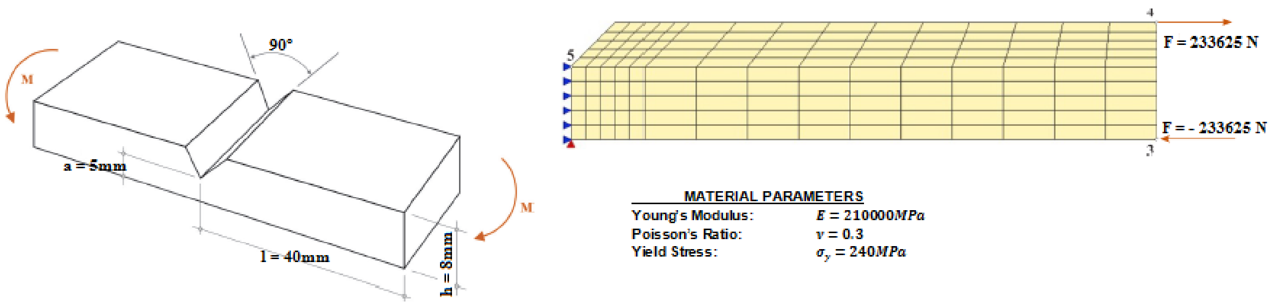

Figure 5. Bending of a V-Notched bar: geometry, loading and boundary conditions and FEM mesh. Adapted from Souza Neto *et al.* (2008)

The analysis was carried out with the adjustment of hardening parameters for the perfect plasticity behavior, adopting zero for isotropic and kinematic modules. The non-linear solution was performed using, the generalized displacement control method (Yang and Shieh, 1990), load factor 2.50×10^{-5} and tolerance of 1×10^{-3}. The results are shown in FIG. 6.

Figure 6 shows the comparison between the analytical normalized bending moment per unit width, M/ca^2 and the numerical results. The normalized moment obtained by the model is $0,630357$, this value is $1,18\%$ higher than the Green's upper limit, allowing to conclude that the model value, can adequately represent perfect plasticity behavior.

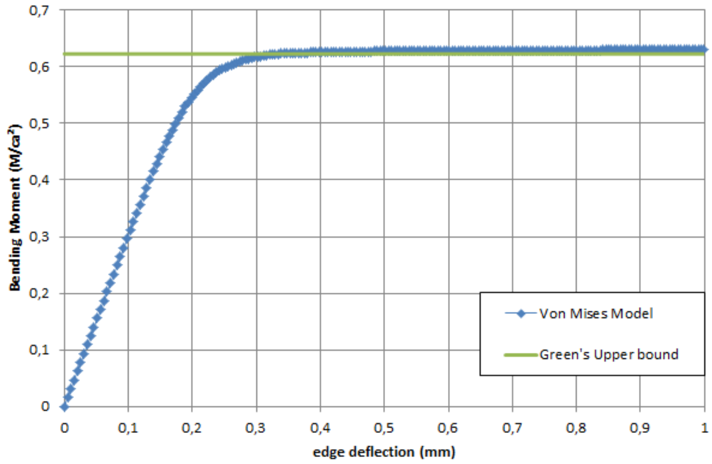

Figure 6. Comparison between the result of Green (1953) and the result obtained by von Mises model.

6. CONCLUSION

The traditional von Mises plasticity model had its formulation presented in this work, according to the Environment Theoretical and Computational for Constitutive Models developed by Penna (2011) verifying the great potential of the constitutive models library of the INSANE system, due to the majority of the models implementations and the several resources for constitutive modeling of the platform.

By numerical examples provided we could observe the validity of the von Mises model. However, it was also observed that the model implemented needs to present a better connection with different inelastic laws, so you can represent, more accurately, a non-linear behavior.

The continuity of this research involves the implementation of new elastoplastic constitutive models for simulation of other materials, such as concrete, soil, etc. Among the more used surfaces break, we can mention Tresca, Mohr Coulomb, Rankine and Drucker Prager criterias.

7. ACKNOWLEDGEMENTS

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