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NEW NUMERICAL METHODS FOR PHOTOELASTIC PARAMETERS EVALUATION WITH PLANE POLARISCOPE

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Abstract. *The purpose of this research is get new numerical equations for the phase shifting technique in plane polariscopes. The numerical equations are an alternative approach to the algebraic techniques in digital photoelasticity. The numerical method has the advantage that the number of images is only limited by the resolution of the polariscopes, therefore this method can get more images than conventional algebraic method. With more equations, the average errors are minimized due to increased observations number. Another advantage is the versatility of the numerical technique, which can be used in any commercially available polariscope. Theoretical tests applied to the numerical equations gave good results, that indicate which the model has high chance of give satisfactory results, with very low possibility of it has being wrong.*

Keywords: *photoelasticity, numerical methods, plane polariscope, phase-shifting technique.*

1. INTRODUCTION

The photoelasticity is an experimental technique to analyze stress fields on models made with polymers that have the temporary birefringent property. Whenever the model is free of stresses, it displays optical isotropy, otherwise, it exhibits optical anisotropy (Dally and Riley, 2005) with the appearance of isochromatic and isoclinic fringes, related to principal stresses difference and the principal directions, respectively. The photoelastic technique can be applied in the analyze of complex geometries, in dynamic analysis, in analysis of large equipment and in many others situations where the aim is know the response of the material to load applied. In this time, the greatest challenge for researchers is develop techniques that permit improve the automation of the photoelastic analysis (Magalhães, 2011). The automation is a pre-requirement to expand the use of the technique in industrial environment, besides allowing the acquisition of a larger number of images, increase the accuracy and decrease the time for get the analysis results.

The phase shifting technique allows the determination of isoclinic and isochromatic fringes from intensity data of three or more optical arrangements. For achieve the phase shifting, the optical elements of the polariscope need to be rotated independently and positioned in determined positions defined by the technique used, but it is not the case of the most commercially available polariscopes, which have limitations of positions in their optical elements due to its constructive characteristics. The main advantage of the numerical method proposed in this research is the capacity of application of phase shifting technique in any set of optical arrangements of the polariscope. Another advantage is use more images than the conventional algebraic approach; therefore, the numerical approach can decrease the measurement uncertainties by increase of the number of observations (Magalhães, 2011).

The shift from obtaining equations for calculating the phase analytically to obtaining them numerically is a significant innovation. It breaks a paradigm that was hitherto used by several authors (Magalhães and Magalhães Jr., 2012). The plane polariscopes are one of the simplest optical arrangements possible in photoelasticity (Ramesh, 2000), they are immune to mismatch of quarter-wave plate and need fewer adjusts than circular polariscopes, because of less number of optical elements.

In this research, a new numerical method for the phase shifting technique in plane polariscopes was developed to overcome the limitation of positioning of the polariscopes widely used in laboratories of industries and universities. To validate the model a theoretical test was performed, this is the same method used in Magalhães and Magalhães Jr. (2012).

2. THE THEORETICAL BASE

The plane polariscope is one of the simplest optical arrangements possible in the photoelastic technique. The Fig. 1 shows the typical arrangement, when P, R and A represent polarizer, retarder (stressed model) and analyzer, respectively. The subscripts written after P and A means the angle between the polarizing axis and the reference axis x, thus, $P_\alpha R_{\theta,\delta} A_\beta$ indicates that the polarizer is at α and analyzer is at an arbitrary angle β with x-axis. The subscripts δ and θ of R indicate the retardation added by stressed sample and whose slow axis is at angle θ with x-axis.

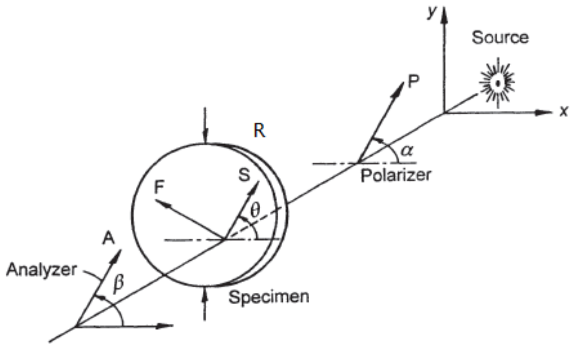

Figure 1. Optical arrangement of a plane polariscope

The optical elements of the polariscopes introduce rotation and retardation on the light waves. With the Jones calculus for any arrangement of the plane polariscope is possible represent these operations as matrices. Then the arrangement $P_\alpha R_{\theta,\delta} A_\beta$, shown in Fig. 1, has its electrical field in light along and perpendicular to the analyzer axis (E_β, $E_{\beta+90°}$) given as (Ramesh, 2000)

$$\begin{pmatrix} E_\beta \\ E_{\beta+90°} \end{pmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\frac{\delta}{2} - i\cdot\sin\frac{\delta}{2}\cos 2\theta & -i\cdot\sin\frac{\delta}{2}\sin 2\theta \\ -i\cdot\sin\frac{\delta}{2}\sin 2\theta & \cos\frac{\delta}{2} + i\cdot\sin\frac{\delta}{2}\cos 2\theta \end{bmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \cdot K e^{i\omega t} \quad (1)$$

where $i = \sqrt[2]{-1}$. The symbols K and ω are the amplitude and the angular frequency of the light vector, respectively. For find an equation that represents the output light intensity the following operation is made (Ramesh, 2000):

$$I = E_\beta E_\beta^* . \quad (2)$$

In Eq. (2), I is the output light intensity and E_β^* is the complex conjugate of E_β. After operation using Eq. (2) the output intensity emerging from analyzer of the polariscope arrangement $P_\alpha R_{\theta,\delta} A_\beta$ is given by (Ramesh, 2000) as

$$I = I_o\left[\left(\cos\frac{\delta}{2}\right)^2 (\cos(\beta-\alpha))^2 + \left(\sin\frac{\delta}{2}\right)^2 (\cos(\beta+\alpha-2\theta))^2\right] \quad (3)$$

where I_o accounts for the amplitude of incident light vector and the proportionality constant.

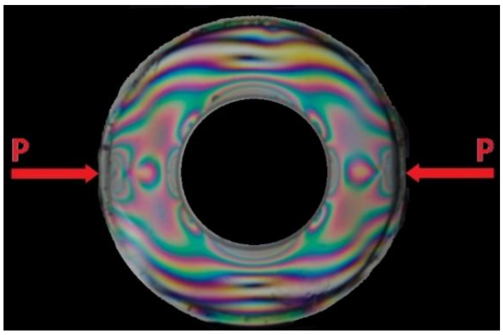

Figure 2. Sample under compression

In the experimental procedure, shown in Fig. 2, the inner diameter, the outer diameter and the thickness of the annular disk are d = 38 mm, D = 75 mm and t = 6 mm, respectively. A diametrically opposite load, p = 50 N is applied on annular disk. An epoxy resin with stress fringe value F = 296 N/mm is used. In photoelastic analysis, the theoretical value of isochromatic δ is related to two principal stress components, σ_1 and σ_2, as in Eq. (4). The theoretical isoclinic angle θ can be calculated using Eq. (5) with the stress components σ_x, σ_y and τ_{xy} (Magalhães and Magalhães Jr., 2012).

$$\delta = \frac{2\pi \cdot t}{F}(\sigma_1 - \sigma_2) \tag{4}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) \tag{5}$$

The exact value of the stress field in polar coordinates, as a function of r and φ with its origin at the center of disk, can be calculated by elasticity theory, Tokovyy, et al. (2010) give the following solution

$$\sigma_r^e = \frac{p}{\pi}\left[-\frac{\rho^2 - k^2}{1-k^2}\frac{2\Theta}{\rho^2} + \sum_{m=1}^{\infty}\frac{(-1)^m R_m(\rho)}{m}\sin(2m\Theta)\cos(2m\varphi)\right] \tag{6}$$

$$\sigma_\varphi^e = \frac{p}{\pi}\left[-\frac{\rho^2 - k^2}{1-k^2}\frac{2\Theta}{\rho^2} + \sum_{m=1}^{\infty}\frac{(-1)^m \Phi_m(\rho)}{m}\sin(2m\Theta)\cos(2m\varphi)\right] \tag{7}$$

$$\tau_{r\varphi}^e = \frac{p}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m S_m(\rho)}{m}\sin(2m\Theta)\cos(2m\varphi) \tag{8}$$

where e means exact result, p is the pressure applied to the ring, Θ is the angle of application of pressure, $\rho = r/R_1$ and $k = R_1/R_2$ with r, R_1, R_2 equal to an arbitrary radius, the internal radius and the external radius of the ring, respectively. The functions of ρ are

$$\begin{aligned} R_m(\rho) &= \left(2(m+1)\rho^{-2m} - (2m+1)k^2\rho^{-2(1+m)} - k^{2(1-2m)}\rho^{-2(1-m)}\right)A_m \\ &-\left(2(m-1)\rho^{2m} - (2m-1)k^2\rho^{-2(1-m)} + k^{2(1+2m)}\rho^{-2(1+m)}\right)B_m \end{aligned} \tag{9}$$

$$\begin{aligned} \Phi_m(\rho) &= \left((2m+1)k^2\rho^{-2(1+m)} - 2(m-1)\rho^{-2m} + k^{2(1-2m)}\rho^{-2(1-m)}\right)A_m \\ &-\left((2m-1)k^2\rho^{-2(1-m)} - 2(m+1)\rho^{2m} - k^{2(1+2m)}\rho^{-2(1+m)}\right)B_m \end{aligned} \tag{10}$$

$$\begin{aligned} S_m(\rho) &= \left(2m\rho^{-2m} - (2m+1)k^2\rho^{-2(1+m)} + k^{2(1-2m)}\rho^{-2(1-m)}\right)A_m \\ &+\left(2m\rho^{2m} - (2m-1)k^2\rho^{-2(1-m)} - k^{2(1+2m)}\rho^{-2(1+m)}\right)B_m \end{aligned} \tag{11}$$

the constants of integration A_m and B_m have the form

$$A_m = \frac{k^{2(1+m)}\left(k^{2m} - k^{-2m}\right) - 2m\left(1-k^2\right)}{4m^2\left(1-k^2\right)^2 - k^2\left(k^{2m} - k^{-2m}\right)^2} \tag{12}$$

$$B_m = \frac{2m\left(1-k^2\right) - k^{2(1+m)}\left(k^{2m} - k^{-2m}\right)}{4m^2\left(1-k^2\right)^2 - k^2\left(k^{2m} - k^{-2m}\right)^2} \tag{13}$$

With these equations σ_1, σ_2 and θ can be obtained as follows

$$\sigma_{1,2}^{e} = \frac{\sigma_r + \sigma_\varphi}{2} \pm \sqrt{\left(\frac{\sigma_r - \sigma_\varphi}{2}\right)^2 + \sigma_{r\varphi}^2} \tag{14}$$

$$\theta^{e} = \frac{1}{2}\tan^{-1}\left(\frac{(\sigma_r - \sigma_\varphi)\sin(2\varphi) + 2\sigma_{r\varphi}\cos(2\varphi)}{(\sigma_r - \sigma_\varphi)\cos(2\varphi) - 2\sigma_{r\varphi}\sin(2\varphi)}\right) \tag{15}$$

The change of polar coordinates to Cartesian coordinates is given by

$$\sigma_{xx}^{e} = \sigma_{rr}\cos^2(\theta) + \sigma_{\theta\theta}\sin^2(\theta) - \tau_{r\theta}\sin(2\theta) \tag{16}$$

$$\sigma_{yy}^{e} = \sigma_{rr}\sin^2(\theta) + \sigma_{\theta\theta}\cos^2(\theta) + \tau_{r\theta}\sin(2\theta) \tag{17}$$

$$\tau_{xy}^{e} = (\sigma_{rr} - \sigma_{\theta\theta})\cos(\theta)\sin(\theta) + \tau_{r\theta}\cos(2\theta) \tag{18}$$

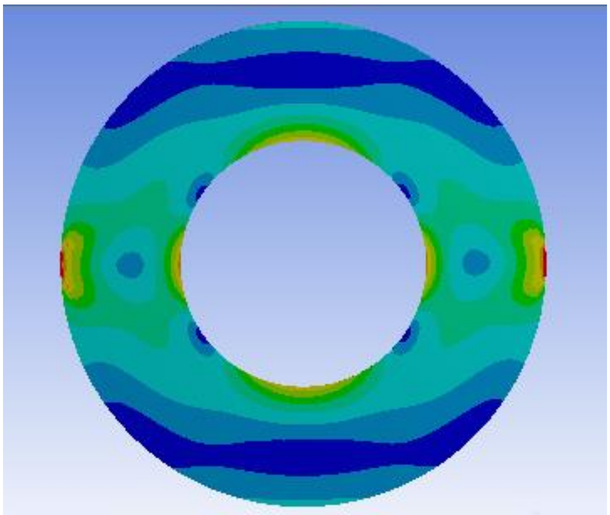

Figure 3. Analytical solution of the isochromatic fringes

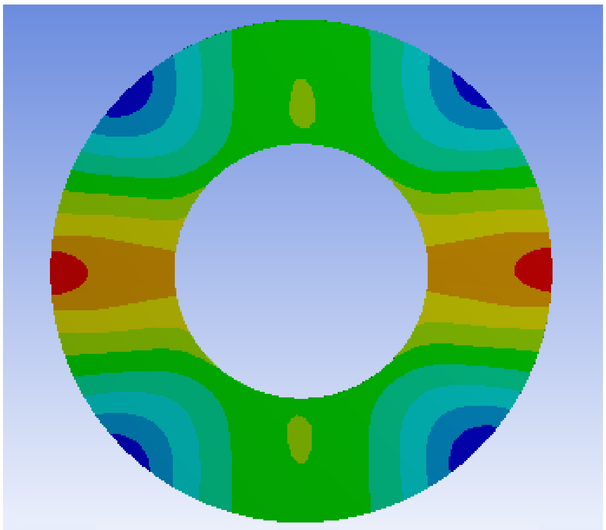

Figure 4. Analytical solution of the isoclinic fringes

Thus, using Eq. (14) and Eq. (15), the exact value of δ and θ can be calculated for each point of the annular disk. How in Magalhães and Magalhães Jr. (2012), we compare the experimental measurements of δ and θ, obtained from the proposed method in this research, with its exact values to test the new equations. The Fig. 3 and 4 shows analytical results applying Eq. (14) and (15) for isochromatic and isoclinic on the ring.

3. THE NEW PROPOSED METHOD

Using the same principle of the mathematical model proposed in Magalhães and Magalhães Jr. (2012), we make a new model for the equations of phase obtained by comparing with the conventional form of obtain equations of phase for the plane polariscope. The general equation for calculate the photoelastic parameters with any number N of images is proposed:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{\sum_{j=1}^{N} b_j I_j}{\sum_{j=1}^{N} c_j I_j}\right) \tag{19}$$

$$\delta = \cos^{-1}\left(\frac{\cos 2(\theta-\alpha)\left(\sin 2\theta \sum_{j=1}^{N} d_j I_j + \cos 2\theta \sum_{j=1}^{N} e_j I_j\right)}{\sin 2(\theta-\alpha)\left(\cos 2\theta \sum_{j=1}^{N} f_j I_j + \sin 2\theta \sum_{j=1}^{N} g_j I_j\right)}\right) \tag{20}$$

where N is the number of images, b_j, d_j and e_j are the coefficients of numerator, c_j, f_j and g_j are coefficients of denominator, and j is the index of the sum. A mathematical model, based on that developed in Magalhães and Magalhães Jr. (2012), was obtained how shown in Eq. (19) for isoclinic angle θ and in Eq. (20) for isochromatic parameter δ.

The mathematical model is easy to solve because it involves quadratic programming and a maximum global solution can be obtained using the Newton’s method with Karush-Kuhn-Tucker (KKT) conditions (Magalhães and Magalhães Jr., 2012). The processing time for the solution of this mathematical model is very fast, even using personal computers, the solution is given in few seconds. After several attempts, the following mathematical model was identified for numerical modeling the problem:

$$
\begin{aligned}
&\textit{Maximize} \quad \sum_{j=1}^{N} b_j + c_j \\
&\textit{subject to} \begin{cases}
1) \quad \tan(2\theta^v)\sum_{j=1}^{N} c_j . I_j^v = \sum_{j=1}^{N} b_j . I_j^v & \text{Quantities} \quad v > 2N \\
2) \quad -1 \le b_j \le 1, \quad -1 \le c_j \le 1 & j = 1..N \\
3) \quad b_j, \ \mathrm{c_j} \text{ are real numbers} & j = 1..N
\end{cases}
\end{aligned}
$$

where for each v:

$$
\begin{cases}
I_j^v = I_o^v . \left[\cos^2\left(\frac{\delta^v}{2}\right) . \cos^2\left(\beta_j - \alpha\right) + \sin^2\left(\frac{\delta^v}{2}\right) . \cos^2\left(\beta_j + \alpha - 2.\theta^v\right) \right], \ j = 1..N \\
I_o^v \in [0;\ 255] \quad \text{random and real} \\
\theta^v \in [-\pi/2;\ \pi/2] \quad \text{random and real} \\
\delta^v \in [0;\ \pi] \quad \text{random and real} \\
\beta_j = \pi\left(\frac{j-1}{Step-1}\right) - \frac{\pi}{2} \qquad , \ j = 1..N
\end{cases}
$$

$$
\begin{aligned}
&step = [2,3,4,5,6,7,10,11,13,16,19,21,31,37,46] \text{ or integer values greater than } 2 \\
&N = \text{number of images from 2 until the step value} \\
&j = 1..N \\
&\beta_j = \text{ analyzer angle} \\
&\alpha = \text{polarizer angle } \ \pi/2 \\
&\theta = \text{isoclinic angle} \\
&\delta = \text{isochromatic parameter}
\end{aligned} \tag{21}
$$

$$Maximize \quad \sum_{j=1}^{N} d_j + e_j + f_j + g_j$$

$$\text{subject to}\begin{cases} 1) & \cos(\delta^v)\sin\left[2.\left(\theta^v-\alpha\right)\right]\left[\cos(2.\theta^v).\sum_{j=1}^{N} f_j.I_j^v+\sin(2.\theta^v).\sum_{j=1}^{N} g_j.I_j^v\right]= \\ & \cos\left[2.\left(\theta^v-\alpha\right)\right]\left[\sin(2.\theta^v).\sum_{j=1}^{N} d_j.I_j^v+\cos(2.\theta^v).\sum_{j=1}^{N} e_j.I_j^v\right] \\ 2) & -1\le b_j\le 1, \quad -1\le c_j\le 1 \\ 3) & d_j,\ \mathrm{e_j},\ \mathrm{f_j},\ \mathrm{g_j}\ \text{ are real numbers} \\ Quantities & v>2N \quad and \quad j=1..N \end{cases}$$

where for each v:

$$\begin{cases} I_j^v = I_o^v.\left[\cos^2\left(\dfrac{\delta^v}{2}\right).\cos^2\left(\beta_j-\alpha\right)+\sin^2\left(\dfrac{\delta^v}{2}\right).\cos^2\left(\beta_j+\alpha-2.\theta^v\right)\right], j=1..N \\ I_o^v\in[0;\ 255] \quad random \quad \text{and} \quad \text{real} \\ \theta^v\in[-\pi/2;\ \pi/2] \quad random \quad \text{and} \quad \text{real} \\ \delta^v\in[0;\ \pi] \quad random \quad \text{and} \quad \text{real} \\ \beta_j=\pi\left(\dfrac{j-1}{Step-1}\right)-\dfrac{\pi}{2} \qquad , j=1..N \end{cases}$$

$$step = [\,2,3,4,5,6,7,10,11,13,16,19,21,31,37,46,61,91,181\,] \ or \ \text{int}\,eger \ values \ greater \ than\ 2$$
$$N = number\ of\ images\ from\ 2\ until\ the\ step\ de\ value$$
$$j=1..N$$
$$\beta_j = \ analyzer\ angle$$
$$\alpha = \ polarizer\ \ angle\ \pi/2$$
$$\theta = \ isoclinic\ \ angle$$
$$\delta = \ isochromatic\ \ parameter \tag{22}$$

The mathematical models above maximize the coefficients of numerator (b_j, d_j, e_j) and the coefficients of denominator (c_j, f_j, g_j) in both equations, thus the obtained values for them are large enough to be significant in Eq. (19) and (20). Step represents values greater than or equal to 2. *N* is the number of images and is an integer number between 2 and the value of step.

The constraints 1 in Eq. (21) and (22) are made so that all coefficients generate correct values for the calculation of θ and δ. To ensure that we have a hyperrestricted problem, Magalhães and Magalhães Jr. (2012) suggest that the number of restrictions must be, at least, equal to the number of variables.

The constraints 2 and 3 in Eq. (21) and (22) indicate that the coefficients are real numbers not greater than one and not smaller than negative one, thus error propagation are avoided. For the evaluation of phase, these limiting factors will increase the values of intensity I_o that contains errors due to noise and improve discretization in pixels and in shades of gray (Magalhães and Magalhães Jr., 2012).

In the model the values of I_o is limited between 0 and 255, θ is limited between -90° and 90° and δ is limited between 0 and π. Step defines the distance between the angles measured in polariscope, only when step is equal to *N* the last angle measured is equal to the first with opposite sign. Table 1 shows the coefficients obtained when $N = 10$, *step* = 10, $\alpha = 0°$ and the angles of the analyzer (β) as follows: -90°, -70°, -50°, -30°, -10°, 10°, 30°, 50°, 70°, 90°.

Table 1. Values of the real coefficients (b_j, c_j, d_j, e_j, f_j, g_j) when N = 10 and *step* = 10.

j	β_j	b_j	c_j	d_j	e_j	f_j	g_j
1	-90°	-1,00000	-0,13175	-0,01917	-0,47294	-1,00000	0,43879
2	-70°	0,15064	1,00000	-0,27668	-1,00000	1,00000	-1,00000
3	-50°	1,00000	1,00000	-0,50723	-1,00000	1,00000	-1,00000
4	-30°	1,00000	1,00000	-0,91845	1,00000	1,00000	0,41157
5	-10°	1,00000	0,12896	-1,00000	1,00000	1,00000	-1,00000
6	10°	1,00000	-0,28419	-0,37822	1,00000	1,00000	0,14963
7	30°	1,00000	-1,00000	0,14626	1,00000	1,00000	1,00000
8	50°	1,00000	-1,00000	1,00000	1,00000	1,00000	1,00000
9	70°	0,15064	0,28698	1,00000	1,00000	1,00000	-1,00000
10	90°	-1,00000	-1,00000	0,95350	-0,08737	-0,87240	1,00000

4. RESULTS AND DISCUSSION

To verify the new equations for calculating phase, a computer program was created and it generated random values of I_o, θ' and δ'. With Eq. (3) are calculated N values of I_j to each value of β_j, then the new phase equations were applied on the values of I_j to determine whether they produced correct values for θ and δ. The calculations of the accuracy was performed thousands of times (at least 100.000 times) for each equation in phase calculation. At least 99,99% of time we obtain the accuracy ($|\theta' - \theta|\ |\delta' - \delta|) \leq 10^{-6}$. The models of Eq. (19) and (20) were tested until step and N equal to 2000, the value at which the increment $\Delta\beta$ would be 0,05°.

5. CONCLUSION

The main advantage of the approach numerical over the conventional method it is be usable in any polariscope commercially available, eliminating the needs for dedicated equipment to determined experimental procedures. Furthermore, the new method facilitates the automation of the analyses; because only the analyzer is rotated for collect the photographic images. The results of the research indicate that the method gives fewer errors and so is applicable to experimental stress or strain analysis. With the new approach, it would be possible to develop a photoelastic system that moves the analyzer of the plane polariscope at a constant speed while a camera takes many pictures at equal intervals of time. However, experimental procedures should be made to validate the new numerical technique.

6. ACKNOWLEDGEMENTS

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