COBEM2015

ABCM Associação Brasileira de Engenharia e Ciências Mecânicas

23rd ABCM International Congress of Mechanical Engineering
December 6-11, 2015, Rio de Janeiro, RJ, Brazil

COMPARISON BETWEEN SOS AND (S)DSOS LYAPUNOV FUNCTIONS FOR NONLINEAR SYSTEMS

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Abstract.*The continuous search for new Lyapunov functions for nonlinear system has brought several approaches and techniques. Generally, dealing with a nonlinear system is a difficult problem. However, during the last decade, the sum of squares (SOS) approach has presented relevant results, transforming the polynomial positivity problem into the search for polynomials that can be represented as a sum of squares of polynomials, which can be more easily treated by means of semidefinite optimization. In this paper, we will present a numerical comparison between different approaches of search for Lyapunov functions while maximizing the region of attraction (ROA) using SOS. For high-dimensional systems, SOS constraints may demand an excessive computational requirement, precluding the use the method. Alternatively, it is possible to relax the semidefinite cone by means of stronger conditions but computationally more tractable optimization problems. With the properties of diagonally dominant matrix theory, we may rewrite SOS constraints as diagonally-dominant-sum-of-squares (DSOS) constraints. Accordingly, for Lyapunov functions of the same order, the ROA obtained will be smaller. In a resembling way, the use of the scaled diagonal dominance (SDSOS) approach presents less conservatism when compared with DSOS conditions, and therefore a better approximation for the ROA. These new approaches compromise conservativeness of the results with less computational effort. Nevertheless, high dimensional systems or high order Lyapunov functions that reached the limits of available solvers are now solvable with these alternatives approaches. For illustration, we use the reverse-time Van der Pol Oscillator as example, presented both in continuous and discrete-time.*

Keywords: *Lyapunov functions, sum of squares, diagonally dominant, region of attraction*

1. INTRODUCTION

Numerical methods for analysis and control of nonlinear systems are generally challenging problems. In recent decades, several approaches based on polynomial non-negativity constraints under a Lyapunov framework have been presented (Peet and Lall, 2004), (Peet and Papachristodoulou, 2012) for control analysis and synthesis. It is known that polynomial nonnegativity is a NP hard problem, and hence unlike to be solved efficiently. Therefore, many researchers looked for alternative, and computationally easier, tests for nonnegativity.

Recently, many approaches described a general methodology to find formulations or relaxations solvable by semidefinite programming (SDP), closely related to what linear matrix inequality (LMI) methods provided for linear systems. One of them uses a new computational relaxation based on sum-of-squares decomposition, which provides potentially effective ways to solve such problems (Parrilo, 2000). Interestingly, a slightly modified version of this relaxation, uses diagonally dominant matrix properties (DSOS) (Majumdar *et al.*, 2014). However, it results in different computational cost and solutions. In this case, the Region-of-Attraction (ROA) is a fair parameter to compare such relaxations.

The algorithm presented here follows the steps presented in (Tan and Packard, 2004), that involves a search between the Lyapunov function and auxiliary SOS polynomials. We modified the original approach in order to add DSOS and SDSOS constraints, with a view to compare with the original SOS constraints. We compare the ROA and the runtime to solve the reverse-time Van der Pol oscillator problem as an example. The Systems Polynomial Optimization Toolbox (SPOT) (Megretski, 2004) was used since it includes a complete implementation to solve problems that utilize the SOS, DSOS and SDSOS techniques.

The notation and abbreviation used are standard and as close as possible to existing common practice. Vectors are denoted by lowercase Roman letters and always represent a column vector. For vectors and matrices ($'$) denotes its transpose. Scalar values are denoted by lowercase Greek letters such as α and β. We define $\mathcal{R}_n$ to be the set of all polynomials with real coefficients in n variables, i.e., if $p(x) \in \mathcal{R}_n$ then $\left\{x \in \mathbb{R}^n,\, \alpha \in \mathbb{Z}^n_+ \,|\, p(x) = \sum_{j=1}^{k} c_j \prod_{i=1}^{n} x_i^{\alpha_i}\right\}$ where $\mathbb{R}^n$ is the set of n-dimensional real numbers. Capital Roman letters denote matrices, with one exception: $V(x)$ or simply V represents the polynomial Lyapunov function.

2. DEFINITIONS AND BASIC PROPERTIES

Some definitions are needed before we show how to search for a Lyapunov function that stabilizes a given system in closed loop.

2.1 Polynomial definitions

We will define various terms that will be used throughout this text:

Let $s(x)$ be a polynomial that can be decomposed as $s(x) = \sum_{i=1}^{k} p_i(x)^2$, $p_i(x) \in \mathcal{R}_n$, In this case, $s(x)$ is called a sum-of-squares (SOS) polynomial, and the set of all SOS polynomials in n variables is defined as Σ_n. Clearly, if $p(x) \in \Sigma_n$ then $p(x) \geq 0$ for all $x \in \mathbb{R}^n$

The set generated by the all finite products of $g_j \in \mathcal{R}_n$ taken from the set $\{g_1, \cdots, g_l\}$, including the empty product, is called the *Multiplicative Monoid* and denoted as $\mathcal{M}(g_1, \cdots, g_l)$.

Now suppose that $\mathcal{P}(p_1, \cdots, p_k)$, $p_j \in \mathcal{R}_n$, is such that:

$$\mathcal{P}(p_1, \cdots, p_k) := \Big\{ s_0 + \sum_{i=1}^{t} s_i b_i \mid t \in \mathbb{Z}_+, s_i \in \Sigma_n, b_i \in \mathcal{M}(p_1, \cdots, p_k) \Big\}. \tag{1}$$

We name $\mathcal{P}(p_1, \cdots, p_k)$ the *Cone* generated by p_i.

Lastly, suppose that a set $\mathcal{I}(h_1, \cdots, h_m)$ is given by:

$$\mathcal{I}(h_1, \cdots, h_m) := \Big\{ \sum_j h_j f_j \mid f_j \in \mathcal{R}_n \Big\} \tag{2}$$

This set is known as the *Ideal* generated by $(h_1, \cdots, h_m)$.

With these definitions, Bochnak *et al.* (1986) states the Positivstellensatz theorem:

Theorem 1 *(Positivstellensatz). Given polynomials $\{p_1, \cdots, p_k\}$, $\{g_1, \cdots, g_l\}$ and $\{h_1, \cdots, h_m\}$ in $\mathcal{R}_n$, the following statements are equivalent:*

1. *The set below is empty:*

$$\Big\{ x \in \mathbb{R}^n \mid p_1(x) \geq 0, \cdots, p_k(x) \geq 0, g_1(x) \neq 0, \cdots, g_l(x) \neq 0, h_1(x) = 0, \cdots, h_m(x) = 0 \Big\} \tag{3}$$

2. *There exist polynomials $p \in \mathcal{P}(p_1, \cdots, p_k), g \in \mathcal{M}(g_1, \cdots, g_l), h \in \mathcal{I}(h_1, \cdots, h_m)$ such that:*

$$p + g^2 + h = 0 \tag{4}$$

2.2 Diagonal dominance and Scaled Diagonal Dominance constraints

Firstly consider $m_\theta(x)$ given by $m_\theta(x) = x^\theta := x_1^{\theta_1} x_2^{\theta_2} ... x_n^{\theta_n}$, $\theta \in \mathbb{Z}_+^n$. This function is called a monomial in n variables and its degree is defined as $\deg(m_\theta) := \sum_{i=1}^{n} \theta_i$. Now, consider a polynomial $s(x) \in \sum_n$. In this case, there exists a positive semidefinite symmetric matrix $Q = Q' \in \mathbb{R}^{n \times n}$ such that:

$$s(x) = v(x)' Q v(x), \tag{5}$$

where $v(x)$ is the vector of all monomials with degree less than or equal to half the degree of $s(x)$ (i.e., $\deg(v_i) \leq \deg(s(x))/2$).

We can replace the semidefinite positive constraint by a stronger, but sufficient, condition, resulting in a smaller but simples space, which can be exploited by the optimization algorithm. A sufficient condition is to require that Q is a diagonal dominant matrix or a scaled diagonal dominant, as shown in (Majumdar *et al.*, 2014).

Define $DSOS_n$ as the set of all diagonally dominant matrix $D \in \mathbb{R}^{n \times n}$. A symmetric matrix $Q \in \mathbb{R}^{n \times n}$ is diagonally dominant ($Q \in DSOS_n$) if $q_{ii} \geq \sum_{i \neq j} |q_{ij}|$ for all i.

Similarly, define $SDSOS_n$ as the set of all scaled diagonally dominant matrix $E \in \mathbb{R}^{n \times n}$. A symmetric matrix $Q \in \mathbb{R}^{n \times n}$ is in $SDSOS_n$ if exist an element-wise positive vector $\gamma \in \mathbb{R}^n$ such that $q_{ii} \geq \sum_{i \neq j} |q_{ij}| \gamma_j \; \forall i$.

The fact that $Q \in DSOS_n$ or $Q \in SDSOS_n$ implies that $Q \geq 0$ follows directly from Gershgorin's circle theorem. These constraints, however, results in different properties for the polynomial given in Eq. (5). Indeed, a polynomial $d(x)$ is a *scaled-diagonally-dominant-sum-of-squares* if $d(x) \in \Upsilon_n$, where Υ_n is the set of all polynomials:

$$d(x) = \sum_i \eta_i m_i^2 + \sum_{i,j} \rho_{ij}^+ (m_i + m_j)^2 + \rho_{ij}^- (m_i - m_j)^2 \tag{6}$$

with constants $\eta_i, \rho_{ij}^+, \rho_{ij}^- \geq 0$ and monomials m_j. Clearly, if $d(x) \in \Upsilon_n$ then $d(x) \in \Sigma_n$, but the converse is not true.

Moreover, we say that $e(x)$ is a *scaled-diagonally-dominant-sum-of-squares* if $e(x) \in \Gamma_n$, Γ_n being the set of all polynomials that can be written as:

$$e(x) = \sum_i \eta_i m_i^2 + \sum_{i,j} (\rho_i^+ m_i + \tau_j^+ m_j)^2 + (\rho_i^- m_i - \tau_j^- m_j)^2 \tag{7}$$

for some monomials m_i, m_j and constants $\eta_i, \rho_i^+, \rho_i^-, \tau_j^+, \tau_j^- \geq 0$. Similarly, $e(x) \in \Gamma_n$ implies that $e(x) \in \Sigma_n$, but again the converse is not true.

Notice that these definitions follows directly from Eq. (5) with a suitable choice of matrix $Q \in DSOS_n$ or $Q \in SDSOS_n$, respectively.

3. COMPUTATIONAL SEARCH FOR LYAPUNOV FUNCTIONS VIA SOS PROGRAMMING

In this section, we show how estimating the ROA can be formulated as a sum-of-squares optimization problem, and sequently modify the original problem in order to derive new conditions using DSOS and SDSOS constraints. This technique is presented for both continuous- and discrete-time systems.

3.1 Continuous-time systems

Consider a continuous-time system with state-space equations given by:

$$\dot{x} = f(x), \tag{8}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $f_i \in \mathcal{R}_n$, $i = 1, \ldots, n$, is the vector field. The existence of a Lyapunov function $V(x) > 0$ such that $\nabla V(x) \cdot f(x) < 0$, $\forall x \in (\mathbb{R}^n - \{0\})$, implies that Eq. (8) is asymptotically globally stable.

For cases where any candidate function $V(x) > 0$ fails to satisfy $\nabla V(x) \cdot f(x) < 0$, $x \in (\mathbb{R}^n - \{0\})$, there exist points such that $\nabla V \cdot f(x) \geq 0$. In these cases, we shall search for a compact set of $\mathbb{R}^n$ that excludes such points.

Consider a level set $\Phi_\alpha = \{x \in \mathbb{R} \mid V(x) \leq \alpha\}$, such that $\nabla V \cdot f(x) < 0$, $\forall x \in \Phi_\alpha$. In this case, the level set Φ_α is a region of attraction for the system. In order to maximize our estimate for the region of attraction, we pick a positive definite $p(x) \in \mathcal{R}_n$. We define the variable sized region $P_\beta := \{x \in \mathbb{R}^n \mid p(x) \leq \beta\}$ such that $P_\beta \subseteq \Phi_\alpha$. By maximizing the parameter β, both Φ_α and P_β are enlarged. These conditions can be rewritten by the equivalent empty set constraints:

$$\{x \in \mathbb{R}^n \mid V(x) \leq \alpha, \nabla V \cdot f(x) \geq 0, x \neq 0\} \text{ is empty} \tag{9}$$

$$\{x \in \mathbb{R} \mid p(x) \leq \beta, V(x) \geq \alpha, V(x) \neq \alpha\} \text{ is empty} \tag{10}$$

By the Positivstellensatz theorem previously presented, we obtain the equivalent optimization problem for the ROA estimation:

$$\max \beta \text{ over } s_1, s_2, s_8 \in \Sigma_n;\ V \in \mathcal{R}_n;$$

such that

$$\begin{aligned} & V - l_1 \in \Sigma_n \\ & -\big((\beta - p(x))s_8 + (V - \alpha)\big) \in \Sigma_n \\ & -\Big\{s_1(\alpha - V) + s_2\big(\nabla V \cdot f(x)\big) + l_2\Big\} \in \Sigma_n \end{aligned} \tag{11}$$

The proof is available in (Tan and Packard, 2004). By maximizing β we maximize our estimate of the ROA.

An important part of this algorithm is the choice for the variables degrees s_1, s_2, s_8 to match the problem dimensions. Moreover, increasing the degrees adds additional variables to the problem, and therefore it is important to choose degrees that make the problem computationally tractable.

3.2 Discrete-time systems

Now suppose a nonlinear discrete time system given by:

$$x(k+1) = f\big(x(k)\big) \tag{12}$$

Applying techniques analogous to the continuous-time case, where $\nabla V \cdot f(x)$ is replaced by $V \circ f\big(x(k)\big) - V$ as shown in (Wloszek, 2003), we obtain the optimization problem:

$\max \beta$ over $s_1, s_2, s_8 \in \Sigma_n;\ V \in \mathcal{R}_n;$

such that

$$\begin{aligned} & V - l_1 \in \Sigma_n \\ & -\Big((\beta - p\big(x(k)\big)s_8 + (V-\alpha)\Big) \in \Sigma_n \\ & -\Big\{s_1(\alpha - V) + s_2\big(V \circ f\big(x(k)\big) - V\big) + l_2\Big\} \in \Sigma_n \end{aligned} \tag{13}$$

where, by maximizing β, we also maximize our estimate for the ROA:

Exchanging the SOS constraints by DSOS or SDSOS in both cases (continuous and discrete time systems) we obtain new stronger conditions.

We will apply these conditions for an illustrative example.

4. REVERSE-TIME VAN DER POL OSCILLATOR EXAMPLE

The Van der Pol oscillator is the object of many studies in theory of analysis for dynamical systems (Barbosa *et al.*, 2007), (Kharchenkoa *et al.*, 2014). The mathematical model is given by the second-order differential equation:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \tag{14}$$

From now on, we will consider $\mu = 1$.

By reversing the time variable in this differential equation, the reverse-time Van der Pol oscillator is obtained. This new system exhibits a globally repulsive limit cycle. The continuous state equations corresponding to the reverse model are given by:

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = (x_1^2 - 1)x_2 + x_1 \end{cases} \tag{15}$$

The equivalent discrete polynomial model is expressed using Kronecker product, given by:

$$X_{k+1} = F_1 X_k + F_3 X_k^{[3]} \tag{16}$$

where $X_k = [X_{1k}\ \ X_{2k}]'$ and $X_k^{[i]}$ express the *i-th* Kronecker power of a discrete state vector X_k.

Bacha *et al.* (2006) demonstrate the algorithm formulation and the discretizing approach of the continuous model. Choosing the sampling period $T = 0.05s$, the values of F_1 and F_3 are:

$$F_1 = \begin{bmatrix} 0.9988 & -0.0488 \\ 0.0488 & 0.9500 \end{bmatrix}, \qquad F_3 = \begin{bmatrix} 0 & -0.0012 & \multirow{2}{*}{$0_{2\times6}$} \\ 0 & 0.0488 & \end{bmatrix} \tag{17}$$

4.1 ROA estimation

We now demonstrate the scalability of SDSOS and DSOS approaches applying the technique in the reverse-time Van der Pol model. We compare the estimated ROA obtained in each case for different Lyapunov functions degree.

We use MOSEK (MOSEK ApS, 2015) as the convex programming solver and the software package SPOT (Megretski, 2004) containing the DSOS and SDSOS methods. The code was executed on a machine with Intel i7-950 processor and 6 GB of RAM.

In order to solve the optimization problems given by Eq. (11) and Eq. (13) with DSOS or SDSOS constraints, we maintained the polynomials s_1, s_2 and s_8 with SOS constraints, replacing only the three main conditions to DSOS or SDSOS. These assumptions were made due to the fact that restricting the search space for s_1, s_2 and s_8 would provide results with a smaller ROA while not resulting in a significant impact on total time of convergence.

Figures 1 and 2 show the results of the estimated ROA, for Lyapunov functions with degree 4, of the continuous- and discrete-time reverse model, respectively.

As expected, in both cases we have a better estimation applying SOS constraints, followed by SDSOS and DSOS. This explained by the that fact.

$$DSOS_n \subseteq SDSOS_n \subseteq SOS_n \tag{18}$$

Notice also that the true ROA was obtained by numerical integration of the Van der Pol Oscillator.

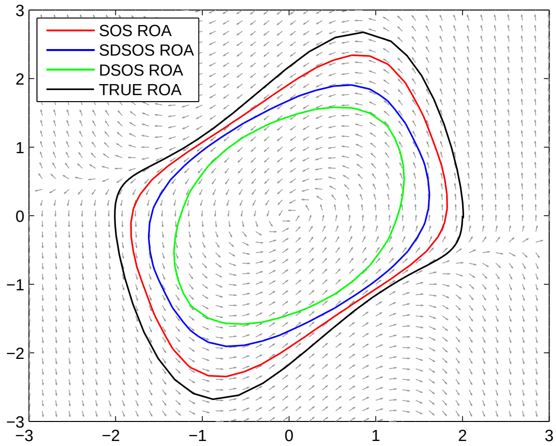

Figure 1: ROA for continuous-time, degree 4

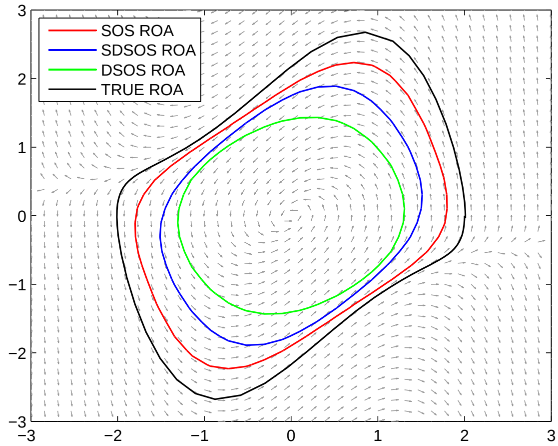

Figure 2: ROA for discrete-time, degree 4

4.2 Performance Analysis

As shown in the previous section, ROA estimation with DSOS and SDSOS constraints provides smaller areas than SOS. However, the computational burden is also largely decreased. As shown in Tab. 1, according to the Lyapunov function degrees, DSOS and SDSOS can be up to 5 times faster than the SOS in continuous-time case. The same pattern occurs in discrete-time, as shown in Tab. 2, where we can observe that SDSOS convergence may occur about 50 times faster than SOS. In both cases the stopping criteria was $\beta_i - \beta_{i-1} < 10^{-3}$.

Table 1: Continuous time.

Degree	Time (seconds)		
	SOS	**DSOS**	**SDSOS**
2	42.34	55.14	44.26
4	957.21	233.00	269.30
6	1570.20	253.50	309.76

Table 2: Discrete time.

Degree	Time (seconds)		
	SOS	**DSOS**	**SDSOS**
4	3074.21	222.22	227.66
8	23989.49	519.51	1073.60

We continued by fixing a Lyapunov function of sixth degree and limiting the time of execution in approximately 253.5 seconds (time required to ensure convergence from DSOS case). We applied SOS, DSOS and SDSOS constraints to the optimization problem, obtaining the estimated ROA as shown in Fig. 3. Clearly, with SDSOS constraints, the ROA obtained is larger than with SOS.

This result suggests that SDSOS, or even DSOS constraints, can lead to better results in cases where time is a critical factor. For instance, in higher dimensional systems, SOS constraints may take weeks or months to converge, whereas DSOS/SDOS could take significant less.

Unfortunately, as increasing the degree also increases variables number, some instances presented an excessive relaxation and also were prone to numerical difficulties. That is the main cause of non convergence in some cases, specially on discrete-time.

Finally, for Lyapunov functions of degree 6, the estimation using SOS constraints was almost equals to the true ROA, but at the cost of high computational time.

5. CONCLUSION

In this work we presented stability analysis for a specific problem, precisely the Reverse-time Van der Pol oscillator. The main goal was to search for a Lyapunov function that guarantees the largest ROA for the nonlinear system. To do that, it was applied different approaches that derive from sum of squares theory (namely, SOS, DSOS and SDSOS).

As expected, each approach presents different results. Using SOS, we obtained the highest accuracy in ROA estimation, however with high computational cost. The results with SDSOS and DSOS present lower computational cost and median accuracy. This results suggests that, in high dimensional systems, SDSOS or DSOS constraints may be more indicated.

In a future work, high dimensional systems should be investigated in order to verify the impact of those constraints in ROA estimation. For the present moment, it was not possible to obtain good results only increasing the Lyapunov function degree.

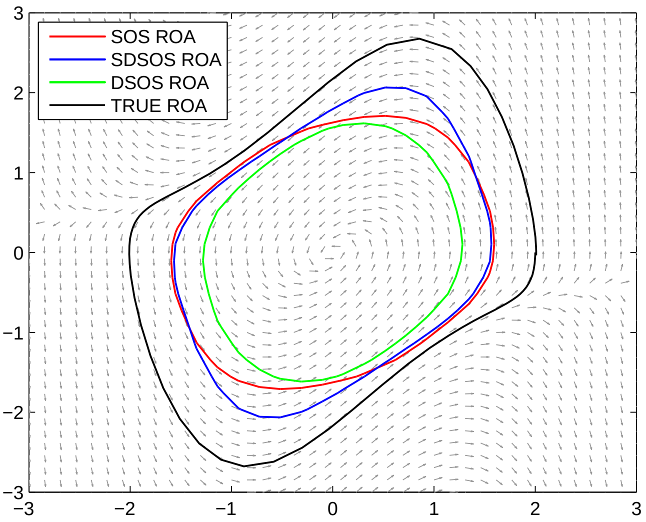

Figure 3: ROA comparison between functions of degree 6 and same computational time

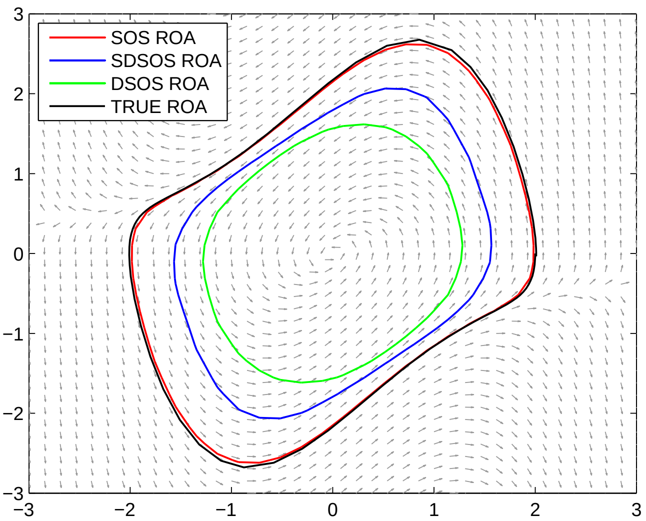

Figure 4: ROA estimation in continuous-time for Lyapunov function of degree 6

Another problem that should be investigated is the synthesis of a controller that maximizes the ROA within the same framework of (S)DSOS optimization. The conditions stated here can be adapted to provide a controller synthesis algorithm, however, its convergence needs to be verified.

6. ACKNOWLEDGMENTS

The authors are grateful to FAPESP, CAPES and FAEPEX for the financial support.

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