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Asymptotic solution of the Orr-Sommerfeld equation for surface waves on an inclined plane.

Bruno Pelisson Chimetta
Erick de Moraes Franklin
University of Campinas - UNICAMP
Faculty of Mechanical Engineering
Rua Mendeleyev, 200 - CEP 13083-860 - Campinas - SP

brunopc@fem.unicamp.br
franklin@fem.unicamp.br

Abstract. *This study presents an analysis of the initial behavior of free-surface liquid flows on inclined planes, where, under some conditions, long-wave instability may appear. These instabilities may evolve to a surface-wave regime, that often appear on thin liquid films. Such knowledge is useful in industry, once liquid films help to remove the heat from solid surfaces, and also reduces the friction between high viscosity fluids and pipe walls, by injecting close to the wall a less viscous fluid. The surface-wave instability is governed by the Orr-Sommerfeld equation. In this work, we present a long-wave solution for the Orr-Sommerfeld equation based on asymptotic expansions. For the long-wave perturbations, the wave number can be treated as a small parameter, and the form of the equations suggests that the velocity and the amplitude of the eigenfunction can be sought as a power series of the wave number. The asymptotic solution gives the growth rate, the celerity, the wavelength and the critical Froude number of the liquid film at the instability threshold. The results are compared with previously published data.*

Keywords: *Surface-wave instability, Orr-Sommerfeld equation, Asymptotic Solution.*

1. Introduction.

A better knowledge of the instabilities of liquid films is central to industrial applications; therefore, the study of liquid films is of importance. Industrial applications such as friction reducing effects in the case of flows of viscous oils, or cooling engines and reactors, are some examples. When dealing with such flows, the appearance of instabilities must be considered. Depending on the system being used, the presence of these instabilities can be (or not) desirable. In the case of oil transport in an annular flow, the film thickness must be constant, so that it is necessary to avoid instability.On the other hand, sometimes we can use instabilities to increase heat and mass transfer, for example, in a reactor cooling process. In this article we present a study on some physical mechanisms of a liquid film falling down on a inclined plane without heat exchange with the wall. First, it will be considered the physical aspects of the problem and some pertinent dimensionless groups, then we use the perturbations in the form of a stream function as done by (Yih, 1963). Considering only the first order perturbations, we have the equation of Orr-Sommerfeld (Orr, 1907; Sommerfeld, 1908). This equation, together with the flow boundary conditions, will produce a generalized eigenvalue problem. These equations are solved by means of an asymptotic analysis (Kevorkian and Cole, 1981), similar to the approach employed by (Smith, 1990), who considered the wave number as a small parameter and dealt with the speed and amplitude of the waves as an expansion in power series. In this work we performed expansions up to the second order, we found the growth rate of the instability, celerity, and critical Froude number.

2. Formulation of the problem.

We consider a film of incompressible Newtonian liquid of thickness h, viscosity μ and density ρ, falling down on a inclined plane with an angle θ with respect to the horizontal (Fig. 1). The interface between the liquid and the gas has a surface tension γ and the pressure applied by the gas on the interface is P_0. Based on the Navier-Stokes equations and considering the boundary conditions of permanent flow, non-slip at solid surface, no shear and constant pressure P_0 at the liquid-gas interface, we can find a steady parallel flow solution corresponding to a planar interface and a parabolic velocity profile:

$$\overline{U}(y) = U_0(1 - \frac{y^2}{h^2}) \tag{1}$$

$$\overline{V}(y) = 0 \tag{2}$$

$$\overline{P}(y) = P_0 - \rho g cos(\theta) y \tag{3}$$

where the velocity of the interface U_0 is given by,

$$U_0 = \frac{\rho g h^2 sin(\theta)}{2\mu} \tag{4}$$

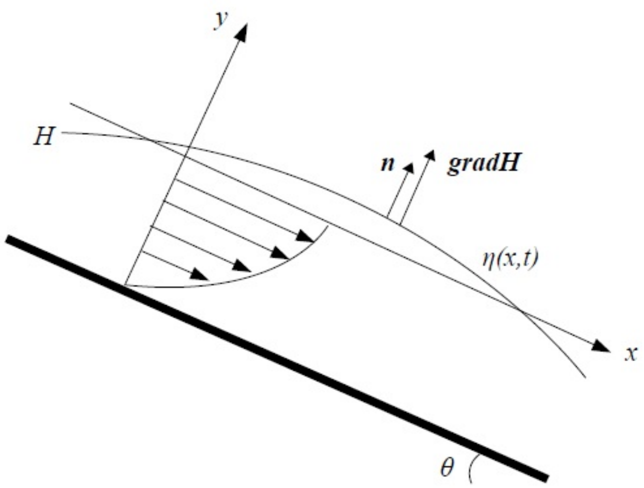

Figure 1. Liquid film falling down on a inclined plane with parabolic velocity profile, planar interface H, where $\vec{n}$ is parallel to the gradient of H (∇H), and perturbed interfacial position $\eta(x,t)$.

The solutions are given in terms of the parameters h, ρ and U_0, where the latter is the speed of the fluid at the interface. The contribution of inertia relative to viscosity, gravity, and surface tension is measured, respectively, by the Reynolds, Froude, and Weber numbers, defined as:

$$Re = \frac{\rho U_0 h}{\mu}, \;\; Fr = \frac{U_0^2}{g h cos(\theta)} = \frac{Re\ \tan(\theta)}{2}, \;\; We = \frac{\rho U_0^2 h}{\gamma} \tag{5}$$

where the Froude number is defined using the gravity component $gcos(\theta)$ normal to the flow. When the interface is perturbed ($\eta(x,t) \neq 0$), the velocity profile no longer has a exactly parabolic behavior. This, combined with inertia, leads to instabilities. Such instabilities will be studied in the next sections based on perturbation methods, in order to find an approximate solution to the problem.

3. The equation of Orr-Sommerfeld.

According to Squire's theorem (Squire, 1933), we know that the first instability on parallel flow of a Newtonian fluid, is always two-dimensional. This last result remains valid for a flow with interface (Hesla *et al.*, 1986), then we can consider only the two-dimensional case of the problem. Keeping these considerations in mind, we will use the Navier-Stokes equations and continuity in the form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6}$$

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \rho g_x \tag{7}$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho g_y \tag{8}$$

Considering the components of velocity as $u = \overline{U} + \hat{u}$, $v = \hat{v}$, where $\hat{u} = \frac{\partial \Psi}{\partial y}$, $\hat{v} = -\frac{\partial \Psi}{\partial x}$, and linearizing the equations, we get, in dimensionless form, the equation:

$$\left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x}\right)\nabla^2\Psi - \frac{\partial^2 \overline{U}}{\partial y^2}\frac{\partial \Psi}{\partial x} = \frac{1}{Re}[\nabla^2(\nabla^2\Psi)] \tag{9}$$

Consider the normal mode $\Psi(x,y,t) = \hat{\Psi}(y)e^{i\alpha(x-ct)}$, where $\alpha = kh \in R$, $c = \frac{\omega}{k} \in C$, k is the wave number and ω is the frequency. Using $\Psi(x,y,t)$ in Eq. (9) and making the contraction of notation for $D = \frac{\partial}{\partial y}$, we obtain the Orr-Sommerfeld Equation in dimensionless form, given by:

$$(D^2-\alpha^2)^2\hat{\Psi}(y) = i\alpha Re[(\overline{U}-c)(D^2-\alpha^2)-D^2\overline{U}]\hat{\Psi}(y) \tag{10}$$

4. Boundary Conditions.

In this section, we present the boundary conditions of the problem, which are: wall condition, kinematic condition of the interface and dynamic condition of the interface. In order to obtain these conditions, the same procedure used in the previous section is used here. The wall condition is given by:

$$u = 0 \ \ ,y = -h \tag{11}$$

$$v = 0 \ \ ,y = -h \tag{12}$$

$$D\hat{\Psi}(-1) = 0 \tag{13}$$

$$\hat{\Psi}(-1) = 0 \tag{14}$$

Kinematic and dynamic conditions are present at the interface. Figure 1 presents the interface $H(x,y,t)$ and the corresponding vectors. For the two-dimensional problem, $H(x,y,t) = y - \eta(x,t)$. The unit vectors normal and tangent to the interface, when linearized, are given by:

$$\vec{n} = -\frac{\partial\eta(x,t)}{\partial x}\vec{e_x} + \vec{e_y} \tag{15}$$

$$\vec{t} = \vec{e_x} + \frac{\partial\eta(x,t)}{\partial x}\vec{e_y} \tag{16}$$

where $\eta(x,t)$ and $\frac{\partial\eta(x,t)}{\partial x}$ are the position and inclination of the interface, respectively. The kinematic condition is,

$$\vec{u}\cdot\vec{n} = \vec{w}\cdot\vec{n} \ \ em \ \ y = \eta(x,t) \tag{17}$$

where,

$$\vec{u}\cdot\vec{n} = -u\frac{\partial\eta(x,t)}{\partial x} + v \tag{18}$$

$$\vec{w}\cdot\vec{n} = \frac{\partial\eta(x,t)}{\partial t} \tag{19}$$

Replacing Eq. (18), Eq. (19) in Eq. (17), using the normal modes of $\Psi(x,y,t)$, $\eta(x,t) = \hat{\eta}e^{i\alpha(x-ct)}$, and linearizing the equation around $y = 0$,

$$\hat{\Psi}(0) - (c-1)\hat{\eta} = 0 \tag{20}$$

The dynamic condition at the interface has two parts. Considering that the surface between liquid and gas has no surfactants or temperature gradient; the first condition is the continuity of the tangential stress, associated with the viscous stress of the fluid. The second one is the continuity of the normal stress associated with the surface tension. The stress in the fluid is given by $\Sigma\cdot\vec{n}$, where Σ is the stress tensor, and reducing the effect of air to a purely normal stress $-P_0\vec{n}$, the dynamic conditions will be given by:

$$\vec{t}\cdot(\Sigma\cdot\vec{n}) = 0 \tag{21}$$

$$P_1 - P_2 = -\gamma(\nabla\cdot\vec{n}) \Rightarrow \vec{n}\cdot(\Sigma\cdot\vec{n}) - \vec{n}\cdot(-P_0\vec{n}) = -\gamma(\nabla\cdot\vec{n}) \ \ ; \ \ y = \eta(x,t) \tag{22}$$

where γ is the surface tension and the mean local curvature $\nabla\cdot\vec{n}$ is defined as,

$$-\nabla\cdot\vec{n} = \frac{\frac{\partial^2\eta(x,t)}{\partial x^2}}{\left[1+\left(\frac{\partial\eta(x,t)}{\partial x}\right)^2\right]^{\frac{3}{2}}} \tag{23}$$

Linearizing Eq.(23) we have,

$$-\nabla \cdot \vec{n} = \frac{\partial^2 \eta(x,t)}{\partial x^2} \tag{24}$$

Inserting the normal modes in Eqs. (21) and (22) and linearizing around $y = 0$, we obtain:

$$D^2\hat{\Psi}(0) + \alpha^2\hat{\Psi}(0) + \hat{\eta}D^2\overline{U}(0) = 0 \tag{25}$$

$$-D^3\hat{\Psi}(0) + [3\alpha^2 - i\alpha Re(c-1)]D\hat{\Psi}(0) + i\alpha Re\left[\frac{1}{Fr} + \frac{\alpha^2}{We}\right]\hat{\eta} = 0 \tag{26}$$

5. Asymptotic Solution.

This section is devoted to the solutions to the equation of Orr-Sommerfeld together with the boundary conditions of the problem. To find these solutions we will expand the eigenfunction $\hat{\Psi}(y)$ and the eigenvalue c in power series of α, from $O(1)$ to $O(\alpha^2)$.

5.1 Solution for $O(1)$

Considering,

$$\alpha << 1; \;\; Re = O(1); \;\; \frac{We}{\alpha^2} = O(1) \tag{27}$$

For a long wave disturbance, the wave number α can be treated as a small parameter. The equations suggest the speed c and the amplitude $\hat{\Psi}$ of eigenfunctions can be treated as a power series of α as follows:

$$\hat{\Psi}(y) = \hat{\Psi}_0(y) + \alpha\hat{\Psi}_1(y) + \alpha^2\hat{\Psi}_2(y) + ... \tag{28}$$

$$c = c_0 + \alpha c_1 + \alpha^2 c_2 + ... \tag{29}$$

In order to achieve an approximate solution we will replace Eqs. (28) and (29) into the Orr-Sommerfeld equation and the terms of the same order will be collected. The very same procedure will be applied on the boundary conditions.

At $O(1)$:

$$D^4\hat{\Psi}_0(y) = 0 \tag{30}$$

$$\hat{\Psi}_0(-1) = 0 \tag{31}$$

$$D\hat{\Psi}_0(-1) = 0 \tag{32}$$

$$\hat{\Psi}_0(0) - (c_0 - 1)\hat{\eta} = 0 \tag{33}$$

$$D^3\hat{\Psi}_0(0) = 0 \tag{34}$$

$$D^2\hat{\Psi}_0(0) - 2\frac{\hat{\Psi}_0(0)}{c_0 - 1} = 0 \tag{35}$$

From equations (30) to (35),

$$\hat{\Psi}_0(y) = \hat{\eta}(y+1)^2; \;\; c_0 = 2 \tag{36}$$

where $\hat{\eta}$ is the amplitude of deformation of the interface. The eigenvalue c_0 is real and independent of wave number; therefore, all disturbances are propagated with the same speed $2U_0$, independent of the wave (non-dispersive). Since the imaginary part of c_0 is zero, the growth rate of instability is zero, so there is no instability at $O(1)$.

5.2 Solution for $O(\alpha)$

At $O(\alpha)$:

$$D^4\hat{\Psi}_1(y) = 4iRe\hat{\eta}y \tag{37}$$

$$\hat{\Psi}_1(-1) = 0 \tag{38}$$

$$D\hat{\Psi}_1(-1) = 0 \tag{39}$$

$$\hat{\Psi}_1(0) = c_1\hat{\eta} \tag{40}$$

$$D^2\hat{\Psi}_1(0) = 0 \tag{41}$$

$$D^3\hat{\Psi}_1(0) = -2iRe\hat{\eta} + iRe\hat{\eta}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right) \tag{42}$$

By integrating Eq. (37), and applying Eq. (38) to Eq. (42) we find:

$$\hat{\Psi}_1(y) = iRe\hat{\eta}\left\{\frac{y^5}{30} + \left[-\frac{1}{3} + \frac{1}{6}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right]y^3 + \left[\frac{5}{6} - \frac{1}{2}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right]y + \frac{8}{15}\left[1 - \frac{5}{8}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right]\right\} \tag{43}$$

$$c_1 = iRe\frac{8}{15}\left[1 - \frac{5}{8}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right] \tag{44}$$

At $O(1)$ the solution is purely imaginary, and does not contribute to the wave speed, but affects the growth rate $\sigma = \alpha c_i = \alpha^2 c_{1i}$ significantly,

$$\sigma = \alpha^2 c_{1i} = \alpha^2 Re\frac{8}{15}\left[1 - \frac{5}{8}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right] = \frac{\alpha^2 Re}{3}\left(\frac{1}{\frac{5}{8}} - \frac{1}{Fr}\right) - \frac{Re}{3We}\alpha^4 \tag{45}$$

Therefore, we can write,

$$\sigma = \frac{\alpha^2 Re}{3}\left(\frac{1}{Fr_c} - \frac{1}{Fr}\right) - \frac{Re}{3We}\alpha^4 \quad \text{where} \quad Fr_c = \tfrac{5}{8} \tag{46}$$

When $Fr < Fr_c$, σ is negative for every α, and the flow of the liquid film is linearly stable. For $Fr > Fr_c$, perturbations of wave number below α_c will be amplified. We can find α_c by,

$$\sigma = 0 \Leftrightarrow \alpha^2\left[\frac{Re}{3}\left(\frac{1}{Fr_c} - \frac{1}{Fr}\right) - \frac{Re}{3We}\alpha^2\right] = 0 \tag{47}$$

Excluding the case $\alpha^2 = 0$ we have,

$$\frac{Re}{3}\left(\frac{1}{Fr_c} - \frac{1}{Fr}\right) - \frac{Re}{3We}\alpha^2 = 0 \Leftrightarrow \alpha_c^2 = We\left(\frac{1}{Fr_c} - \frac{1}{Fr}\right) \text{ with } Fr_c = \tfrac{5}{8} \tag{48}$$

Perturbations with wave number $\alpha > \alpha_c$ are attenuated due to the combined effect of surface tension and viscosity. The number $Fr_c = \frac{5}{8}$ is the critical Froude number above which the liquid film is unstable.

5.3 Solution for $O(\alpha^2)$

At $O(\alpha^2)$:

$$D^4\hat{\Psi}_2(y) = 4\hat{\eta} - Re^2\hat{\eta}\left\{-\frac{3}{5}y^5 + \left[\frac{2}{3} - \frac{2}{3}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right]y^3 + \left[\frac{11}{3} - 2\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right]y\right\} \tag{49}$$

$$\hat{\Psi}_2(-1) = 0 \tag{50}$$

$$D\hat{\Psi}_2(-1) = 0 \tag{51}$$

$$\hat{\Psi}_2(0) = \hat{\eta} c_2 \tag{52}$$

$$D^2\hat{\Psi}_2(0) = -\hat{\eta} \tag{53}$$

$$D^3\hat{\Psi}_2(0) = 6\hat{\eta} + Re^2\hat{\eta}\left[\frac{121}{30} - \frac{5}{2}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right] \tag{54}$$

Integrating Eq. (49) and replacing Eq. (50) - Eq. (54) we obtain,

$$\hat{\Psi}_2(y) = \frac{\hat{\eta}}{6}y^4 - \frac{Re^2\hat{\eta}}{60}\left\{-\frac{1}{84}y^9 + \frac{1}{21}\left[1 - \left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right]y^7 + \left[\frac{11}{6} - \left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right]y^5\right\} + \frac{A}{6}y^3 + \frac{B}{2}y^2 + Cy + D \tag{55}$$

$$c_2 = -2 - \frac{128}{105}Re^2\left[1 - \frac{5}{8}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right] \tag{56}$$

where A, B, C and D are, respectively,

$$A = 6\hat{\eta} + Re^2\hat{\eta}\left[\frac{121}{30} - \frac{5}{2}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right] \tag{57}$$

$$B = -\hat{\eta} \tag{58}$$

$$C = -\frac{10\hat{\eta}}{3} - Re^2\hat{\eta}\left[\frac{625}{336} - \frac{209}{180}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right] \tag{59}$$

$$D = \hat{\eta} c_2 \tag{60}$$

Performing the calculations at $O(\alpha^2)$ we found a correction for the real part of eigenvalue c. This correction only affects the wave speed, therefore, at $O(\alpha^2)$, long wavelengths are weakly dispersive (Benney, 1966).

6. Discussion and Results.

As we had seen, no instabilities appear at $O(1)$. Since the ratio of amplitudes $\hat{\eta}$ and $\hat{\Psi}_0$ are real, the interface η and the stream function Ψ are in phase. The speed perturbations $u = \frac{\partial \Psi}{\partial y}$ and $v = -\frac{\partial \Psi}{\partial x}$ are, respectively, in and out of phase with the interface. For smaller orders the flow does not manifest any instability for long-wave disturbances, (Charru, 2011). At $O(\alpha)$, we found the correction of eigenvalue $c_1 = iRe\frac{8}{15}[1 - \frac{5}{8}(\frac{1}{Fr} + \frac{\alpha^2}{We})]$, which affects the growth rate of instability. This correction generates as a result a critical Froude number equal to $\frac{5}{8}$, which from Eq. (5) provides the critical Reynolds number $Re_c = \frac{5}{4}cot(\theta)$; therefore, the same condition discussed at the end of section (5.2) can be applied for the critical Reynolds number. This results agrees with Benney's results (Benney, 1966) for the $O(\alpha)$, which is given by $c_1 = iRe(Re - \frac{5}{4}cot(\theta)) = iRe^2(1 - \frac{5}{8}\frac{1}{Fr})$. Benney disregarded the contributions of Weber number until $O(\alpha^3)$, however the same criteria for the onset of instabilities was obtained. The solution for $O(\alpha^2)$ was $c_2 = -2 - \frac{128}{105}Re^2[1 - \frac{5}{8}(\frac{1}{Fr} + \frac{\alpha^2}{We})]$ which affects the phase velocity. The value found by Benney was $c_2 = -2 - \frac{32}{63}Re(Re - \frac{5}{4}cot(\theta)) = -2 - \frac{32}{63}Re^2(1 - \frac{5}{8}\frac{1}{Fr})$ which shows a difference of 2.4, approximately, in the multiplicative constant. This difference is due to the contribution of $\frac{\alpha^2}{We}$ terms that are associated with the surface tension.

We have plot the growth rate $\sigma(\alpha)$ (see Fig. 2), and the stability diagram $\alpha(Fr)$ (see Fig. 3), given by Eq. (45) and Eq. (48). We use the properties of water, with a 0.1 mm thickness, as reference in these plots. In Fig. (2) it was used $\frac{\pi}{8} < \theta < \frac{\pi}{5.8}$ in order to obtain the $Fr < Fr_c$ and $Fr > Fr_c$. Figure 3 presents the stability diagram with the marginal stability curve ($\sigma = 0$, and $0 < \theta < \frac{\pi}{2.5}$), which separates the stable and unstable domains. This diagram shows that the width of the unstable band tends to zero at the threshold $Fr = Fr_c$.

7. Conclusion.

The asymptotic analysis is a useful method to provide a good physical sense of the problem. Performing the analysis at $O(1)$ to $O(\alpha)$ we found the celerity and the growth rate of instability, both in good agreement with Benney (1966). At the $O(\alpha^2)$, our result for c_2 differs from Benney's result by a factor 2.4. This difference is due to the fact that in our resolution, we consider the contributions of $\frac{\alpha^2}{We}$ at $O(\alpha^2)$, while Benney only consider the contributions of Weber number starting from $O(\alpha^3)$. With the correction c_1 at $O(\alpha)$, it was possible to find the critical Froude number, which determines a critical condition between the effects of inertia and gravity: when $Fr < \frac{5}{8}$ the gravity effect dominates, and the liquid film is stable, and when $Fr > \frac{5}{8}$ the inertial effects dominates the flow and the liquid film become unstable. In possession of the critical Froude number we developed an expression for the instability curves, as well as the marginal stability diagram.

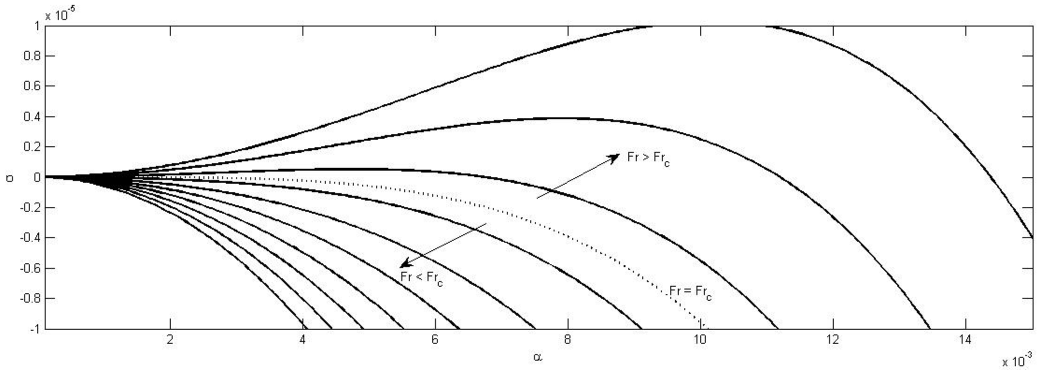

Figure 2. Behavior of the growth rate $\sigma(\alpha)$ for $Fr < Fr_c$ and $Fr > Fr_c$.

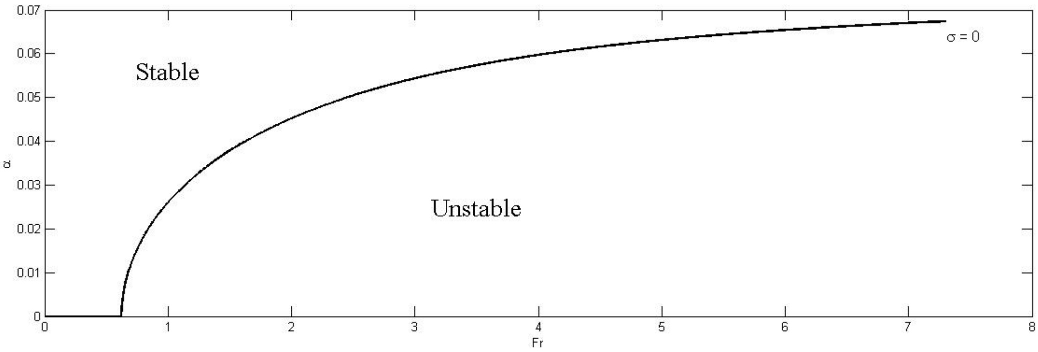

Figure 3. Stability diagram.

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